ON A CLASS OF SEMILINEAR ELLIPTIC EQUATIONS WITH BOUNDARY CONDITIONS AND POTENTIALS WHICH CHANGE SIGN

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We study the existence of nontrivial solutions for the problem $\Delta u = u$, in a bounded smooth domain $\Omega \subset \mathbb{R}^N$, with a semilinear boundary condition given by $\frac{\partial u}{\partial \nu} = \lambda u - W(x)g(u)$, on the boundary of the domain, where $W$ is a potential changing sign, $g$ has a superlinear growth condition, and the parameter $\lambda \in ]0, \lambda_1]$; $\lambda_1$ is the first eigenvalue of the Steklov problem. The proofs are based on the variational and min-max methods.

1. Introduction

In this paper, we study the existence of nontrivial solutions of the following problem:

$$(P_\lambda) \quad \begin{array}{ll}
\Delta u = u & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = \lambda u - W(x)g(u) & \text{on } \partial \Omega,
\end{array}$$

where $\Omega$ is a bounded domain set of $\mathbb{R}^N$, $N \geq 3$ with smooth boundary $\partial \Omega$, $\Delta u = \nabla \cdot (\nabla u)$ is the Laplacian and $\partial/\partial \nu$ is the outer normal derivative; the parameter $\lambda \in ]0, \lambda_1]$, where $\lambda_1$ is the first eigenvalue of the Steklov problem (see [5]), $W \in C(\overline{\Omega})$ different from zero almost everywhere and changes sign, while $g(u)$ is a continuous and superlinear function (see (G1), (G2), (G3)) below.

In the case of $W \equiv 0$, $(P_\lambda)$ becomes a linear eigenvalue problem and it is known as the Steklov problem studied in [5], which proved the existence, the simplicity, and the isolation of the first eigenvalue $\lambda_1$.

The study of the similar problem when the nonlinear term is placed in the equation, that is, when one considers problem of the form $-\Delta u = \lambda u + W(x)g(u)$ with Dirichlet boundary condition, there is more work; hence, in the case where $g$ behaves as a power near 0 and infinity, Alama and Tarantello in [2] showed the existence of a positive solution, provided that $f$ is odd, and found that a necessary and sufficient condition to obtain...
such a solution is

$$\int_{\Omega} W(x)e_1^p dx < 0,$$

where $e_1$ denotes a positive eigenfunction of Laplacian related to the first eigenvalue, with $p \in ]2,2^*[$, $2^* = 2N/(N-2)$ if $N > 2$, $2^* = +\infty$ if $N = 2$. Also, in [3], it was proved that (1.2) is a necessary and sufficient condition to obtain a positive solution; recently, Mar-gone in [14], proved some results of existence in case that $0 < \lambda < \lambda_1$, close to $\lambda_1$; by using mountain pass lemma (see [4]) and linking-type theorem (see [15]). Finally, in [1], Alama and Delpino proved under some restriction on the sign of $W(x)$ the existence of nontrivial solution, by using two different approach: one involving min-max methods, the other Morse theory methods.

However, nonlinear boundary conditions have only been considered in recent years, for the Laplacian with boundary conditions, see, for example [6, 7, 8, 12, 13, 16], where the authors discussed mountain pass theorem on an order interval with Dirichlet boundary condition. For elliptic systems with nonlinear boundary conditions, see [9, 10]. The main purpose of this work is to study one problem of Neumman boundary value, in the case $\lambda = \lambda_1$ because if $\lambda < \lambda_1$, it is easy to prove that the functional $\Phi_\lambda$ has a condition of mountain pass structure. We show two results of existence obtained as critical points of the functional related at $(P_\lambda)$, by using mountain pass lemma introduced in [4] and linking-type theorem introduced in [15].

The rest of this paper is organized as follows: in Section 2, we cite the main results and in Section 3, we prove the main results.

2. Main results

In the sequel, we consider the following functional:

$$G(u) = \int_0^u g(t)dt.$$

Then, we show the following existence results for $(P_\lambda)$.

**Theorem 2.1.** Let $g$ be a continuous real-valued function on $\mathbb{R}$ such that the following assumptions hold:

1. $g(u)u \geq 0$ for all $u \in \mathbb{R}$,
2. $|g(u)| \leq C|u|^{r-1}$ for all $u \in \mathbb{R}$, and for some $r \in ]2,2(N-1)/(N-2)[$,
3. $g(u)u \geq (s+1)G(u)$ for $u > R$, $R$ sufficiently large, and for some $s \in ]1, N/(N-2)[$,
4. $\lim_{u \to 0}(g(u)/|u|^{r-2}u) = a > 0$,
5. $g(u)u \geq c|u|^{s+1}$ for $|u| > R$, and $R$ sufficiently large,
6. $W^- (g(u)u - (s+1)G(u)) \leq \gamma |u|^2$, $|u| > R$, for some

$$\gamma \in \left[0, \left(\frac{s+1}{2} - 1\right)(\lambda_2 - \lambda_1)\right],$$

where $\lambda_2$ is the second eigenvalue of the Steklov problem, and $W^- (x) = -\min \{W(x), 0\}$, $W^- = \max_{x \in \partial \Omega} W^-(x)$; moreover, let
Theorem 2.3. Let \( g \) satisfy conditions (G1)–(G3), (G5), (G6), and (W0). If \( W \) verifies the further assumptions,

\[(W_2) \int_{\partial \Omega} W(x)G(te_1)\,d\sigma > 0, \text{ for all } t \in \mathbb{R} \setminus \{0\},\]
\[(W_3) \int_D W(x)G(te_1)\,d\sigma > c, \text{ for all } t \in \mathbb{R} \text{ and for some } c \in \mathbb{R}, \text{ where } D \text{ is a nonempty open subset in } \partial \Omega \text{ such that } \text{supp } W^- \subset D,\]

then \((P_\lambda)\) has a nontrivial solution.

Remark 2.4. Note that the solution found in Theorem 2.3 is surely not always positive because \((W_1)\) does not hold. Moreover, condition \((W_2)\), which appears in Theorem 2.3, is in some sense complementary to \((W_1)\) if \( g \) is a power.

3. Proof of the main results

It is well known that the solutions of \((P_\lambda)\) are critical points of the functional

\[
\Phi_\lambda(u) = \frac{1}{2} \left( \| \nabla u \|_2^2 + \| u \|_2^2 - \lambda \int_{\partial \Omega} |u|^2\,d\sigma \right) - \int_{\partial \Omega} W(x)G(u)\,d\sigma, \quad u \in H^1(\Omega). \tag{3.1}
\]

In order to prove the main results, we apply the mountain pass theorem (see [4]) and a suitable version of the linking-type theorem (see [15]) to the functional \( \Phi_\lambda \).

The following lemma is the key for proving our theorems, in which we consider \( \lambda = \lambda_1 \) because if \( \lambda < \lambda_1 \), the argument is the same.

Lemma 3.1. Under assumptions \((W_0)\), (G2), (G3), (G5), (G6), the functional \( \Phi_\lambda(u) \) satisfies the Palais-Smale condition on \( H^1(\Omega) \). That is, any sequence \((u_n)_n \) in \( H^1(\Omega) \), such that

\[
(\Phi_\lambda(u_n))_n \text{ is bounded and } \Phi'_\lambda(u_n) \to 0 \tag{3.2}
\]

possesses a converging subsequence.

Proof. Let \((u_n)_n \subset H^1(\Omega) \) be a Palais-Smale sequence, namely, there exist \( c_1 \) and \( c_2 \) such that

\[
c_1 \leq \frac{1}{2} \left( \| \nabla u_n \|_2^2 + \| u_n \|_2^2 - \lambda_1 \int_{\partial \Omega} |u_n|^2\,d\sigma \right) - \int_{\partial \Omega} W(x)G(u_n)\,d\sigma \leq c_2, \tag{3.3}
\]

\[
\sup_{\| \phi \|_{H^1(\Omega)} = 1} \left\{ \int_{\Omega} \nabla u_n \nabla \phi + u_n \phi \,dx - \lambda_1 \int_{\partial \Omega} u_n \phi\,d\sigma \right. \nonumber
\]
\[
- \left. \int_{\partial \Omega} W(x)g(u_n)\phi\,d\sigma \right\} \to 0 \quad \text{as } n \to +\infty. \tag{3.4}
\]
We are going to show that \((u_n)_n\) is bounded in \(H^1(\Omega)\). By assumptions \((G3)\) and \((G6)\), and from (3.3) and (3.4), we get for some constant \(c_R > 0\) depending on the number \(R\) of \((G3),\)
\[
\int_\Omega (|\nabla u_n|^2 + u_n^2) \, dx = \lambda_1 \int_{\partial \Omega} u_n^2 \, d\sigma - \int_{\partial \Omega} W(x) g(u_n) u_n \, d\sigma + \epsilon_n ||u_n||_{1,2}
\geq \lambda_1 \int_{\partial \Omega} u_n^2 \, d\sigma + \int_{\partial \Omega} W^+(x) g(u_n) u_n \, d\sigma
- \int_{\partial \Omega} W^-(x) g(u_n) u_n \, d\sigma + c_R \epsilon_n ||u_n||_{1,2}
\geq \lambda_1 \int_{\partial \Omega} u_n^2 \, d\sigma + (s+1) \int_{\partial \Omega} W^+(x) G(u_n) \, d\sigma - \gamma \int_{\partial \Omega \cap \{|u| > R\}} |u_n|^2 \, d\sigma
- (s+1) \int_{\partial \Omega \cap \{|u| > R\}} W^-(x) G(u_n) \, d\sigma + c_R \epsilon_n ||u_n||_{1,2}
\geq \lambda_1 \int_{\partial \Omega} u_n^2 \, d\sigma + (s+1) \left[ \frac{1}{2} ||u_n||_{1,2}^2 - \frac{\lambda_1}{2} \int_{\partial \Omega} u_n^2 \, d\sigma - c_1 \right]
- \gamma \int_{\partial \Omega} u_n^2 \, d\sigma + c_R \epsilon_n ||u_n||_{1,2}.
\]
(3.5)

Set \(X_1 = \text{vect}(e_1)\), then, there exist \(k_n \in \mathbb{R}\) such that \(u_n = k_n e_1 + v_n\), where \(v_n \in X_1^\perp\).

Using the variational characterization of \(\lambda_2\), (3.5) becomes
\[
\left( \frac{s+1}{2} - 1 \right) \left( 1 - \frac{\lambda_1}{\lambda_2} \right) ||v_n||_{1,2}^2 + \epsilon_n ||v_n||_{1,2} \leq \gamma \int_{\partial \Omega} (k_n e_1 + v_n)^2 \, d\sigma + c,
\]
(3.6)
where \(\epsilon_n\) is an infinitesimal sequence of positive numbers.

On the other hand, using variational characterization of \(\lambda_1\), it follows that
\[
\left[ \left( \frac{s+1}{2} - 1 \right) \left( 1 - \frac{\lambda_1}{\lambda_2} \right) - \frac{\gamma}{\lambda_2} \right] ||v_n||_{1,2}^2 + \epsilon_n ||v_n||_{1,2} \leq c + yk_n^2 \int_{\partial \Omega} e_1^2 \, d\sigma.
\]
(3.7)

On the other side, by (2.2) and taking into account that \(\epsilon_n \to 0\), we deduce that
\[
||v_n||_{1,2} \leq \text{const} \left( 1 + k_n^2 \right),
\]
(3.8)
hence, it suffices to prove that \((|k_n|)_n\) is bounded. So, if \(|k_n| \to +\infty\) (at least a subsequence), therefore \((v_n/|k_n|)_n\) is bounded in \(H^1(\Omega)\), so a subsequence, also called \((v_n/|k_n|)_n\), weakly converges in \(H^1(\Omega)\) at some \(f\) and that
\[
f(x) + e_1(x) \neq 0 \quad \text{a.e. in } \overline{\Omega}.
\]
(3.9)

Indeed, if (3.9) is false, taking into account that
\[
\int_{\Omega} \left( \nabla \left( \frac{v_n}{|k_n|} \right) \right) \nabla e_1 + \frac{v_n}{|k_n|} e_1 \, dx = 0 \quad \forall n \in \mathbb{N}
\]
(3.10)
as \( n \to +\infty \), we obtain \( \|e_1\|_{L^2,\Omega}^2 = \lambda_1 \int_{\partial \Omega} e_1^2 = 0 \), which is an absurdum as we know that \( e_1 \) is the principal eigenvector related with \( \lambda_1 \).

From (3.4), we obtain

\[
\int_{\Omega} (\nabla u_n \nabla \phi + u_n \phi) \, dx - \lambda_1 \int_{\partial \Omega} u_n \phi \, d\sigma - \int_{\partial \Omega} W(x)g(u_n) \phi \, d\sigma = \eta_n \tag{3.11}
\]

with \( \lim_{n \to +\infty} \eta_n = 0 \) in \( \mathbb{R} \).

Let \( \phi_n = (k_n e_1 + v_n)|k_n|^{-1} \phi \), where \( \phi \) is a regular function with support compact in \( \overline{\Omega} \) and \( \text{meas}(\text{supp} \phi \cap \partial \Omega) \neq 0 \); then

\[
\int_{\Omega} (\nabla (k_n e_1 + v_n) \nabla \phi_n + (k_n e_1 + v_n) \phi_n) \, dx \\
- \lambda_1 \int_{\partial \Omega} (k_n e_1 + v_n) \phi_n \, d\sigma - \int_{\partial \Omega} W(x)g(k_n e_1 + v_n) \phi_n \, d\sigma = \eta_n, \tag{3.12}
\]

hence

\[
\frac{1}{|k_n|} \int_{\Omega} [\nabla v_n \nabla \phi_n + v_n \phi_n] \, dx - \frac{\lambda_1}{|k_n|} \int_{\partial \Omega} v_n \phi_n \, d\sigma
= \frac{1}{|k_n|} \int_{\partial \Omega} W(x)g(k_n e_1 + v_n) \phi_n \, d\sigma + o(1) \tag{3.13}
\]

for \( n \) large enough.

So, Hölder inequality and (3.8) imply that \((1/|k_n|) \int_{\Omega} (\nabla v_n \nabla \phi_n + v_n \phi_n) \, dx\) and \((\lambda_1/|k_n|) \int_{\partial \Omega} v_n \phi_n \, d\sigma\) are bounded.

On the other side, combining (\( W_0 \)) and (3.9), it follows that either

\[
\int_{\text{supp} \, W^+} |h(x) + e_1(x)|^{s+1} \, d\sigma > 0 \quad \text{or} \quad \int_{\text{supp} \, W^-} |h(x) + e_1(x)|^{s+1} \, d\sigma > 0. \tag{3.14}
\]

In the first case, we take \( \phi \) regular nonnegative function with \( \text{meas}(\text{supp} \phi \cap \text{supp} \, W^+) \neq 0 \) such that

\[
\int_{\text{supp} \, W^+} W^+(x) \phi(x) |h(x) + e_1(x)|^{s+1} \, d\sigma > 0, \tag{3.15}
\]

then, by \((G6)\) and (3.15), we get for some positive constant \( c \),

\[
\frac{1}{|k_n|} \int_{\partial \Omega} W(x)g(k_n e_1 + v_n) \phi_n \, d\sigma \geq \frac{c}{|k_n|} \int_{\text{supp} \, W^+} W^+(x) |k_n e_1 + v_n|^{s+1} \phi \, d\sigma - c
\geq c|k_n|^{-1} \int_{\text{supp} \, W^+} W^+(x) \left| e_1 + \frac{v_n}{k_n} \right|^{s+1} \phi \, d\sigma - c \to +\infty. \tag{3.16}
\]

This and formula (3.13) contradict the bound of \((1/|k_n|) \int_{\Omega} (\nabla v_n \nabla \phi_n + v_n \phi_n) \, d\sigma\) and \((\lambda_1/|k_n|) \int_{\partial \Omega} v_n \phi_n \, d\sigma\).
For the second case, it suffices to take $\phi$ nonnegative function with $\text{meas}(\text{supp} \phi \cap \text{supp} W^-) \neq 0$ such that

$$\int_{\text{supp} W^-} W^-(x) \phi(x) |h(x) + e_1(x)|^{s+1} \, d\sigma > 0. \quad (3.17)$$

Finally, we have proved that $(u_n)_n$ is bounded, this implies the existence of a subsequence weakly converging in $H^1(\Omega)$. On the other side, thanks to $(G2)$ and the compact embedding $H^1(\Omega) \hookrightarrow L^r(\partial \Omega)$ for $r \in ]2, 2(N - 1)/(N - 2)[$, we have the strong convergence. □

**Lemma 3.2.** The origin is a strict locale minimizer of $\Phi_\lambda$.

**Proof.** First, remark that each $u \in H^1(\Omega)$ can be written as $u = te_1 + v$, where $t \in \mathbb{R}$, and $v \in X^\perp$, then

$$\int_\Omega (|\nabla u|^2 + |u|^2) \, dx = t^2 \lambda_1 \int_{\partial \Omega} e_1^2 \, d\sigma + \|v\|^2_{1,2}. \quad (3.18)$$

Choosing $e_1$ such that $\int_{\partial \Omega} e_1^2 \, d\sigma = 1/\lambda_1$, one gets, for all $u$ satisfying $\|u\|_{1,2} \leq 1/2 \|e_1\|_\infty$,

$$t^2 < \|u\|^2_{1,2} < \frac{1}{4\|e_1\|_\infty^2}. \quad (3.19)$$

Hence, by variational characterization of the eigenvalues of the Laplacian with boundary conditions and for a suitable function $F(t, v)$, we obtain

$$\Phi_\lambda(u) \geq \frac{1}{2} \left( 1 - \frac{\lambda_1}{\lambda_2} \right) \|v\|^2_{1,2} - \int_{\partial \Omega} W(x) G(te_1 + v) \, d\sigma$$
$$\geq \frac{1}{2} \left( 1 - \frac{\lambda_1}{\lambda_2} \right) \|v\|^2_{1,2} - |t|^r \int_{\partial \Omega} W(x) e_1 \, d\sigma + F(t, v), \quad (3.20)$$

where by $(G4)$,

$$F(t, v) = \int_{\partial \Omega} W(x) \left[ |te_1|^r - G(te_1) \right] \, d\sigma + \int_{\partial \Omega} W(x) \left[ G(te_1) - G(te_1 + v) \right] \, d\sigma$$
$$= \int_{\partial \Omega} W(x) \left[ G(te_1) - G(te_1 + v) \right] \, d\sigma + o(|t|^r). \quad (3.21)$$

On the other hand, using arrangement-finite theorem, there exists a function $0 < \theta \equiv \theta(x, t, v) < 1$ such that

$$|G(te_1 + v) - G(te_1)| = |g(te_1 + \theta v(x)) v(x)| \quad (3.22)$$
In case that \(|te_1 + \theta \nu(x)| \geq 1\), by (3.19), we deduce
\[
|\theta \nu(x)| \geq 2|t||e_1|_\infty - |t||e_1|_\infty \geq |t||e_1|_\infty,
\]
so by (G2),
\[
|g(te_1 + \theta \nu(x)) \nu(x)| \leq C|te_1 + \theta \nu(x)|^{r-1} |\nu(x)| \\
\leq 2^{r-2}C|\theta \nu(x)|^{r-1} |\nu(x)| \leq 2^{r-1}C|\nu(x)|^r,
\]
while, if \(|te_1 + \theta \nu(x)| \leq 1\), using again (G2), one obtains
\[
|W(x)| |g(te_1 + \theta \nu(x)) \nu(x)| \leq C|te_1 + \theta \nu(x)|^{r-1} |\nu(x)| \\
\leq C\left[ |te_1|^{r-1} + |\nu(x)|^r \right] \leq \epsilon |te_1|^{r} + C_\epsilon |\nu(x)|^r,
\]
where \(\epsilon, C_\epsilon\) are two positive constants.

Set \(A = -\int_{\partial \Omega} W(x)e_1' d\sigma > 0\). Combining (3.21), (3.24), and (3.25), and using (W1), (3.20) becomes
\[
\Phi_{\lambda_1}(u) \geq \frac{1}{2} \left( 1 - \frac{\lambda_1}{\lambda_2} \right) \|v\|_{1,2}^2 - t \int_{\partial \Omega} W(x)e_1' - |F(t, \nu)| \\
\geq \frac{1}{2} \left( 1 - \frac{\lambda_1}{\lambda_2} \right) \|v\|_{1,2}^2 + t' A - 2^{r-1}C \int_{\partial \Omega \cap \{|u| > 1\}} |W(x)| |\nu(x)|^r d\sigma \\
- \int_{\partial \Omega \cap \{|u| \leq 1\}} \left[ \epsilon |te_1|^{r} + C_\epsilon |\nu(x)|^{r} \right] + \theta(|t|^{r}) \\
\geq \frac{1}{2} \left( 1 - \frac{\lambda_1}{\lambda_2} \right) \|v\|_{1,2}^2 + t' (A - C_1 \epsilon) - C_2 \|v\|_{1,2}^r + o(|t|^{r}),
\]
where \(C_1, C_2\) are two positive constants.

Hence, using Sobolev trace embedding, for \(\epsilon < A/C_1\), we deduce
\[
\Phi_{\lambda_1}(u) \geq \frac{1}{2} \left( 1 - \frac{\lambda_1}{\lambda_2} \right) \|v\|_{1,2}^2 + C_3 t' - C_4 \|v\|_{1,2}^r + o(|t|^{r}).
\]

For \(r > 2\), the least expression is strictly positive as \(\|v\|_{1,2}\) is close to 0. \(\square\)

Proof of Theorem 2.1. We will study only the case \(\lambda = \lambda_1\) because if \(\lambda < \lambda_1\), it is easily proved that the functional \(\Phi_\lambda\) has a condition of mountain pass structure.

Now, it suffices to prove that there exist \(\pi \in H^1(\Omega)\) such that \(\|\pi\|_{1,2} > \rho, \rho\) large enough satisfying \(\Phi_\lambda(\pi) < 0\) which completes the proof of Theorem 2.3.

Let \(t \in \mathbb{R}\) and \(\phi \in C_0^\infty(\text{supp} \ W^+)\), where \(W^+(x) = \max(W(x), 0)\) (note that \(\phi\) is well defined, thanks to (W0)).

Using (G4), we obtain
\[
\Phi_{\lambda_1}(t\phi) = \frac{t^2}{2} \left( \|\phi\|_{1,2}^2 - \lambda_1 \int_{\partial \Omega} \phi^2 d\sigma \right) - \int_{\partial \Omega} W(x)G(t\phi) d\sigma \\
\leq \frac{t^2}{2} \|\phi\|_{1,2}^2 - Ct' \int_{\text{supp} \ W^+} W^+(x)|\phi|^r d\sigma \rightarrow -\infty \quad \text{as} \ t \rightarrow +\infty.
\]
Then, there exists $t_0 > 0$ large enough, such that $u = t_0 \phi$. Hence, using mountain pass lemma, there exists a critical point $u$ of $\Phi_{\lambda_1}$ at the level

$$c = \inf_{y \in \Gamma} \max_{v \in y([0,1])} \Phi_{\lambda_1}(v) > 0,$$

where $\Gamma = \{ y \in C([0,1], H^1(\Omega)) : y(0) = 0, y(\bar{\Omega}) = 1 \}$ is the class of the path joining the origin to $\bar{\Omega}$.

The positivity of $u$ can be checked by a standard argument based on (3.29) (which yields the nonnegativity of $u$) and by the strong maximum principle of Vazquez [17] (which yields the strict positivity of $u$). □

The proof of Theorem 2.3 is based on Lemma 3.1 and the following version of the linking theorem, see [15].

**Proposition 3.3.** Let $E$ be a real Banach space with $E = X_1 \oplus X_2$, where $X_1$ is finite dimensional. Suppose $J \in C^1(E, \mathbb{R})$ satisfies the Palais-Smale condition and

(J1) there are two constants $\rho, \alpha > 0$ such that $J(u) \geq \alpha$, for all $u \in X_2$: $\|u\|_E = \rho$,

(J2) there exists $\bar{x} \in X_2$ with $\|\bar{x}\| = 1$ and $R > \rho$ such that, if

$$Q = \{ u \in E : u = w + \delta \bar{x} \text{ with } w \in X_1, \|w\| \leq R, \delta \in (0, R) \},$$

then $J|_{\partial Q} \leq 0$.

Then $J$ possesses a critical value $c \geq \alpha$.

**Proof of Theorem 2.3.** Set $E = H^1(\Omega)$ and $J = \Phi_{\lambda}$ in Proposition 3.3.

First, thanks to Lemma 3.1, $\Phi_{\lambda}$ satisfies Palais-Smale condition.

We take $X_1 = \{ t e_1 / t \in \mathbb{R} \}$, then $X_2 = \{ v \in H^1(\Omega)/\int_\Omega v e_1 dx = 0 \}$ and let $v \in X_2$, $\|v\|_{1,2} = \rho$, then

$$\Phi_{\lambda_1}(v) = \frac{1}{2} \int_\Omega (|\nabla v|^2 + |v|^2) dx - \frac{\lambda_1}{2} \int_{\partial \Omega} v^2 d\sigma - \int_{\partial \Omega} W(x) G(u) d\sigma$$

$$\geq \frac{1}{2} \left( 1 - \frac{\lambda_1}{\lambda_2} \right) \|v\|_{1,2}^2 - C \sup_{\partial \Omega} W(x) \int_{\partial \Omega} |v'| d\sigma$$

$$\geq \frac{1}{2} \left( 1 - \frac{\lambda_1}{\lambda_2} \right) \rho^2 - C \rho^r.$$

Then, for $\rho$ small enough, we have $\Phi_{\lambda_1}(v) \geq \alpha$, so (J1) is verified.

As for the proof of (J2), first of all, we note that, as also observed in [15], it is enough to prove the following two properties:

(a) $\Phi_{\lambda_1}(te_1) \leq 0$ for all $t \in \mathbb{R}$;

(b) there exist $\bar{v} \in X_2 \setminus \{0\}$ and $\rho_0 > \rho$ such that $\Phi_{\lambda_1}(u) \leq 0$ for all $u \in X_1 \oplus [\bar{v}]$ and $\|u\| \geq \rho_0$.

For (a), we have

$$\Phi_{\lambda_1}(te_1) = - \int_{\partial \Omega} W(x) G(te_1)$$

which is not positive by (W2), and (a) follows.
On the other side, let $\overline{v}$ be a sufficiently regular function in $X_2 \setminus \{0\}$ such that $\text{supp} \overline{v} \subset \overline{\Omega} \setminus D$ and $\text{meas}(\text{supp} \overline{v} \cap \partial \Omega) \neq 0$. Hence, for $u \in X_1 \oplus [\overline{v}] = \{te_1 + \delta \overline{v}, (t, \delta) \in \mathbb{R}^2\}$, we obtain

$$
\Phi_{\lambda_1}(u) = \frac{\delta^2}{2} \left[ \int_{\Omega} \left( |\nabla \overline{v}|^2 + |\overline{v}|^2 \right) dx - \lambda_1 \int_{\partial \Omega} |\overline{v}|^2 d\sigma \right] - \int_{\partial \Omega} W(x) G(te_1 + \delta \overline{v}) d\sigma \\
\leq \frac{\delta^2}{2} \int_{\Omega} \left( |\nabla \overline{v}|^2 + |\overline{v}|^2 \right) dx - \int_{\partial \Omega \setminus D} W^+(x) G(te_1 + \delta \overline{v}) d\sigma - \int_D W(x) G(te_1) d\sigma + c,
$$

(3.33)

therefore, by $(W_3)$, one gets

$$
\Phi_{\lambda_1}(te_1 + \delta \overline{v}) \leq c(t^2 + \delta^2) - c \int_{\partial \Omega \setminus D} W^+(x) |te_1 + \delta \overline{v}|^{s+1} d\sigma + c.
$$

(3.34)

We observe now that the map

$$
te_1 + \delta \overline{v} \in X_1 \oplus [\overline{v}] \longrightarrow (t, \delta) \in \mathbb{R}^2
$$

is an isomorphism and that

$$
te_1 + \delta \overline{v} \longrightarrow \left( \int_{\partial \Omega \setminus D} W^+(x) |te_1 + \delta \overline{v}|^{s+1} d\sigma \right)^{1/(s+1)}
$$

(3.36)

yields a norm from $X_1 \oplus [\overline{v}]$ as it easily can be deduced from the fact that $-te_1(x) \neq \delta \overline{v}(x)$ in $\overline{\Omega} \setminus D$ if $\delta^2 + t^2 \neq 0$ (indeed $e_1(x) > 0$ everywhere on $\overline{\Omega}$, while $\overline{v}$ has a compact support in $\overline{\Omega} \setminus D$) therefore, as all the norms are equivalents in a finite dimensional space, we get, for some positive constant $c$, 

$$
\Phi_{\lambda_1}(te_1 + \delta \overline{v}) \leq c(t^2 + \delta^2) - c(t^{s+1} + \delta^{s+1}) + c
$$

(3.37)

then,

$$
\lim_{t^2 + \delta^2 \to +\infty} \Phi_{\lambda_1}(te_1 + \delta \overline{v}) = -\infty,
$$

(3.38)

hence, $\Phi_1$ satisfies the assumptions of Proposition 3.3, which completes the proof of Theorem 2.3.

□

References


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