A NOTE ON THE DIFFERENCE SCHEMES FOR HYPERBOLIC-ELLIPTIC EQUATIONS

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The nonlocal boundary value problem for hyperbolic-elliptic equation $d^2u(t)/dt^2 + Au(t) = f(t)$, $(0 \le t \le 1)$, $-d^2u(t)/dt^2 + Au(t) = g(t)$, $(-1 \le t \le 0)$, $u(0) = \varphi$, $u(1) = u(-1)$ in a Hilbert space H is considered. The second order of accuracy difference schemes for approximate solutions of this boundary value problem are presented. The stability estimates for the solution of these difference schemes are established.

1. Introduction

It is known (see [14, 15, 19, 20]) that various boundary value problems for the hyperbolic-elliptic equations can be reduced to the nonlocal boundary value problem

$$\begin{gathered}\frac{d^2u(t)}{dt^2} + Au(t) = f(t) \quad (0 \le t \le 1),\\ -\frac{d^2u(t)}{dt^2} + Au(t) = g(t) \quad (-1 \le t \le 0),\\ u(0) = \varphi, \qquad u(1) = u(-1)\end{gathered} \tag{1.1}$$

for differential equation in a Hilbert space H, with the self-adjoint positive definite operator A.

A function $u(t)$ is called a solution of problem (1.1) if the following conditions are satisfied.

(i) $u(t)$ is twice continuously differentiable in the region $[-1,0)\bigcup(0,1]$ and continuously differentiable on the segment $[-1,1]$. The derivative at the endpoints of the segment are understood as the appropriate unilateral derivatives.

(ii) The element $u(t)$ belongs to $D(A)$ for all $t \in [-1,1]$, and the function $Au(t)$ is continuous on $[-1,1]$.

(iii) $u(t)$ satisfies the equation and boundary value conditions (1.1).

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Theorem 1.1 [13]. *Suppose that $\varphi \in D(A)$, and let $f(t)$ be continuously differentiable on $[0,1]$ and $g(t)$ be continuously differentiable on $[-1,0]$ functions. Then there is a unique solution of the problem (1.1) and the stability inequalities*

$$\begin{aligned}
&\max_{-1\le t\le 1}\|u(t)\|_H \le M\left[\|\varphi\|_H + \max_{-1\le t\le 0}\|A^{-1/2}g(t)\|_H + \max_{0\le t\le 1}\|A^{-1/2}f(t)\|_H\right],\\
&\max_{-1\le t\le 1}\left\|\frac{du}{dt}\right\|_H + \max_{-1\le t\le 1}\|A^{1/2}u(t)\|_H\\
&\qquad \le M\left[\|A^{1/2}\varphi\|_H + \int_{-1}^{0}\|g(t)\|_H dt + \int_0^1\|f(t)\|_H dt\right],\\
&\max_{-1\le t\le 1}\left\|\frac{d^2u}{dt^2}\right\|_H + \max_{-1\le t\le 1}\|Au(t)\|_H\\
&\qquad \le M\left[\|A\varphi\|_H + \|g(0)\|_H + \|f(0)\|_H + \int_{-1}^{0}\|g'(t)\|_H dt + \int_0^1\|f'(t)\|_H dt\right],
\end{aligned} \tag{1.2}$$

hold, where M does not depend on $f(t)$, $t\in[0,1]$, $g(t)$, $t\in[-1,0]$ and φ.

In the paper [13] the first order of accuracy difference scheme for approximately solving the boundary value problem (1.1)

$$\begin{gathered}
\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2} + Au_{k+1} = f_k, \quad f_k = f(t_{k+1}),\ t_{k+1} = (k+1)\tau,\ 1\le k\le N-1,\ N\tau = 1,\\
-\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2} + Au_k = g_k, \quad g_k = g(t_k),\ t_k = k\tau,\ -N+1\le k\le -1,\\
u_0 = \varphi, \quad u_N = u_{-N}, \quad u_1 - u_0 = u_0 - u_{-1}
\end{gathered} \tag{1.3}$$

was investigated.

Theorem 1.2 [6]. *Let $\varphi\in D(A)$. Then for the solution of the difference scheme (1.3) obey the stability inequalities*

$$\begin{aligned}
&\max_{-N\le k\le N}\|u_k\|_H \le M\left[\|\varphi\|_H + \max_{-N+1\le k\le -1}\|A^{-1/2}g_k\|_H + \max_{1\le k\le N-1}\|A^{-1/2}f_k\|_H\right],\\
&\max_{-N+1\le k\le N}\left\|\frac{u_k-u_{k-1}}{\tau}\right\|_H + \max_{-N\le k\le N}\|A^{1/2}u_k\|_H\\
&\qquad \le M\left[\|A^{1/2}\varphi\|_H + \sum_{k=-N+1}^{-1}\tau\|g_k\|_H + \sum_{k=1}^{N-1}\tau\|f_k\|_H\right],
\end{aligned}$$

$$\max_{-N+1\le k\le N-1}\left\|\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}\right\|_H+\max_{-N\le k\le N}\|Au_k\|_H$$
$$\le M\left[\|A\varphi\|_H+\|g_{-1}\|_H+\|f_1\|_H+\sum_{k=-N+1}^{-1}\|g_k-g_{k-1}\|_H+\sum_{k=2}^{N-1}\|f_k-f_{k-1}\|_H\right], \tag{1.4}$$

where M does not depend on τ, φ, and f_k, $1\le k\le N-1$, g_k, $-N+1\le k\le -1$.

Methods for numerical solutions of the nonlocal boundary value problems for partial differential equations have been studied extensively by many researches (see [1, 2, 5, 3, 4, 7–9, 11, 12, 16–18, 21, 22] and the references therein).

In present paper the second order of accuracy difference schemes approximately solving the boundary-value problem (1.1) are presented. The stability estimates for the solution of these difference schemes are established.

2. The second order of accuracy difference schemes

Applying the second order of accuracy difference schemes of paper [10] for hyperbolic equations and the second order of accuracy difference scheme for elliptic equations we will construct the following second order of accuracy difference schemes for approximately solving the boundary value problem (1.1):

$$\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+Au_k+\frac{\tau^2}{4}A^2u_{k+1}=f_k,\quad f_k=f(t_k),\ t_k=k\tau,\ 1\le k\le N-1,\ N\tau=1,$$
$$-\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+Au_k=g_k,\quad g_k=g(t_k),\ t_k=k\tau,\ -N+1\le k\le -1,$$
$$u_0=\varphi,\quad u_N=u_{-N},\quad u_1-u_0-\frac{\tau^2}{2}(f_0-Au_0)=u_0-u_{-1}-\frac{\tau^2}{2}(g_0-Au_0),$$
$$g_0=g(0),\quad f_0=f(0), \tag{2.1}$$

$$\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+\frac{1}{2}Au_k+\frac{1}{4}(Au_{k+1}+Au_{k-1})=f_k,$$
$$f_k=f(t_k),\ t_k=k\tau,\ 1\le k\le N-1,\ N\tau=1,$$
$$-\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+Au_k=g_k,\quad g_k=g(t_k),\ t_k=k\tau,\ -N+1\le k\le -1,\ u_0=\varphi,$$
$$u_N=u_{-N},\quad \left(I+\frac{\tau^2A}{4}\right)(u_1-u_0)-\frac{\tau^2}{2}(f_0-Au_0)=u_0-u_{-1}-\frac{\tau^2}{2}(g_0-Au_0),$$
$$g_0=g(0),\quad f_0=f(0). \tag{2.2}$$

Theorem 2.1. *Let $\varphi \in D(A)$. Then for the solution of the difference scheme (2.1) obey the stability inequalities*

$$\begin{aligned}
&\max_{-N\le k\le N}\|u_k\|_H \le M\Big[\|\varphi\|_H + \max_{-N+1\le k\le 0}\|A^{-1/2}g_k\|_H + \max_{0\le k\le N-1}\|A^{-1/2}f_k\|_H\Big],\\
&\max_{-N+1\le k\le N}\left\|\frac{u_k-u_{k-1}}{\tau}\right\|_H + \max_{-N\le k\le N}\|A^{1/2}u_k\|_H\\
&\qquad\le M\Big[\|A^{1/2}\varphi\|_H + \sum_{k=-N+1}^{0}\tau\|g_k\|_H + \sum_{k=0}^{N-1}\tau\|f_k\|_H\Big],\\
&\max_{-N+1\le k\le N-1}\left\|\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}\right\|_H + \max_{-N\le k\le N}\|Au_k\|_H\\
&\qquad\le M\Big[\|A\varphi\|_H + \|g_0\|_H + \|f_0\|_H + \sum_{k=-N+1}^{0}\|g_k-g_{k-1}\|_H + \sum_{k=1}^{N-1}\|f_k-f_{k-1}\|_H\Big],
\end{aligned}\tag{2.3}$$

where M does not depend on τ, φ, and f_k, $0\le k\le N-1$, g_k, $-N+1\le k\le 0$.

The proof of Theorem 2.1 follows the scheme of the proof of Theorem 1.2 is based on the formulas

$$\begin{aligned}
u_k &= \left(D(\tau A^{1/2}) - D(-\tau A^{1/2})\right)^{-1}\\
&\quad\times\left[\left(D(-\tau A^{1/2}) - I\right)D^{k-1}(\tau A^{1/2}) + \left(I - D(\tau A^{1/2})\right)D^{k-1}(-\tau A^{1/2})\right]u_0\\
&\quad+\left(D(\tau A^{1/2}) - D(-\tau A^{1/2})\right)^{-1}\left(D^k(\tau A^{1/2}) - D^k(-\tau A^{1/2})\right)(u_0-u_{-1})\\
&\quad+\frac{\tau^2}{2}\left(D(\tau A^{1/2}) - D(-\tau A^{1/2})\right)^{-1}\left(D^k(\tau A^{1/2}) - D^k(-\tau A^{1/2})\right)(f_0-g_0)\\
&\quad-\sum_{s=1}^{k-1}\frac{\tau}{2i}A^{-1/2}\left[D^{k-s}(\tau A^{1/2}) - D^{k-s}(-\tau A^{1/2})\right]f_s,\\
&\qquad 1\le k\le N-1,\ D(\pm\tau A^{1/2}) = \left(1\pm i\tau A^{1/2} - \frac{\tau^2 A}{2}\right)^{-1},\\
u_k &= R^{-k}u_0 + \left(I-R^{2N}\right)^{-1}\left(R^{N-k}-R^{N+k}\right)\left[R^N u_0 - u_{-N}\right]\\
&\quad+\left(I-R^{2N}\right)^{-1}\left(R^{N-k}-R^{N+k}\right)\sum_{s=-N+1}^{-1}B^{-1}\left[R^{N-s}-R^{N+s}\right]R^{-1}(2+\tau B)^{-1}g_s\tau\\
&\quad+\sum_{s=-N+1}^{-1}B^{-1}\left(R^{-(k+s)}-R^{|s-k|}\right)(2+\tau B)^{-1}R^{-1}g_s\tau,\\
&\qquad -N+1\le k\le -1,\ R=(1+\tau B)^{-1},\ B=\frac{A\tau + A^{1/2}\sqrt{\tau^2A+4}}{2},
\end{aligned}$$

$$
\begin{aligned}
u_{-N} = T\Big\{ & \left(D(\tau A^{1/2}) - D(-\tau A^{1/2})\right)^{-1} \\
& \times \left[\left(D(-\tau A^{1/2}) - I\right) D^{N-1}(\tau A^{1/2}) + \left(I - D(\tau A^{1/2})\right) D^{N-1}(-\tau A^{1/2}) \right] u_0 \\
& + \left(D(\tau A^{1/2}) - D(-\tau A^{1/2})\right)^{-1} \left(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2})\right) u_0 \\
& - \left(D(\tau A^{1/2}) - D(-\tau A^{1/2})\right)^{-1} \Big(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2}) \Big) \\
& \times \Big\{ R u_0 + \left(I - R^{2N}\right)^{-1} \left(R^{N+1} - R^{N-1}\right) R^N u_0 + \left(I - R^{2N}\right)^{-1} \left(R^{N+1} - R^{N-1}\right) \\
& \quad \times \sum_{s=-N+1}^{-1} B^{-1} \left[R^{N-s} - R^{N+s}\right] R^{-1} (2 + \tau B)^{-1} g_s \tau \\
& \quad + \sum_{s=-N+1}^{-1} B^{-1} \left(R^{1-s} - R^{1+s}\right) (2 + \tau B)^{-1} R^{-1} g_s \tau \Big\} \\
& + \frac{\tau^2}{2} \left(D(\tau A^{1/2}) - D(-\tau A^{1/2})\right)^{-1} \left(D^k(\tau A^{1/2}) - D^k(-\tau A^{1/2})\right) (f_0 - g_0) \\
& - \sum_{s=1}^{N-1} \frac{\tau}{2i} A^{-1/2} \left[D^{N-s}(\tau A^{1/2}) - D^{N-s}(-\tau A^{1/2})\right] f_s \Big\}, \\
T = \Big(I - & \left(I - R^{2N}\right)^{-1} \left(R^{N+1} - R^{N-1}\right) \left(D(\tau A^{1/2}) - D(-\tau A^{1/2})\right)^{-1} \left(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2})\right) \Big)^{-1}
\end{aligned}
\tag{2.4}
$$

and on the estimates

$$
\left\| D(\pm \tau A^{1/2}) \right\|_{H \to H} \le 1, \qquad \tau \left\| A^{1/2} D(\pm \tau A^{1/2}) \right\|_{H \to H} \le 2, \tag{2.5}
$$

$$
\left\| (k\tau B)^{\alpha} R^k \right\|_{H \to H} \le M (1 + \delta \tau)^{-k}, \quad k \ge 1,\ 0 \le \alpha \le 1,\ \delta > 0,\ M > 0, \tag{2.6}
$$

and on the following lemmas.

Lemma 2.2. *The estimate holds:*

$$
\left\| \left[D^N(\pm \tau A^{1/2}) - \exp\{\mp i A^{1/2}\} \right] A^{-1} \right\|_{H \to H} \le \frac{\tau}{2}. \tag{2.7}
$$

Proof. We use the identity

$$
D^N(\pm \tau A^{1/2}) - \exp\{\mp i A^{1/2}\} = \int_0^1 \Psi'(s \tau A^{1/2}) ds, \tag{2.8}
$$

where

$$\Psi(s\tau A^{1/2}) = D^N(\pm s\tau A^{1/2})\exp\{\mp i(1-s)A^{1/2}\}. \tag{2.9}$$

The derivative $\Psi'(s\tau A^{1/2})$ is given by

$$\Psi'(s\tau A^{1/2}) = D^{N+1}(\mp s\tau A^{1/2})\left(\mp i\frac{\tau^2 s^2 A^{3/2}}{2}\right)\exp\{\mp i(1-s)A^{1/2}\}. \tag{2.10}$$

Thus,

$$\begin{aligned} &D^N(\pm \tau A^{1/2}) - \exp\{\mp iA^{1/2}\} \\ &\quad = \mp\int_0^1 D^{N+1}(\pm s\tau A^{1/2})(iA^{3/2})\frac{1}{2}\tau^2 s^2 \exp\{\mp i(1-s)A^{1/2}\}ds. \end{aligned} \tag{2.11}$$

Using the last identity and estimates (2.6) and

$$\|\exp\{\mp i(1-s)A^{1/2}\}\| \le 1, \tag{2.12}$$

we obtain

$$\begin{aligned} &\left\|[D^N(\pm \tau A^{1/2}) - \exp\{\mp iA^{1/2}\}]A^{-1}\right\|_{H\to H} \\ &\quad \le \frac{1}{2}\int_0^1 \left\|D^N(\pm s\tau A^{1/2})\right\|_{H\to H}\tau s\|\tau s A^{1/2}D(\pm s\tau A^{1/2})\|_{H\to H} \\ &\qquad \times \|\exp\{\mp i(1-s)A^{1/2}\}\|_{H\to H}ds \\ &\quad \le \tau\int_0^1 s\,ds = \frac{\tau}{2}. \end{aligned} \tag{2.13}$$

□

Lemma 2.3. *The following estimate holds:*

$$\|T\|_{H\to H} \le M, \tag{2.14}$$

where M does not depend on τ.

Proof. Since

$$\begin{aligned} T &= (I - R^{2N})(I - R^{2N} + (R^{N+1} - R^{N-1}) \\ &\quad \times (D(\tau A^{1/2}) - D(-\tau A^{1/2}))^{-1}(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2})))^{-1}, \end{aligned} \tag{2.15}$$

$$\begin{aligned} &\widetilde{T} - \{I - \exp\{-2A^{1/2}\} + 2A^{1/2}s(1)\exp\{-A^{1/2}\}\}^{-1} \\ &\quad = \widetilde{T}\{I - \exp\{-2A^{1/2}\} + 2A^{1/2}s(1)\exp\{-A^{1/2}\}\}^{-1} \\ &\qquad \times \{R^{2N} - \exp\{-2A^{1/2}\} + 2A^{1/2}s(1)\exp\{-A^{1/2}\} - (R^{N+1} - R^{N-1}) \\ &\qquad\quad \times (D(\tau A^{1/2}) - D(-\tau A^{1/2}))^{-1}(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2}))\}, \end{aligned} \tag{2.16}$$

$$\|\{I - \exp\{-2A^{1/2}\} + 2A^{1/2}s(1)\exp\{-A^{1/2}\}\}^{-1}\|_{H\to H} \le M, \tag{2.17}$$

to prove (2.14) it suffices to establish the estimate

$$\begin{aligned} &\Big\|R^{2N} - \exp\{-2A^{1/2}\} + 2A^{1/2}s(1)\exp\{-A^{1/2}\} - (R^{N+1} - R^{N-1}) \\ &\quad \times (D(\tau A^{1/2}) - D(-\tau A^{1/2}))^{-1}(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2}))\Big\|_{H\to H} \le M\tau. \end{aligned} \tag{2.18}$$

Here

$$\begin{aligned} \widetilde{T} &= (I - R^{2N} + (R^{N+1} - R^{N-1}) \\ &\quad \times (D(\tau A^{1/2}) - D(-\tau A^{1/2}))^{-1}(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2})))^{-1}, \\ s(1) &= A^{-1/2}\frac{e^{iA^{1/2}} - e^{-iA^{1/2}}}{2i}. \end{aligned} \tag{2.19}$$

The estimate (2.17) was proved in [19]. Finally, using the identity

$$\begin{aligned} &R^{2N} - \exp\{-2A^{1/2}\} + 2A^{1/2}s(1)\exp\{-A^{1/2}\} - (R^{N+1} - R^{N-1}) \\ &\quad \times (D(\tau A^{1/2}) - D(-\tau A^{1/2}))^{-1}(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2})) \\ &\quad = R^{2N} - \exp\{-2A^{1/2}\} \\ &\qquad + \Big[2A^{1/2}s(1) - \frac{1}{i}(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2}))\Big]\exp\{-A^{1/2}\} \\ &\qquad + \frac{1}{i}(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2}))[\exp\{-A^{1/2}\} - R^N] \\ &\qquad + \frac{1}{i}(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2})) \\ &\qquad \times [R^N - (R^{N+1} - R^{N-1})(D(\tau A^{1/2}) - D(-\tau A^{1/2}))^{-1}] \end{aligned} \tag{2.20}$$

and the estimates (2.5), (2.6), and (2.7), we obtain the estimate (2.18). □

Theorem 2.4. *Let $\varphi \in D(A^{3/2})$. Then for the solution of the difference scheme (2.2) obey the stability inequalities*

$$\max_{-N\le k\le N}\|u_k\|_H \le M\left[\left\|\left(I\pm\frac{1}{2}i\tau A^{1/2}\right)\varphi\right\|_H+\max_{-N+1\le k\le 0}\|A^{-1/2}g_k\|_H+\max_{0\le k\le N-1}\|A^{-1/2}f_k\|_H\right],$$

$$\max_{-N+1\le k\le N}\left\|\frac{u_k-u_{k-1}}{\tau}\right\|_H+\max_{-N\le k\le N}\|A^{1/2}u_k\|_H$$

$$\le M\left[\left\|A^{1/2}\left(I\pm\frac{1}{2}i\tau A^{1/2}\right)\varphi\right\|_H+\sum_{k=-N+1}^{0}\tau\|g_k\|_H+\sum_{k=0}^{N-1}\tau\|f_k\|_H\right],$$

$$\max_{-N+1\le k\le N-1}\left\|\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}\right\|_H+\max_{-N\le k\le N}\|Au_k\|_H$$

$$\le M\left[\left\|A\left(I\pm\frac{1}{2}i\tau A^{1/2}\right)\varphi\right\|_H+\|g_0\|_H+\|f_0\|_H+\sum_{k=-N+1}^{0}\|g_k-g_{k-1}\|_H+\sum_{k=1}^{N-1}\|f_k-f_{k-1}\|_H\right], \tag{2.21}$$

where M does not depend on τ, φ, and f_k, $0\le k\le N-1$, g_k, $-N+1\le k\le 0$.

The proof of Theorem 2.4 follows the scheme of the proof of Theorem 1.2 is based on the formulas

$$\begin{aligned}
u_k &= \left(D(\tau A^{1/2})-D(-\tau A^{1/2})\right)^{-1}\\
&\quad\times\left[(I-D(-\tau A^{1/2}))D^{k-1}(-\tau A^{1/2})+(D(\tau A^{1/2})-I)D^{k-1}(\tau A^{1/2})\right]u_0\\
&\quad+\left(D(\tau A^{1/2})-D(-\tau A^{1/2})\right)^{-1}\left(D^k(\tau A^{1/2})-D^k(-\tau A^{1/2})\right)\left(I+\frac{\tau^2A}{4}\right)^{-1}(u_0-u_{-1})\\
&\quad+\frac{\tau^2}{2}\left(D(\tau A^{1/2})-D(-\tau A^{1/2})\right)^{-1}\left(D^k(\tau A^{1/2})-D^k(-\tau A^{1/2})\right)\left(I+\frac{\tau^2A}{4}\right)^{-1}(f_0-g_0)\\
&\quad+\sum_{s=1}^{k-1}\left(I+\frac{\tau^2A}{4}\right)^{-1}\left(D(\tau A^{1/2})-D(-\tau A^{1/2})\right)^{-1}\left[D^{k-s}(\tau A^{1/2})-D^{k-s}(-\tau A^{1/2})\right]f_s,\\
&\qquad 1\le k\le N-1,\ D(\pm\tau A^{1/2})=\left(1\mp\frac{i\tau A^{1/2}}{2}\right)\left(I\pm\frac{i\tau A^{1/2}}{2}\right)^{-1},
\end{aligned}$$

$$
\begin{aligned}
u_k = {} & R^{-k}u_0 + \left(I - R^{2N}\right)^{-1}\left(R^{N-k} - R^{N+k}\right)\left[R^N u_0 - u_{-N}\right] \\
& + \left(I - R^{2N}\right)^{-1}\left(R^{N-k} - R^{N+k}\right)\sum_{s=-N+1}^{-1} B^{-1}\left[R^{N-s} - R^{N+s}\right]R^{-1}(2+\tau B)^{-1}g_s\tau \\
& + \sum_{s=-N+1}^{-1} B^{-1}\left(R^{-(k+s)} - R^{|s-k|}\right)(2+\tau B)^{-1}R^{-1}g_s\tau, \\
& \qquad -N+1 \le k \le -1,\ R = (1+\tau B)^{-1},\ B = \frac{A\tau + A^{1/2}\sqrt{\tau^2 A + 4}}{2}, \\
u_{-N} = {} & T\Big\{\left(D(\tau A^{1/2}) - D(-\tau A^{1/2})\right)^{-1} \\
& \times\left[\left(I - D(-\tau A^{1/2})\right)D^{N-1}(-\tau A^{1/2}) + \left(D(\tau A^{1/2}) - I\right)D^{N-1}(\tau A^{1/2})\right]u_0 \\
& + \left(D(\tau A^{1/2}) - D(-\tau A^{1/2})\right)^{-1}\left(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2})\right)\left(I + \frac{\tau^2 A}{4}\right)^{-1}u_0 \\
& - \left(D(\tau A^{1/2}) - D(-\tau A^{1/2})\right)^{-1}\left(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2})\right) \\
& \times\Big\{Ru_0 + \left(I - R^{2N}\right)^{-1}\left(R^{N+1} - R^{N-1}\right)R^N u_0 + \left(I - R^{2N}\right)^{-1}\left(R^{N+1} - R^{N-1}\right) \\
& \quad \times\sum_{s=-N+1}^{-1} B^{-1}\left[R^{N-s} - R^{N+s}\right]R^{-1}(2+\tau B)^{-1}g_s\tau \\
& \quad + \sum_{s=-N+1}^{-1} B^{-1}\left(R^{1-s} - R^{1+s}\right)(2+\tau B)^{-1}R^{-1}g_s\tau\Big\} \\
& + \frac{\tau^2}{2}\left(D(\tau A^{1/2}) - D(-\tau A^{1/2})\right)^{-1}\left(D^k(\tau A^{1/2}) - D^k(-\tau A^{1/2})\right) \\
& \times\left(I + \frac{\tau^2 A}{4}\right)^{-1}(f_0 - g_0) - \sum_{s=1}^{N-1}\frac{\tau}{2i}A^{-1/2}\left[D^{N-s}(\tau A^{1/2}) - D^{N-s}(-\tau A^{1/2})\right]f_s\Big\}, \\
T = {} & \Bigg(I - \left(I - R^{2N}\right)^{-1}\left(R^{N+1} - R^{N-1}\right) \\
& \times\left(D(\tau A^{1/2}) - D(-\tau A^{1/2})\right)^{-1}\left(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2})\right)\left(I + \frac{\tau^2 A}{4}\right)^{-1}\Bigg)^{-1}
\end{aligned}
\tag{2.22}
$$

and on the estimates (2.6) and

$$\|D(\pm\tau A^{1/2})\|_{H\to H}\le 1,\qquad \tau\left\|A^{1/2}\left(I\pm\frac{i\tau A^{1/2}}{2}\right)^{-1}\right\|_{H\to H}\le 2, \tag{2.23}$$

and on the following lemmas.

Lemma 2.5. *The estimate holds:*

$$\left\|\left[D^N(\pm\tau A^{1/2})-\exp\{\mp iA^{1/2}\}\right]A^{-1}\right\|_{H\to H}\le\frac{\tau}{4}. \tag{2.24}$$

Proof. We use the identity

$$D^N(\pm\tau A^{1/2})-\exp\{\mp iA^{1/2}\}=\int_0^1\Psi'(s\tau A^{1/2})\,ds, \tag{2.25}$$

where

$$\Psi(s\tau A^{1/2})=D^N(\pm s\tau A^{1/2})\exp\{\mp i(1-s)A^{1/2}\}. \tag{2.26}$$

The derivative $\Psi'(s\tau A^{1/2})$ is given by

$$\begin{aligned}\Psi'(s\tau A^{1/2})&=D^{N-1}(\pm s\tau A^{1/2})(\pm iA^{1/2})\\&\quad\times\left(-\frac{1}{4}\tau^2s^2A\right)\left(I\pm\frac{1}{2}i\tau A^{1/2}\right)^{-2}\exp\{\mp i(1-s)A^{1/2}\}.\end{aligned} \tag{2.27}$$

Thus,

$$\begin{aligned}&D^N(\pm\tau A^{1/2})-\exp\{\mp iA^{1/2}\}\\&\quad=\mp\int_0^1D^{N-1}(\pm s\tau A^{1/2})(iA^{3/2})\frac{1}{4}\tau^2s^2\left(I\pm\frac{1}{2}i\tau A^{1/2}\right)^{-2}\exp\{\mp i(1-s)A^{1/2}\}ds.\end{aligned} \tag{2.28}$$

Using the last identity and the estimates (2.23) and (2.12), we obtain

$$\begin{aligned}&\left\|[D^N(\pm\tau A^{1/2})-\exp\{\mp iA^{1/2}\}]A^{-1}\right\|_{H\to H}\\&\quad\le\frac{\tau}{2}\int_0^1\left\|D^{N-1}(\pm s\tau A^{1/2})\right\|_{H\to H}s\\&\qquad\times\left\|isA^{1/2}\frac{1}{2}\tau\left(I\pm\frac{1}{2}i\tau A^{1/2}\right)^{-2}\right\|_{H\to H}\\&\qquad\times\left\|\exp\{\mp i(1-s)A^{1/2}\}\right\|_{H\to H}ds\\&\quad\le\frac{\tau}{2}\int_0^1s\,ds=\frac{\tau}{4}.\end{aligned} \tag{2.29}$$

□

Lemma 2.6. *The following estimate holds:*

$$||T||_{H\to H} \le M, \tag{2.30}$$

where M does not depend on τ.

Proof. Since

$$\begin{aligned}
T &= (I - R^{2N})\Big(I - R^{2N} + (R^{N+1} - R^{N-1})(D(\tau A^{1/2}) - D(-\tau A^{1/2}))^{-1} \\
&\qquad \times (D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2}))\Big(I + \frac{\tau^2 A}{4}\Big)^{-1}\Big)^{-1}, \\
&\widetilde{T} - \{I - \exp\{-2A^{1/2}\} + 2A^{1/2}s(1)\exp\{-A^{1/2}\}\}^{-1} \\
&\quad = \widetilde{T}\{I - \exp\{-2A^{1/2}\} + 2A^{1/2}s(1)\exp\{-A^{1/2}\}\}^{-1} \\
&\qquad \times \Big\{R^{2N} - \exp\{-2A^{1/2}\} + 2A^{1/2}s(1)\exp\{-A^{1/2}\} - (R^{N+1} - R^{N-1}) \\
&\qquad\quad \times (D(\tau A^{1/2}) - D(-\tau A^{1/2}))^{-1}(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2}))\Big(I + \frac{\tau^2 A}{4}\Big)^{-1}\Big\}
\end{aligned} \tag{2.31}$$

and (2.17) to prove (2.30) it suffices to establish the estimate

$$\begin{aligned}
&\Big\|R^{2N} - \exp\{-2A^{1/2}\} + 2A^{1/2}s(1)\exp\{-A^{1/2}\} - (R^{N+1} - R^{N-1}) \\
&\times (D(\tau A^{1/2}) - D(-\tau A^{1/2}))^{-1}(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2}))\Big(I + \frac{\tau^2 A}{4}\Big)^{-1}\Big\|_{H\to H} \le M\sqrt{\tau}.
\end{aligned} \tag{2.32}$$

Here

$$\begin{aligned}
\widetilde{T} = \Big(&I - R^{2N} + (R^{N+1} - R^{N-1})(D(\tau A^{1/2}) - D(-\tau A^{1/2}))^{-1} \\
&\times (D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2}))\Big(I + \frac{\tau^2 A}{4}\Big)^{-1}\Big)^{-1}.
\end{aligned} \tag{2.33}$$

Finally, using the identity

$$
\begin{aligned}
&R^{2N} - \exp\{-2A^{1/2}\} + 2A^{1/2}s(1)\exp\{-A^{1/2}\} - (R^{N+1} - R^{N-1}) \\
&\quad \times \left(D(\tau A^{1/2}) - D(-\tau A^{1/2})\right)^{-1}\left(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2})\right)\left(I + \frac{\tau^2 A}{4}\right)^{-1} \\
&= R^{2N} - \exp\{-2A^{1/2}\} + \left[2A^{1/2}s(1) - \frac{1}{i}\left(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2})\right)\right]\exp\{-A^{1/2}\} \\
&\quad + \frac{1}{i}\left(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2})\right)\left[\exp\{-A^{1/2}\} - R^N\right] \\
&\quad + \frac{1}{i}\left(D^N(\tau A^{1/2}) - D^N(-\tau A^{1/2})\right) \\
&\quad \times \left[R^N - (R^{N+1} - R^{N-1})\left(D(\tau A^{1/2}) - D(-\tau A^{1/2})\right)^{-1}\left(I + \frac{\tau^2 A}{4}\right)^{-1}\right]
\end{aligned}
\tag{2.34}
$$

and the estimates (2.6), (2.23), and (2.24), we obtain the estimate (2.32). □

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