ON THE RATE OF THE VOLUME GROWTH FOR SYMMETRIC VISCOUS HEAT-CONDUCTING GAS FLOWS WITH A FREE BOUNDARY

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The system of quasilinear equations for symmetric flows of a viscous heat-conducting gas with a free external boundary is considered. For global in time weak solutions having nonstrictly positive density, the linear in time two-sided bounds for the gas volume growth are established.

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1. Introduction

We consider the system of quasilinear equations describing symmetric flows of a viscous heat-conducting perfect polytropic gas [1]

$$\eta_t = (r^k v)_x, \quad (1.1)$$

$$v_t = r^k \sigma_x, \quad (1.2)$$

$$c_V \theta_t = (r^k \pi)_x + \sigma (r^k v)_x - 2k\mu(r^{k-1} v^2)_x, \quad (1.3)$$

$$r_t = v, \quad (1.4)$$

$$\sigma = \nu \rho (r^k v)_x - R \rho \theta, \quad \pi = \kappa \rho r^k \theta_x, \quad \rho = \frac{1}{\eta}, \quad (1.5)$$

in the domain $Q := \Omega \times \mathbb{R}^+ = (0, M) \times (0, \infty)$. The system is supplemented with the boundary and initial conditions

$$v|_{x=0} = 0, \quad \left(\sigma - 2k\mu \frac{v}{r} \right) \Big|_{x=M} = 0, \quad (r^k \pi)|_{x=0, M} = 0 \quad \text{for } t > 0, \quad (1.6)$$

$$\{\eta, v, \theta, r\}|_{t=0} = \{\eta^0(x), v^0(x), \theta^0(x), r^0(x)\} \quad \text{for } x \in \Omega. \quad (1.7)$$

The parameter k takes the values 1 or 2 accordingly to the cylindrical or spherical symmetry.

2 The rate of the volume growth

The unknown functions $\eta > 0$, v , $\theta \geq 0$ and $r \geq r_0$ depend on the Lagrangian mass coordinates (x, t) and denote the specific volume, the velocity, the absolute temperature and the Eulerian coordinate that is the radius of a gas particle. The functions ρ , σ and $-\pi$ are the density, the stress and the heat flux. We consider flows around a hard core so that $r_0 > 0$ is its radius, and the internal boundary ($x = 0$) is one with the core. The external boundary ($x = M$) is free; both boundaries are thermally isolated, see (1.6).

The quantities $\nu > 0$, $\mu, R > 0$, $c_V > 0$ and $\varkappa > 0$ are physical constants; $M > 0$ is the total mass of the gas. We impose the standard condition on the viscosity coefficients ν and μ

$$\nu_1 := \nu - \frac{2k}{k+1}\mu > 0. \quad (1.8)$$

The initial function r^0 is not arbitrary but rather connected to η^0 by the physical relation

$$(r^0)^{k+1}(x) = r_0^{k+1} + (k+1) \int_0^x \eta^0(\xi) d\xi \quad \text{for } x \in \overline{\Omega}. \quad (1.9)$$

In the simpler case of the planar symmetry ($k = 0$), the asymptotic behavior of solutions was studied in detail in [5] and more recently in [6, 7] for other boundary conditions. In the case of the spherical symmetry, some results on the growth of the (scaled) gas volume $V(t) := \int_{\Omega} \eta(x, t) dx$ as $t \rightarrow \infty$ are available in [2].

We prove the sharp result establishing the linear growth of V both in the cases of the spherical and cylindrical symmetries like that for the planar one. In contrast to [2, 5–7], we treat essentially more general global in time *weak* solutions to the problem whose density is non-strictly positive only.

2. Results

We introduce the integration operators

$$Iz(x) := \int_0^x z(\xi) d\xi, \quad I^*z(x) := \int_x^M z(\xi) d\xi \quad \text{for } z \in L^1(\Omega). \quad (2.1)$$

They are connected by the identity

$$\int_{\Omega} (Iz_1)z_2 dx = \int_{\Omega} z_1 I^*z_2 dx \quad \text{for any } z_1, z_2 \in L^1(\Omega). \quad (2.2)$$

Let $V_q(Q_T)$ be the space of functions $w \in L^{q,\infty}(Q_T)$ having the derivative $w_x \in L^q(Q_T)$, for $q = 1, 2$ and $Q_T := \Omega \times (0, T)$; recall that $\|w\|_{L^{q,s}(Q_T)} = \|\|w\|_{L^q(\Omega)}\|_{L^s(0,T)}$, for $q, s \in [1, \infty]$.

We study *global in time weak solution* to the problem (1.1)–(1.7) such that:

(1) the properties

$$\begin{aligned} \eta, \eta_t &\in L^{1,2}(Q_T), & \frac{1}{\eta} &\in L^\infty(Q_T), & v &\in V_2(Q_T), \\ \theta &\in V_1(Q_T), & r, r_x, r_t &\in L^{1,\infty}(Q_T) \end{aligned} \quad (2.3)$$

together with $\eta > 0$, $\theta \geq 0$, $r \geq r_0$ (almost everywhere in Q_T) and $v|_{x=0} = 0$ are valid;

(2) equations (1.1) and (1.4) together with the initial conditions $\eta|_{t=0} = \eta^0$ and $r|_{t=0} = r^0$ are satisfied;

(3) the integral identities

$$\int_{Q_T} \left\{ -v\varphi_t + \sigma(r^k\varphi)_x \right\} dx dt = \int_{\Omega} v^0\varphi|_{t=0} dx + 2k\mu \int_0^T (vr^{k-1})|_{x=M}\varphi|_{x=M} dt, \quad (2.4)$$

for any $\varphi \in H^1(Q_T)$ with $\varphi|_{x=0} = 0$ and $\varphi|_{t=T} = 0$, as well as

$$\int_{Q_T} \left\{ -c_V\theta\psi_t + r^k\pi\psi_x - [\sigma(r^k v)_x - 2k\mu(r^{k-1}v^2)_x]\psi \right\} dx dt = \int_{\Omega} c_V\theta^0\psi|_{t=0} dx, \quad (2.5)$$

for any $\psi \in C^1(\overline{Q_T})$ with $\psi|_{t=T} = 0$, are valid, where relations (1.5) are assumed to hold.

Hereafter $T > 0$ is arbitrary and it is assumed that $\eta^0 \in L^1(\Omega)$, $v^0 \in L^2(\Omega)$, $\theta^0 \in L^1(\Omega)$ as well as $\eta^0 > 0$ and $\theta^0 \geq 0$ (almost everywhere in Ω).

We have to justify correctness of the definition of the weak solution. First notice that actually $\eta \in L^{1,\infty}(Q_T)$ and $r \in L^\infty(Q_T)$ according to properties (2.3). Next, we recall that (1.1) and (1.4) together with relation (1.9) imply the following relation between r and η

$$r^{k+1} = r_0^{k+1} + (k+1)I\eta. \quad (2.6)$$

In particular, actually $r \geq r_0$ and $\rho r_x = r^{-k}$. Consequently

$$\sigma = \rho(vr^k v_x - R\theta) + vkr^{-1}v \in L^2(Q_T), \quad (2.7)$$

where the embedding $V_1(Q_T) \subset L^2(Q_T)$ is taken into account. Moreover, for any $\varphi \in V_2(Q_T)$, we have

$$\sigma(r^k\varphi)_x = \sigma r^k\varphi_x + kr^{-1}(vr^k v_x - R\theta)\varphi + vk^2 r^{k-2} r_x v\varphi, \quad (2.8)$$

and since $V_2(Q_T) \subset L^{\infty,4}(Q_T)$ [4], we obtain

$$\sigma(r^k\varphi)_x \in L^1(Q_T). \quad (2.9)$$

If in addition $\varphi_x \in L^{2,\infty}(Q_T)$, then

$$\sigma(r^k\varphi)_x \in L^{1,2}(Q_T). \quad (2.10)$$

Furthermore

$$(r^{k-1}v^2)_x = 2r^{k-1}vv_x + (k-1)r^{k-2}r_x v^2 \in L^{1,2}(Q_T). \quad (2.11)$$

Consequently identities (2.4) and (2.5) are well-defined.

Notice also that

$$\sigma\eta = vr^k v_x - R\theta + vkr^{-1}v\eta \in L^{1,2}(Q_T). \quad (2.12)$$

Concerning the existence of strong and weak solutions, see in particular [1, 3, 8].

4 The rate of the volume growth

We will need the energy conservation law. Let us set $\sigma_\Gamma := 2k\mu(v/r)|_{x=M}$; notice that $\sigma_\Gamma \in L^4(0, T)$.

LEMMA 2.1. *The total kinetic energy $(1/2) \int_\Omega v^2 dx$ and the total internal energy $\int_\Omega c_V \theta dx$ are absolutely continuous functions on $[0, T]$ for any $T > 0$ having the derivatives*

$$\frac{d}{dt} \frac{1}{2} \int_\Omega v^2 dx = - \int_\Omega (\sigma - \sigma_\Gamma) (r^k v)_x dx, \quad \frac{d}{dt} \int_\Omega c_V \theta dx = \int_\Omega (\sigma - \sigma_\Gamma) (r^k v)_x dx. \quad (2.13)$$

Consequently the total energy conservation law holds

$$\mathcal{E} := \int_\Omega \left(\frac{1}{2} v^2 + c_V \theta \right) dx \equiv \mathcal{E}^0 \quad \text{on } \mathbb{R}^+, \quad (2.14)$$

where $\mathcal{E}^0 := \int_\Omega ((1/2)(v^0)^2 + c_V \theta^0) dx$ is the total initial energy.

Proof. Though results of the stated type are known, we prefer to present an independent proof.

(1) We first notice that if a function $w \in L^2(Q_T)$ has the derivatives $w_x, (I^* w)_t \in L^2(Q_T)$ and $w|_{x=0} = 0$, then the function $\int_\Omega w^2 dx$ is absolutely continuous on $[0, T]$ and has the derivative

$$\frac{d}{dt} \int_\Omega w^2 dx = 2 \int_\Omega (I^* w)_t w_x dx. \quad (2.15)$$

Actually, under the additional condition $w_t \in L^2(Q_T)$, by exploiting identity (2.2) we have

$$2 \int_{t_1}^{t_2} \int_\Omega (I^* w)_t w_x dx dt = \int_\Omega w^2 dx \Big|_{t_1}^{t_2} \quad (2.16)$$

for all $0 \leq t_1 \leq t_2 \leq T$. In the general case, by applying (2.16) for w mollified with respect to t and passing to the limit there, we establish (2.16) for almost all t_1 and t_2 such that $0 \leq t_1 \leq t_2 \leq T$. This leads to (2.15).

(2) We rewrite identity (2.4) in the form

$$\int_{Q_T} \left\{ -v \varphi_t + (\sigma - \sigma_\Gamma) (r^k \varphi)_x \right\} dx dt = \int_\Omega v^0 \varphi|_{t=0} dx. \quad (2.17)$$

Since $(r^k \varphi)_x = r^k \varphi_x + (r^k)_x \varphi$, by choosing $\varphi := I\zeta$ with $\zeta \in C^1(\overline{Q_T})$ having $\zeta|_{t=0, T} = 0$ and applying (2.2), we get

$$\int_{Q_T} \left\{ - (I^* v) \zeta_t + (\sigma - \sigma_\Gamma) r^k \zeta + \{ I^* [(\sigma - \sigma_\Gamma) (r^k)_x] \} \zeta \right\} dx dt = 0. \quad (2.18)$$

Thus by definition there exists the weak derivative

$$(I^* v)_t = -(\sigma - \sigma_\Gamma) r^k - I^* \left[(\sigma - \sigma_\Gamma) (r^k)_x \right] \in L^2(Q_T), \quad (2.19)$$

see properties (2.7) and (2.10) for $\varphi \equiv 1$. By integrating over Ω this equality multiplied by v_x we have

$$\int_{\Omega} (I^* v_t) v_x dx = - \int_{\Omega} \left\{ (\sigma - \sigma_\Gamma) r^k v_x + (\sigma - \sigma_\Gamma) (r^k)_x v \right\} dx = - \int_{\Omega} (\sigma - \sigma_\Gamma) (r^k v)_x dx, \quad (2.20)$$

where property (2.9) for $\varphi = v$ is also taken into account. This together with formula (2.15) imply the first formula (2.13).

The second formula (2.13) arises simpler after choosing $\psi \in C^1[0, T]$ with $\psi|_{t=0, T} = 0$ in identity (2.5). $\square$

Let us establish the key equality in the paper. We set $V_0 := r_0^{k+1}/(k+1)$.

LEMMA 2.2. *The following equality holds*

$$\frac{dW}{dt} = \int_{\Omega} \left\{ \frac{1}{k+1} \left[1 + k \left(\frac{r_0}{r} \right)^{k+1} \right] v^2 + R\theta \right\} dx, \quad (2.21)$$

where the function

$$W := v_1 V + \frac{2k}{k+1} \mu V_0 \log(V_0 + V) + \int_{\Omega} \frac{v}{r^k} I \eta dx \quad (2.22)$$

is absolutely continuous on $[0, T]$ for any $T > 0$.

Proof. Equation (1.1) and the definition of σ imply

$$v \eta_t = \sigma \eta + R\theta = \sigma_\Gamma \eta + (\sigma - \sigma_\Gamma) \eta + R\theta. \quad (2.23)$$

By integrating this equality over Ω we get

$$v \frac{dV}{dt} = \sigma_\Gamma V + \int_{\Omega} (\sigma - \sigma_\Gamma) \eta dx + \int_{\Omega} R\theta dx. \quad (2.24)$$

Let us transform the first and second summands in the right-hand side. By integrating (1.1) over Ω we get

$$\frac{dV}{dt} = (r^k v)|_{x=M}. \quad (2.25)$$

Using this equality together with (2.6) for $x = M$, we obtain

$$\sigma_\Gamma V = 2k\mu \frac{(r^k v)|_{x=M}}{r^{k+1}|_{x=M}} V = \frac{2k}{k+1} \mu \frac{V}{V_0 + V} \frac{dV}{dt} = \frac{2k}{k+1} \mu \frac{d}{dt} [V - V_0 \log(V_0 + V)]. \quad (2.26)$$

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Let $\zeta \in C^1(\overline{Q}_T)$ and $\zeta|_{t=0,T} = 0$. By choosing $\varphi := I\zeta/r^k$ in identity (2.17), using the formula

$$\left(\frac{I\zeta}{r^k}\right)_t = \frac{I\zeta_t}{r^k} - k\frac{I\zeta}{r^{k+1}}v \quad (2.27)$$

(see (1.4)) and applying identity (2.2), we find

$$\int_{Q_T} \left\{ -\left(I^* \frac{v}{r^k}\right)\zeta_t + k\left(I^* \frac{v^2}{r^{k+1}}\right)\zeta + (\sigma - \sigma_\Gamma)\zeta \right\} dx dt = 0. \quad (2.28)$$

This means that there exists the derivative

$$\left(I^* \frac{v}{r^k}\right)_t = -kI^*\left(\frac{v^2}{r^{k+1}}\right) - (\sigma - \sigma_\Gamma) \in L^2(Q_T). \quad (2.29)$$

Moreover, $(I^*(v/r^k))_t \eta \in L^{1,2}(Q_T)$ according to property (2.12). By integrating over Ω the last equality multiplied by η we have

$$\int_{\Omega} (\sigma - \sigma_\Gamma) \eta dx = -\frac{d}{dt} \int_{\Omega} \left(I^* \frac{v}{r^k}\right) \eta dx + \int_{\Omega} \left(I^* \frac{v}{r^k}\right) \eta_t dx - \int_{\Omega} kI^*\left(\frac{v^2}{r^{k+1}}\right) \eta dx. \quad (2.30)$$

Therefore by applying identity (2.2), equalities $I\eta_t = r^k v$ and $I\eta = (r^{k+1}/(k+1)) - (r_0^{k+1}/(k+1))$, see (1.1) and (2.6), we obtain

$$\int_{\Omega} (\sigma - \sigma_\Gamma) \eta dx = -\frac{d}{dt} \int_{\Omega} \frac{v}{r^k} I\eta dx + \int_{\Omega} \left\{ \frac{1}{k+1} v^2 + \frac{k}{k+1} v^2 \left(\frac{r_0}{r}\right)^{k+1} \right\} dx. \quad (2.31)$$

Inserting equality (2.26) together with the last one into (2.24), we complete the proof. $\square$

Now we are in a position to prove the main result. Let $V^0 := \int_{\Omega} \eta^0 dx$ be the initial volume.

PROPOSITION 2.3. *The following two-sided bounds for the gas volume hold*

$$\alpha_{1\varepsilon} \mathcal{E}^0 t + \beta_{1\varepsilon} \leq V(t) \leq \alpha_{2\varepsilon} \mathcal{E}^0 t + \beta_{2\varepsilon} \quad \text{for any } t \geq 0, \quad (2.32)$$

with any $0 < \varepsilon < \nu_1$ and

$$\begin{aligned} \alpha_{1\varepsilon} &:= \frac{\min\{2/(k+1), R/c_V\}}{\nu_1 + \varepsilon}, & \alpha_{2\varepsilon} &:= \frac{\max\{2, R/c_V\}}{\nu_1 - \varepsilon}, \\ \beta_{i\varepsilon} &= \beta_{i\varepsilon}(V^0, \mathcal{E}^0, \nu, \mu, M, V_0), & i &= 1, 2. \end{aligned} \quad (2.33)$$

Proof. By virtue of the energy conservation law we have

$$\min \left\{ \frac{2}{k+1}, \frac{R}{c_V} \right\} \mathcal{E}^0 \leq \int_{\Omega} \left\{ \frac{1}{k+1} \left[1 + k \left(\frac{r_0}{r} \right)^{k+1} \right] v^2 + R\theta \right\} dx \leq \max \left\{ 2, \frac{R}{c_V} \right\} \mathcal{E}^0, \quad (2.34)$$

$$\|v\|_{\Omega} \leq \sqrt{2\mathcal{E}^0}. \quad (2.35)$$

The latter bound and equality (2.6) together with the Young inequality imply

$$\begin{aligned} \left| \int_{\Omega} \frac{v}{r^k} I\eta dx \right| &\leq \|v\|_{L^1(\Omega)} \left\| \left(\frac{I\eta}{r^{k+1}} \right)^{k/(k+1)} \right\|_{C(\bar{\Omega})} V^{1/(k+1)} \\ &\leq \sqrt{2M\mathcal{E}^0} \frac{1}{(k+1)^{k/(k+1)}} V^{1/(k+1)} \\ &\leq \frac{1}{k+1} (\varepsilon_0 V + c^0 \varepsilon_0^{-1/k}), \end{aligned} \quad (2.36)$$

with $c^0 := c_{0k}(M\mathcal{E}^0)^{(k+1)/(2k)}$ and $c_{0k} > 0$ depending on k only, for any $\varepsilon_0 > 0$. Therefore

$$|W - v_1 V| \leq \frac{1}{k+1} \left[(2k|\mu|\varepsilon_1 + \varepsilon_0) V + 2k|\mu| V_0 (|\log V_0| + c_{\varepsilon_1}) + c^0 \varepsilon_0^{-1/k} \right], \quad (2.37)$$

with $c_{\varepsilon_1} := \log(\varepsilon_1^{-1}) + \varepsilon_1 - 1$, for any $\varepsilon_1 > 0$. This inequality remains valid for W and V replaced by $W(0)$ and V^0 .

By integrating the key equality (2.21) and applying inequalities (2.34) and (2.37) with suitable ε_0 and ε_1 together with condition (1.8), we obtain the two-sided bounds (2.32). $\square$

Notice that the assumption $r_0 > 0$ has been not so crucial, the quantities $\beta_{i\varepsilon}$ in (2.32) are bounded as $r_0 \rightarrow 0$ and thus the case without core, that is, $r_0 = 0$, could be also covered (at least for classical solutions) but we would not like to come into these details here.

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