We consider a biharmonic equation under the Navier boundary condition and with a nearly critical exponent \((P_\varepsilon)\): \(\Delta^2 u = u^{p-\varepsilon}, \ u > 0 \text{ in } \Omega \) and \(u = \Delta u = 0 \text{ on } \partial \Omega\), where \(\Omega\) is a smooth bounded domain in \(\mathbb{R}^5, \varepsilon > 0\). We study the asymptotic behavior of solutions of \((P_\varepsilon)\) which are minimizing for the Sobolev quotient as \(\varepsilon\) goes to zero. We show that such solutions concentrate around a point \(x_0 \in \Omega\) as \(\varepsilon \to 0\), moreover \(x_0\) is a critical point of the Robin’s function. Conversely, we show that for any nondegenerate critical point \(x_0\) of the Robin’s function, there exist solutions of \((P_\varepsilon)\) concentrating around \(x_0\) as \(\varepsilon \to 0\).

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1. Introduction and results

Let us consider the following biharmonic equation under the Navier boundary condition

\[
\begin{align*}
\Delta^2 u &= u^{p-\varepsilon}, \quad u > 0 \text{ in } \Omega \\
\Delta u &= u = 0 \text{ on } \partial \Omega,
\end{align*}
\]

\((Q_\varepsilon)\)

where \(\Omega\) is a smooth bounded domain in \(\mathbb{R}^n, n \geq 5\), \(\varepsilon\) is a small positive parameter, and \(p + 1 = 2n/(n-4)\) is the critical Sobolev exponent of the embedding \(H^2(\Omega) \cap H^1_0(\Omega) \hookrightarrow L^{2n/(n-4)}(\Omega)\).

It is known that \((Q_\varepsilon)\) is related to the limiting problem \((Q_0)\) (when \(\varepsilon = 0\)) which exhibits a lack of compactness and gives rise to solutions of \((Q_\varepsilon)\) which blow up as \(\varepsilon \to 0\). The interest of the limiting problem \((Q_0)\) grew from its resemblance to some geometric equations involving Paneitz operator and which have widely been studied in these last years (for details one can see [4, 6, 10, 12–14, 17] and references therein).

Several authors have studied the existence and behavior of blowing up solutions for the corresponding second order elliptic problem (see, e.g., [1, 3, 9, 18, 21, 22, 24–26] and references therein). In sharp contrast to this, very little is known for fourth order elliptic equations. In this paper we are mainly interested in the asymptotic behavior and
the existence of solutions of \((Q_\varepsilon)\) which blow up around one point, and the location of this blow up point as \(\varepsilon \to 0\).

The existence of solutions of \((Q_\varepsilon)\) for all \(\varepsilon \in (0, p - 1)\) is well known for any domain \(\Omega\) (see, e.g., [16]). For \(\varepsilon = 0\), the situation is more complex, Van Der Vorst showed in [28] that if \(\Omega\) is starshaped \((Q_0)\) has no solution whereas Ebobisse and Ould Ahmedou proved in [15] that \((Q_0)\) has a solution provided that some homology group of \(\Omega\) is nontrivial. This topological condition is sufficient, but not necessary, as examples of contractible domains \(\Omega\) on which a solution exists show [19].

In view of this qualitative change in the situation when \(\varepsilon = 0\), it is interesting to study the asymptotic behavior of the subcritical solution \(u_\varepsilon\) of \((Q_\varepsilon)\) as \(\varepsilon \to 0\). Chou and Geng [11], and Geng [20] made a first study, when \(\Omega\) is strictly convex. The convexity assumption was needed in their proof in order to apply the method of moving planes (MMP for short) in proving a priori estimate near the boundary. Notice that in the Laplacian case (see [21]), the MMP has been used to show that blow up points are away from the boundary of the domain. The process is standard if domains are convex. For nonconvex regions, the MMP still works in the Laplacian case through the applications of Kelvin transformations [21]. For \((Q_\varepsilon)\), the MMP also works for convex domains [11]. However, for nonconvex domains, a Kelvin transformation does not work for \((Q_\varepsilon)\) because the Navier boundary condition is not invariant under the Kelvin transformation of biharmonic operator. In [5], Ben Ayed and El Mehdi removed the convexity assumption of Chou and Geng for higher dimensions, that is \(n \geq 6\). The aim of this paper is to prove that the results of [5] are true in dimension 5. In order to state precisely our results, we need to introduce some notations.

We consider the following problem

\[
\begin{align*}
\Delta^2 u &= u^{9-\varepsilon}, \quad u > 0 \text{ in } \Omega \\
\Delta u &= u = 0 \text{ on } \partial\Omega,
\end{align*}
\tag{P_\varepsilon}
\]

where \(\Omega\) is a smooth bounded domain in \(\mathbb{R}^5\) and \(\varepsilon\) is a small positive parameter.

Let us define on \(\Omega\) the following Robin’s function

\[
\varphi(x) = H(x,x), \quad \text{with } H(x,y) = |x - y|^{-1} - G(x,y), \quad \text{for } (x,y) \in \Omega \times \Omega,
\tag{1.1}
\]

where \(G\) is the Green’s function of \(\Delta^2\), that is,

\[
\begin{align*}
\forall x \in \Omega & \quad \Delta^2 G(x,\cdot) = c\delta_x \quad \text{in } \Omega \\
\Delta G(x,\cdot) &= G(x,\cdot) = 0 \quad \text{on } \partial\Omega,
\end{align*}
\tag{1.2}
\]

where \(\delta_x\) denotes the Dirac mass at \(x\) and \(c = 3\omega_5\), with \(\omega_5\) is the area of the unit sphere of \(\mathbb{R}^5\). For \(\lambda > 0\) and \(a \in \mathbb{R}^5\), let

\[
\delta_{a,\lambda}(x) = \frac{c_0 \lambda^{1/2}}{(1 + \lambda^2|x - a|^2)^{1/2}}, \quad c_0 = (105)^{1/8}.
\tag{1.3}
\]

It is well known (see [23]) that \(\delta_{a,\lambda}\) are the only solutions of

\[
\Delta^2 u = u^9, \quad u > 0 \text{ in } \mathbb{R}^5
\tag{1.4}
\]
and are also the only minimizers of the Sobolev inequality on the whole space, that is

\[ S = \inf \{ |\Delta u|^2_{L^2(\mathbb{R}^5)} |u|^2_{L^2(\mathbb{R}^5)}, \text{ s.t. } \Delta u \in L^2, \ u \in L^{10}, \ u \neq 0 \}. \]  

We denote by \( P_{\delta a} \), the projection of \( \delta a \) on \( \mathcal{H}(\Omega) := H^2(\Omega) \cap H^1_0(\Omega) \), defined by

\[ \Delta^2 P_{\delta a} = \Delta^2 \delta a \in \Omega, \quad \Delta P_{\delta a} = P_{\delta a} = 0 \text{ on } \partial \Omega. \]  

Let

\[ \theta_{a, \lambda} = \delta a - P_{\delta a}, \quad \| u \| = \left( \int_{\Omega} |\Delta u|^2 \right)^{1/2}, \quad \langle u, v \rangle = \int_{\Omega} \Delta u \Delta v, \quad u, v \in H^2(\Omega) \cap H^1_0(\Omega) \]  

Thus we have the following result.

**Theorem 1.1.** Let \((u_\varepsilon)\) be a solution of \((P_\varepsilon)\), and assume that

\[ \| u_\varepsilon \|^2 \| u_\varepsilon \|_{10-\varepsilon}^2 \to S \quad \text{as } \varepsilon \to 0, \]

where \( S \) is the best Sobolev constant in \( \mathbb{R}^5 \) defined by (1.5). Then (up to a subsequence) there exist \( a_\varepsilon \in \Omega, \lambda_\varepsilon > 0, \alpha_\varepsilon > 0 \) and \( v_\varepsilon \) such that \( u_\varepsilon \) can be written as

\[ u_\varepsilon = \alpha_\varepsilon P_{\delta a_{\lambda_\varepsilon}} + v_\varepsilon \]

with \( \alpha_\varepsilon \to 1, \| v_\varepsilon \| \to 0, a_\varepsilon \in \Omega \) and \( \lambda_\varepsilon d(a_\varepsilon, \partial \Omega) \to +\infty \text{ as } \varepsilon \to 0. \)

In addition, \( a_\varepsilon \) converges to a critical point \( x_0 \in \Omega \) of \( \varphi \) and we have

\[ \lim_{\varepsilon \to 0} \varepsilon \| u_\varepsilon \|^2_{L^\infty(\Omega)} = (c_1 c_0^2/c_2) \varphi(x_0), \]

where \( c_1 = c_0^{10} \int_{\mathbb{R}^5} (dx/(1 + |x|^2)^{9/2}), \ c_2 = c_0^{10} \int_{\mathbb{R}^5} (\log(1 + |x|^2)(1 - |x|^2)/(1 + |x|^2)^6) dx \) and \( c_0 = (105)^{1/8}. \)

Our next result provides a kind of converse to Theorem 1.1.

**Theorem 1.2.** Assume that \( x_0 \in \Omega \) is a nondegenerate critical point of \( \varphi \). Then there exists an \( \varepsilon_0 > 0 \) such that for each \( \varepsilon \in (0, \varepsilon_0) \), \((P_\varepsilon)\) has a solution of the form

\[ u_\varepsilon = \alpha_\varepsilon P_{\delta a_{\lambda_\varepsilon}} + v_\varepsilon \]

with \( \alpha_\varepsilon \to 1, \| v_\varepsilon \| \to 0, a_\varepsilon \to x_0 \) and \( \lambda_\varepsilon d(a_\varepsilon, \partial \Omega) \to +\infty \text{ as } \varepsilon \to 0. \)

Our strategy to prove the above results is the same as in higher dimensions. However, as usual in elliptic equations involving critical Sobolev exponent, we need more refined estimates of the asymptotic profiles of solutions when \( \varepsilon \to 0 \) to treat the lower dimensional case. Such refined estimates, which are of self interest, are highly nontrivial and use in a crucial way careful expansions of the Euler-Lagrange functional associated to \((P_\varepsilon)\), and its gradient near a small neighborhood of highly concentrated functions. To perform such
expansions we make use of the techniques developed by Bahri [2] and Rey [25, 27] in the framework of the Theory of critical points at infinity.

The outline of the paper is the following: in Section 2 we perform some crucial estimates needed in our proofs and Section 3 is devoted to the proof of our results.

2. Some crucial estimates

In this section, we prove some crucial estimates which will play an important role in proving our results. We first recall some results.

**Proposition 2.1** [8]. Let \( a \in \Omega \) and \( \lambda > 0 \) such that \( \lambda d(a, \partial \Omega) \) is large enough. For \( \theta(a, \lambda) = \delta(a, \lambda) - P \delta(a, \lambda) \), we have the following estimates

\[
0 \leq \theta(a, \lambda) \leq \delta(a, \lambda), \quad \theta(a, \lambda) = c_0 \lambda^{-1/2} H(a, \cdot) + f(a, \lambda),
\]

where \( f(a, \lambda) \) satisfies

\[
f(a, \lambda) = O\left(\frac{1}{\lambda^{5/2} d^3}\right), \quad \lambda \frac{\partial f(a, \lambda)}{\partial \lambda} = O\left(\frac{1}{\lambda^{3/2} d^3}\right), \quad \frac{1}{\lambda} \frac{\partial f(a, \lambda)}{\partial a} = O\left(\frac{1}{\lambda^{7/2} d^4}\right),
\]

where \( d \) is the distance \( d(a, \partial \Omega) \),

\[
|\theta(a, \lambda)|_{L^{10}} = O((\lambda d)^{-1/2}), \quad \|\theta(a, \lambda)\| = O((\lambda d)^{-1/2}), \quad \left|\lambda \frac{\partial \theta(a, \lambda)}{\partial \lambda}\right|_{L^{10}} = O\left(\frac{1}{\lambda^{7/2} d^3}\right), \quad \frac{1}{\lambda} \left|\frac{\partial \theta(a, \lambda)}{\partial a}\right|_{L^{10}} = O\left(\frac{1}{\lambda^{5/2} d^2}\right). \tag{2.3}
\]

**Proposition 2.2** [5]. Let \( u_\varepsilon \) be a solution of \((P_\varepsilon)\) which satisfies \((H)\). Then, there exist \( a_\varepsilon \in \Omega, \alpha_\varepsilon > 0, \lambda_\varepsilon > 0 \) and \( v_\varepsilon \) such that

\[
u_\varepsilon = \alpha_\varepsilon P \delta_{a_\varepsilon \lambda_\varepsilon} + v_\varepsilon \tag{2.4}
\]

with \( \alpha_\varepsilon \to 1, \lambda_\varepsilon d(a_\varepsilon, \partial \Omega) \to \infty, c_0^{-\varepsilon} ||u_\varepsilon||^2_{\infty} / \lambda_\varepsilon \to 1, ||u_\varepsilon||_{\infty} \to 1 \) and \( ||v_\varepsilon|| \to 0 \). Furthermore, \( v_\varepsilon \in E(a_\varepsilon, \lambda_\varepsilon) \) which is the set of \( v \in \mathcal{H}(\Omega) \) such that

\[
\langle v, P \delta_{a_\varepsilon \lambda_\varepsilon} \rangle = \langle v, \partial P \delta_{a_\varepsilon \lambda_\varepsilon} / \partial \lambda_\varepsilon \rangle = 0, \quad \langle v_\varepsilon, \partial P \delta_{a_\varepsilon \lambda_\varepsilon} / \partial a \rangle = 0. \tag{V_0}
\]

**Lemma 2.3** [5]. \( \delta_\varepsilon^\varepsilon = 1 + o(1) \) as \( \varepsilon \) goes to zero implies that

\[
\delta_\varepsilon^\varepsilon - c_0^{-\varepsilon} \lambda_\varepsilon^{(4-n)/2} = O(\varepsilon \log(1 + \lambda_\varepsilon \varepsilon^2 |x - a_\varepsilon|^2)) \text{ in } \Omega, \tag{2.5}
\]

where \( \delta_\varepsilon = \delta_{a_\varepsilon \lambda_\varepsilon} \) and \( d_\varepsilon = d(a_\varepsilon, \partial \Omega) \).

**Proposition 2.4** [5]. Let \((u_\varepsilon)\) be a solution of \((P_\varepsilon)\) which satisfies \((H)\). Then \( v_\varepsilon \) occurring in Proposition 2.2 satisfies

\[
||v_\varepsilon|| \leq C(\varepsilon + (\lambda_\varepsilon d_\varepsilon)^{-1}), \tag{2.6}
\]

where \( C \) is a positive constant independent of \( \varepsilon \).
Now, we are going to state and prove the crucial estimates needed in the proof of our theorems. In order to simplify the notations, we set \( \delta_\varepsilon = \delta_{a, \lambda_\varepsilon} \), \( P\delta_\varepsilon = P\delta_{a, \lambda_\varepsilon} \), \( \theta_\varepsilon = \theta_{a, \lambda_\varepsilon} \) and \( d_\varepsilon = d(a_\varepsilon, \partial \Omega) \).

**Lemma 2.5.** For \( \varepsilon \) small, we have the following estimates

(i) \( \int_\Omega \delta_\varepsilon^9 (1/\lambda_\varepsilon) (\partial P\delta_\varepsilon/\partial a) = -(c_1/2\lambda_\varepsilon^2) (\partial H/\partial a)(a_\varepsilon, a_\varepsilon) + O(1/\lambda_\varepsilon d_\varepsilon)^3) \),

(ii) \( \int_\Omega P\delta_\varepsilon^{9+\varepsilon} (1/\lambda_\varepsilon) (\partial P\delta_\varepsilon/\partial a) = -(c_1/\lambda_\varepsilon^2) (\partial H/\partial a)(a_\varepsilon, a_\varepsilon) + O(1/\lambda_\varepsilon d_\varepsilon)^3 + \varepsilon/(\lambda_\varepsilon d_\varepsilon)^2) \),

where \( c_1 \) is the constant defined in Theorem 1.1.

**Proof.** Notice that

\[
\int_\Omega \delta_\varepsilon^{10} = O\left( \frac{1}{(\lambda_\varepsilon d_\varepsilon)^5} \right). \tag{2.7}
\]

Thus, we have, for \( 1 \leq k \leq 5 \)

\[
\int_\Omega \delta_\varepsilon^9 \frac{1}{\lambda_\varepsilon} \frac{\partial P\delta_\varepsilon}{\partial a_k} = \int_\Omega \delta_\varepsilon^9 \frac{1}{\lambda_\varepsilon} \frac{\partial \delta_\varepsilon}{\partial a_k} - \int_\Omega \delta_\varepsilon^9 \frac{1}{\lambda_\varepsilon} \frac{\partial \theta_\varepsilon}{\partial a_k} = - \int_{B_\varepsilon} \delta_\varepsilon^9 \frac{1}{\lambda_\varepsilon} \frac{\partial \theta_\varepsilon}{\partial a_k} + O\left( \frac{1}{(\lambda_\varepsilon d_\varepsilon)^5} \right), \tag{2.8}
\]

where \( B_\varepsilon = B(a_\varepsilon, d_\varepsilon) \). Expanding \( \partial \theta_\varepsilon/\partial a_k \) around \( a_\varepsilon \) and using Proposition 2.1, we obtain

\[
\int_{B_\varepsilon} \delta_\varepsilon^9 \frac{1}{\lambda_\varepsilon} \frac{\partial \theta_\varepsilon}{\partial a_k} = c_0 \frac{1}{2\lambda_\varepsilon^3} \frac{\partial H(a_\varepsilon, a_\varepsilon)}{\partial a} \int_{B_\varepsilon} \delta_\varepsilon^9 + O\left( \frac{1}{(\lambda_\varepsilon d_\varepsilon)^3} \right). \tag{2.9}
\]

Estimating the integral on the right-hand side in (2.9) and using (2.8), we easily derive claim (i). To prove claim (ii), we write

\[
\int_\Omega P\delta_\varepsilon^{9+\varepsilon} \frac{1}{\lambda_\varepsilon} \frac{\partial P\delta_\varepsilon}{\partial a_k} = \int_\Omega \delta_\varepsilon^{9+\varepsilon} \frac{1}{\lambda_\varepsilon} \frac{\partial \delta_\varepsilon}{\partial a_k} - (9 + \varepsilon) \int_\Omega \delta_\varepsilon^{8+\varepsilon} \frac{1}{\lambda_\varepsilon} \frac{\partial \theta_\varepsilon}{\partial a_k} \frac{\partial \delta_\varepsilon}{\partial a_k} + \frac{(9 - \varepsilon)(8 - \varepsilon)}{2} \int_\Omega \delta_\varepsilon^{7+\varepsilon} \frac{1}{\lambda_\varepsilon} \frac{\partial \delta_\varepsilon}{\partial a_k} + O\left( \int_\Omega \delta_\varepsilon^{8+\varepsilon} \frac{1}{\lambda_\varepsilon} \frac{\partial \theta_\varepsilon}{\partial a_k} \frac{\partial \delta_\varepsilon}{\partial a_k} \int_{B_\varepsilon} \delta_\varepsilon^9 + \int_\Omega \delta_\varepsilon^{7+\varepsilon} \theta_\varepsilon^3 \right) \tag{2.10}
\]

and we have to estimate each term on the right-hand side of (2.10).

Using Proposition 2.1 and Lemma 2.3, we have

\[
\int_\Omega \delta_\varepsilon^{7+\varepsilon} \theta_\varepsilon^3 \leq c \|	heta_\varepsilon^3\|_\infty \int_\Omega \delta_\varepsilon^7 = O\left( \frac{1}{(\lambda_\varepsilon d_\varepsilon)^3} \right), \tag{2.11}
\]

\[
\int_\Omega \delta_\varepsilon^{8+\varepsilon} \frac{1}{\lambda_\varepsilon} \frac{\partial \theta_\varepsilon}{\partial a_k} \frac{\partial \delta_\varepsilon}{\partial a_k} \leq c \|	heta_\varepsilon\|_\infty \left\| \frac{1}{\lambda_\varepsilon} \frac{\partial \theta_\varepsilon}{\partial a_k} \right\|_\infty \int_\Omega \delta_\varepsilon^8 = O\left( \frac{1}{(\lambda_\varepsilon d_\varepsilon)^3} \right). \tag{2.11}
\]

We also have

\[
\int_\Omega \delta_\varepsilon^{9+\varepsilon} \frac{1}{\lambda_\varepsilon} \frac{\partial \delta_\varepsilon}{\partial a_k} = \int_{\Omega \setminus B_\varepsilon} \delta_\varepsilon^{9+\varepsilon} \frac{1}{\lambda_\varepsilon} \frac{\partial \delta_\varepsilon}{\partial a_k} = O\left( \frac{1}{(\lambda_\varepsilon d_\varepsilon)^5} \right). \tag{2.12}
\]
Expanding $\theta_e$ around $a_e$ and using Proposition 2.1 and Lemma 2.3, we obtain

$$\int_{\Omega} \delta^{2-\epsilon} \frac{1}{\lambda_e} \frac{\partial \delta_e}{\partial a_k} w_e = O\left( \frac{\epsilon}{(\lambda_e d_e)^3} \right),$$

(2.13)

In the same way, we find

$$\int_\Omega \delta^{2-\epsilon} \frac{1}{\lambda_e} \frac{\partial \theta_e}{\partial a_k} = O\left( \frac{1}{(\lambda_e d_e)^3} \right).$$

(2.14)

Combining (2.10)–(2.14), we obtain claim (ii). $\square$

To improve the estimates of the integrals involving $v_e$, we use an idea of Rey [27], namely we write

$$v_e = \Pi v_e + w_e,$$

(2.15)

where $\Pi v_e$ denotes the projection of $v_e$ onto $H^2 \cap H^1_0(B_e)$, that is

$$\Delta^2 \Pi v_e = \Delta^2 v_e \quad \text{in } B_e; \quad \Delta \Pi v_e = \Pi v_e = 0 \quad \text{on } \partial B_e,$$

(2.16)

where $B_e = B(a_e, d_e)$. We split $\Pi v_e$ in an even part $\Pi v^e$ and an odd part $\Pi v^o$ with respect to $(x - a_e)_k$, thus we have

$$v_e = \Pi v^e + \Pi v^o + w_e \quad \text{in } B_e \quad \text{with } \Delta^2 w_e = 0 \text{ in } B_e.$$ 

(2.17)

Notice that it is difficult to improve the estimate (2.6) of the $v_e$-part of solutions. However, it is sufficient to improve the integrals involving the odd part of $v_e$ with respect to $(x - a_e)_k$, for $1 \leq k \leq 5$ and to know the exact contribution of the integrals containing the $w_e$-part of $v_e$. Let us start by the terms involving $w_e$.

**Lemma 2.6.** For $\epsilon$ small, we have that

$$\int_{B_e} \delta^{8} \left( \delta^{2-\epsilon} - \frac{1}{c_0^2 \lambda_e^{1/2}} \right) \frac{1}{\lambda_e} \frac{\partial \delta_e}{\partial a_k} w_e = O\left( \frac{\epsilon \|v_e\|}{\lambda_e d_e^{1/2}} \right).$$

(2.18)

**Proof.** Let $\psi$ be the solution of

$$\Delta^2 \psi = \delta^{6} \left( \delta^{2-\epsilon} - \frac{1}{c_0^2 \lambda_e^{1/2}} \right) \frac{1}{\lambda_e} \frac{\partial \delta_e}{\partial a_k} \quad \text{in } B_e; \quad \Delta \psi = \psi = 0 \quad \text{on } \partial B_e.$$ 

(2.19)

Thus we have

$$I_e := \int_{B_e} \Delta^2 \psi w_e = \int_{\partial B_e} \frac{\partial \psi}{\partial n} \Delta w_e + \int_{\partial B_e} \frac{\partial \Delta \psi}{\partial n} w_e.$$ 

(2.20)
Let $G_\varepsilon$ be the Green’s function for the biharmonic operator on $B_\varepsilon$ with the Navier boundary conditions, that is,
\[ \Delta^2 G_\varepsilon(x, \cdot) = c \delta_x \text{ in } B_\varepsilon; \quad \Delta G_\varepsilon(x, \cdot) = G(x, \cdot) = 0 \text{ on } \partial B_\varepsilon, \] (2.21)
where $c = 3w_5$. Therefore $\psi$ is given by
\[ \psi(y) = \int_{B_\varepsilon} G_\varepsilon(x, y) \delta^8_\varepsilon \left( \delta^{-\varepsilon} - \frac{1}{c_0 \lambda_\varepsilon^{1/2}} \right) \frac{1}{\lambda_\varepsilon} \frac{\partial \delta_\varepsilon}{\partial a_k}, \quad y \in B_\varepsilon \] (2.22)
and its normal derivative by
\[ \frac{\partial \psi}{\partial \nu}(y) = \int_{B_\varepsilon} \frac{\partial G_\varepsilon}{\partial \nu}(x, y) \delta^8_\varepsilon \left( \delta^{-\varepsilon} - \frac{1}{c_0 \lambda_\varepsilon^{1/2}} \right) \frac{1}{\lambda_\varepsilon} \frac{\partial \delta_\varepsilon}{\partial a_k}, \quad y \in \partial B_\varepsilon. \] (2.23)
Notice that for $y \in \partial B_\varepsilon$ we have the following estimates: for $x \in B_\varepsilon \setminus B(y, d_\varepsilon/2)$, we have
\[ \left| \frac{\partial G_\varepsilon}{\partial \nu}(x, y) \right| = O \left( \frac{1}{d_\varepsilon^2} \right); \quad \left| \frac{\partial \Delta G_\varepsilon}{\partial \nu}(x, y) \right| = O \left( \frac{1}{d_\varepsilon^4} \right) \] (2.24)
for $x \in B_\varepsilon \cap B(y, d_\varepsilon/2)$, we have
\[ \left| \frac{\partial G_\varepsilon}{\partial \nu}(x, y) \right| \leq \frac{c}{|x - y|^2}; \quad \left| \frac{\partial \Delta G_\varepsilon}{\partial \nu}(x, y) \right| \leq \frac{c}{|x - y|^4} \] (2.25)
for $x \in B_\varepsilon \cap B(y, d_\varepsilon/2)$, we have
\[ \delta^8_\varepsilon \left( \delta^{-\varepsilon} - \frac{1}{c_0 \lambda_\varepsilon^{1/2}} \right) \frac{1}{\lambda_\varepsilon} \frac{\partial \delta_\varepsilon}{\partial a_k} = O \left( \frac{\varepsilon \log \lambda_\varepsilon d_\varepsilon}{(\lambda_\varepsilon d_\varepsilon)^9} \right), \] (2.26)
for $x \in B_\varepsilon \setminus B(y, d_\varepsilon/2)$, we have
\[ \delta^8_\varepsilon \left( \delta^{-\varepsilon} - \frac{1}{c_0 \lambda_\varepsilon^{1/2}} \right) \frac{1}{\lambda_\varepsilon} \frac{\partial \delta_\varepsilon}{\partial a_k} = O \left( \delta^8_\varepsilon \varepsilon \log (1 + \lambda_\varepsilon^2 |x - a_\varepsilon|^2) \right). \] (2.27)
Therefore
\[ \left| \frac{\partial \psi}{\partial \nu}(y) \right| = O \left( \frac{\varepsilon}{\lambda_\varepsilon^{1/2} d_\varepsilon^2} \right). \] (2.28)
In the same way, we have
\[ \left| \frac{\partial \Delta \psi}{\partial \nu}(y) \right| = O \left( \frac{\varepsilon}{\lambda_\varepsilon^{1/2} d_\varepsilon^4} \right). \] (2.29)
Using (2.20), (2.28), (2.29), we obtain
\[ I_\varepsilon = O \left( \frac{\varepsilon}{\lambda_\varepsilon^{1/2} d_\varepsilon^4} \right) \int_{\partial B_\varepsilon} |\Delta w_\varepsilon| \, d\nu + \frac{\varepsilon}{\lambda_\varepsilon^{1/2} d_\varepsilon^4} \int_{\partial B_\varepsilon} |w_\varepsilon| \, d\nu. \] (2.30)
To estimate the right-hand side of (2.30), we introduce the following function
\[ \bar{w}(X) = d_1 \varepsilon w_\varepsilon(a_\varepsilon + d_\varepsilon X), \quad \bar{v}(X) = d_1 \varepsilon v_\varepsilon(a_\varepsilon + d_\varepsilon X) \quad \text{for } X \in B(0,1). \] (2.31)

\( \bar{w} \) satisfies
\[ \Delta^2 \bar{w} = 0 \quad \text{in } B := B(0,1); \quad \Delta \bar{w} = \Delta \bar{v}, \quad \bar{w} = \bar{v} \quad \text{on } \partial B. \] (2.32)

We deduce that
\[ \int_{\partial B} |\Delta \bar{w}| + \int_{\partial B} |\Delta \bar{v}| \leq C \left( \int_B |\Delta v_\varepsilon|^2 \right)^{1/2}. \] (2.33)

But, we have
\[ \int_{\partial B} |\Delta \bar{w}| + \int_{\partial B} |\Delta \bar{v}| = \left( \frac{1}{d_\varepsilon} \right)^{3/2} \int_{\partial B_\varepsilon} |\Delta w_\varepsilon| + \left( \frac{1}{d_\varepsilon} \right)^{7/2} \int_{\partial B_\varepsilon} |w_\varepsilon|. \] (2.34)

Using (2.30), (2.33) and (2.34), the lemma follows. \( \square \)

**Lemma 2.7.** For \( \varepsilon \) small, we have
(i) \( \int_{B_\varepsilon} \Delta \left( \frac{1}{\lambda_\varepsilon} (\partial \Pi \delta_\varepsilon / \partial a_k) \right) \Delta w_\varepsilon = O(\|v_\varepsilon\|/(\lambda_\varepsilon d_\varepsilon)^{3/2}) \),
(ii) \( \int_{B_\varepsilon} \delta_\varepsilon^{8-\varepsilon} \Pi v_\varepsilon^w \Delta w_\varepsilon = O(\|v_\varepsilon\|\|\Pi v_\varepsilon^w\|/(\lambda_\varepsilon d_\varepsilon)^{1/2}) \).

**Proof.** Using (2.17), we obtain
\[ \int_{B_\varepsilon} \Delta \left( \frac{1}{\lambda_\varepsilon} \frac{\partial \Pi \delta_\varepsilon}{\partial a_k} \right) \Delta w_\varepsilon = \int_{\partial B_\varepsilon} \frac{\partial \psi_\varepsilon}{\partial \nu} \Delta w_\varepsilon, \quad \text{with } \psi_\varepsilon = \frac{1}{\lambda_\varepsilon} \frac{\partial \Pi \delta_\varepsilon}{\partial a_k}. \] (2.35)

Using an integral representation for \( \psi_\varepsilon \) as in (2.23), we obtain for \( y \in \partial B_\varepsilon \),
\[ \frac{\partial \psi}{\partial \nu}(y) = \int_{B_\varepsilon} \frac{\partial G_\varepsilon}{\partial \nu}(x,y) \Delta^2 \psi_\varepsilon, \] (2.36)
where \( G_\varepsilon \) is the Green’s function defined in (2.21). Clearly, we have
\[ \Pi \delta_\varepsilon(x) = \delta_\varepsilon(x) - \frac{c_\varepsilon \lambda_\varepsilon^{1/2}}{1 + \lambda_\varepsilon^2 d_\varepsilon^2} \frac{c_\varepsilon(a_\varepsilon, d_\varepsilon)}{10} (|x - a_\varepsilon|^2 - d_\varepsilon^2), \] (2.37)
with \( c_\varepsilon(a_\varepsilon, d_\varepsilon) = \Delta \delta_\varepsilon \). Thus we deduce that
\[ \frac{\partial \psi}{\partial \nu}(y) = 9 \int_{B_\varepsilon} \frac{\partial G_\varepsilon}{\partial \nu}(x,y) \delta_\varepsilon^{8-\varepsilon} \frac{1}{\lambda_\varepsilon} \frac{\partial \delta_\varepsilon}{\partial a_k}. \] (2.38)

In \( B_\varepsilon \setminus B(a_\varepsilon, d_\varepsilon/2) \), we argue as in (2.28) and (2.25), we obtain
\[ \int_{B_\varepsilon} \frac{\partial G_\varepsilon}{\partial \nu}(x,y) \delta_\varepsilon^{8-\varepsilon} \frac{1}{\lambda_\varepsilon} \frac{\partial \delta_\varepsilon}{\partial a_k} = O \left( \frac{1}{\lambda_\varepsilon^6 \varepsilon^6} \right). \] (2.39)
Furthermore, since
\[
\left| \nabla \frac{\partial G_\varepsilon}{\partial y}(x, y) \right| = O \left( \frac{1}{d_\varepsilon^2} \right) \quad \text{for } (x, y) \in B(a_\varepsilon, d_\varepsilon/2) \times \partial B_\varepsilon,
\]
we obtain
\[
\left| \int_{B(a_\varepsilon, d_\varepsilon/2)} \frac{\partial G_\varepsilon}{\partial y}(x, y) \delta_\varepsilon^8 \frac{\partial \delta_\varepsilon}{\partial a_k} \right| \leq \frac{c}{d_\varepsilon^2} \int_{B(a_\varepsilon, d_\varepsilon/2)} \delta_\varepsilon^9 \left| x - a_\varepsilon \right| = O \left( \frac{1}{\lambda_\varepsilon^{3/2} d_\varepsilon^3} \right),
\]
where we have used the evenness of \( \delta_\varepsilon \) and the oddness of its derivative. Thus
\[
\frac{\partial \psi_k}{\partial y}(y) = O \left( \frac{1}{\lambda_\varepsilon^{1/2} d_\varepsilon^4} \right).
\]
Using (2.35) and (2.42), we obtain
\[
\int_{B_\varepsilon} \delta_\varepsilon^{8-\varepsilon} \Pi \nu_\varepsilon^{a_k} w_\varepsilon = \int_{\partial B_\varepsilon} \frac{\partial \Delta \psi}{\partial y} w_\varepsilon + \int_{\partial B_\varepsilon} \frac{\partial \psi}{\partial y} \Delta w_\varepsilon.
\]
As before, we prove that, for \( y \in \partial B_\varepsilon \)
\[
\frac{\partial \psi}{\partial y}(y) = O \left( \frac{\| \Pi \nu_\varepsilon^{a_k} \|}{\lambda_\varepsilon^{1/2} d_\varepsilon^2} \right), \quad \frac{\partial \Delta \psi}{\partial y}(y) = O \left( \frac{\| \Pi \nu_\varepsilon^{a_k} \|}{\lambda_\varepsilon^{1/2} d_\varepsilon} \right).
\]
Therefore
\[
\int_{B_\varepsilon} \delta_\varepsilon^{8-\varepsilon} \Pi \nu_\varepsilon^{a_k} w_\varepsilon \leq \frac{c\| \Pi \nu_\varepsilon^{a_k} \|}{\lambda_\varepsilon^{1/2} d_\varepsilon^4} \left( \frac{1}{\delta_\varepsilon^{3/2}} \int_{\partial B_\varepsilon} \left| w_\varepsilon \right| + \frac{1}{\delta_\varepsilon^{3/2}} \int_{\partial B_\varepsilon} \left| \Delta w_\varepsilon \right| \right) \leq \frac{c\| \nu_\varepsilon^{a_k} \| \| \Pi \nu_\varepsilon^{a_k} \|}{(\lambda_\varepsilon d_\varepsilon)^{1/2}}.
\]
The proof of the lemma is completed.

**Lemma 2.8.** For \( \varepsilon \) small, we have
\[
\text{(i) } \int_{B_\varepsilon} \delta_\varepsilon^{8-\varepsilon} \nu_\varepsilon(1/\lambda_\varepsilon)(\partial \delta_\varepsilon/\partial a_k) = O(\| \Pi \nu_\varepsilon^{a_k} \|/\lambda_\varepsilon^{1/2} + \| \nu_\varepsilon \|/\lambda_\varepsilon d_\varepsilon^{3/2}),
\]
\[
\text{(ii) } \int_{B_\varepsilon} \delta_\varepsilon^{8-\varepsilon} \theta_\varepsilon \nu_\varepsilon(1/\lambda_\varepsilon)(\partial \delta_\varepsilon/\partial a_k) = O(\| \Pi \nu_\varepsilon^{a_k} \|/\lambda_\varepsilon d_\varepsilon + \| \nu_\varepsilon \|/\lambda_\varepsilon d_\varepsilon^{3/2}).
\]

**Proof.** Claim (i) can be proved in the same way as Lemma 2.6, so we omit its proof. Claim (ii) follows from Proposition 2.1 and claim (i).
Let us now compute the contribution of the following integral which involves $v^2_\varepsilon$.

**Lemma 2.9.** For $\varepsilon$ small, we have

\[
\int_{B_\varepsilon} \delta^{7-\varepsilon}_\varepsilon v^2_\varepsilon \frac{1}{\lambda_\varepsilon} \frac{\partial \delta_\varepsilon}{\partial a_k} = O \left( \frac{\|v_\varepsilon\|}{\lambda_\varepsilon d_\varepsilon^{1/2}} \right). \tag{2.48}
\]

**Proof.** Using (2.17) and the fact that the even part of $v^2_\varepsilon$ has no contribution to the integrals, we obtain

\[
\int_{B_\varepsilon} \delta^{7-\varepsilon}_\varepsilon v^2_\varepsilon \frac{1}{\lambda_\varepsilon} \frac{\partial \delta_\varepsilon}{\partial a_k} = \int_{B_\varepsilon} \delta^{7-\varepsilon}_\varepsilon \frac{1}{\lambda_\varepsilon} \frac{\partial \delta_\varepsilon}{\partial a_k} (2v_\varepsilon - w_\varepsilon) w_\varepsilon + O(||\Pi v_\varepsilon^o||||v_\varepsilon||). \tag{2.49}
\]

Let $\Psi$ be the solution of

\[
\Delta^2 \Psi = \delta^{7-\varepsilon}_\varepsilon \frac{1}{\lambda_\varepsilon} \frac{\partial \delta_\varepsilon}{\partial a_k} (2v_\varepsilon - w_\varepsilon) \quad \text{in} \quad B_\varepsilon; \quad \Delta \Psi = \Psi = 0\quad \text{on} \quad \partial B_\varepsilon. \tag{2.50}
\]

Thus, as in the proof of Lemma 2.6, we obtain for $y \in \partial B_\varepsilon$

\[
\frac{\partial \Psi}{\partial \nu}(y) = O \left( \frac{\|v_\varepsilon\|}{\lambda_\varepsilon^{1/2} d_\varepsilon} \right), \quad \frac{\partial \Delta \Psi}{\partial \nu}(y) = O \left( \frac{\|v_\varepsilon\|}{\lambda_\varepsilon^{1/2} d_\varepsilon^2} \right) \tag{2.51}
\]

and therefore

\[
\int_{B_\varepsilon} \delta^{7-\varepsilon}_\varepsilon \frac{1}{\lambda_\varepsilon} \frac{\partial \delta_\varepsilon}{\partial a_k} (2v_\varepsilon - w_\varepsilon) w_\varepsilon = O \left( \frac{\|v_\varepsilon\|^2}{(\lambda_\varepsilon d_\varepsilon)^{1/2}} \right). \tag{2.52}
\]

Thus our lemma follows. $\Box$

Next we are going to estimate the integrals involving the odd part of $v_\varepsilon$ with respect to $(x - a_\varepsilon)_k$, for $1 \leq k \leq 5$.

**Lemma 2.10.** For $\varepsilon$ small, we have

\[
\int_{B_\varepsilon} u^{9-\varepsilon}_\varepsilon \Pi v_\varepsilon^o = 9 \int_{B_\varepsilon} \delta^8_\varepsilon (\Pi v_\varepsilon^o)^2 + o(||\Pi v_\varepsilon^o||^2) + O \left( ||\Pi v_\varepsilon^o|| \left( \varepsilon^{3/2} + \frac{1}{(\lambda_\varepsilon d_\varepsilon)^{3/2}} \right) \right). \tag{2.53}
\]

**Proof.** We have

\[
\int_{B_\varepsilon} u^{9-\varepsilon}_\varepsilon \Pi v_\varepsilon^o = \alpha^{9-\varepsilon}_\varepsilon \int_{B_\varepsilon} P \delta^{9-\varepsilon}_\varepsilon \Pi v_\varepsilon^o + (9 - \varepsilon) \alpha^{9-\varepsilon}_\varepsilon \int_{B_\varepsilon} P \delta^{8-\varepsilon}_\varepsilon v_\varepsilon \Pi v_\varepsilon^o
\]

\[
+ O \left( \int_{B_\varepsilon} P \delta^{7-\varepsilon}_\varepsilon |v_\varepsilon|^2 |\Pi v_\varepsilon^o| + \int_{B_\varepsilon} |v_\varepsilon|^{9-\varepsilon} |\Pi v_\varepsilon^o| \right) \tag{2.54}
\]

\[
= \alpha^{9-\varepsilon}_\varepsilon \int_{B_\varepsilon} P \delta^{9-\varepsilon}_\varepsilon \Pi v_\varepsilon^o + (9 - \varepsilon) \alpha^{9-\varepsilon}_\varepsilon \int_{B_\varepsilon} P \delta^{8-\varepsilon}_\varepsilon v_\varepsilon \Pi v_\varepsilon^o + O(||v_\varepsilon||^2 ||\Pi v_\varepsilon^o||). \]
We estimate the two integrals on the right-hand side in (2.54). First, using Proposition 2.1 and the Holder inequality, we have

\[
\int_{B_\epsilon} P \delta^{8-\epsilon}_{\epsilon} v_{\epsilon} \Pi v_{\epsilon}^0 = \int_{B_\epsilon} \delta^{8-\epsilon}_{\epsilon} v_{\epsilon} \Pi v_{\epsilon}^0 + O\left(\frac{\|v_{\epsilon}\|\|\Pi v_{\epsilon}^0\|}{\lambda_{\epsilon} d_{\epsilon}}\right) = \int_{B_\epsilon} \delta^{8-\epsilon}_{\epsilon} \Pi v_{\epsilon}^0 + \int_{B_\epsilon} \delta^{8-\epsilon}_{\epsilon} \Pi v_{\epsilon}^0 w_{\epsilon},
\]

(2.55)

where we have used in the last equality the evenness of \(\delta_{\epsilon}\) and \(\Pi v_{\epsilon}^0\) and the oddness of \(\Pi v_{\epsilon}^0\). By Lemmas 2.3 and 2.7 we obtain

\[
\int_{B_\epsilon} P \delta^{8-\epsilon}_{\epsilon} v_{\epsilon} \Pi v_{\epsilon}^0 = \int_{B_\epsilon} \delta^{8-\epsilon}_{\epsilon} \Pi v_{\epsilon}^0 + O\left(\frac{\|v_{\epsilon}\|\|\Pi v_{\epsilon}^0\|}{(\lambda_{\epsilon} d_{\epsilon})^{1/2}}\right).
\]

(2.56)

Secondly, we write

\[
\int_{B_\epsilon} P \delta^{9-\epsilon}_{\epsilon} \Pi v_{\epsilon}^0 = \int_{B_\epsilon} \delta^{9-\epsilon}_{\epsilon} \Pi v_{\epsilon}^0 - (9 - \epsilon) \int_{B_\epsilon} \delta^{8-\epsilon}_{\epsilon} \theta_{\epsilon} \Pi v_{\epsilon}^0 + O\left(\int_{B_\epsilon} \delta^{7-\epsilon}_{\epsilon} \theta_{\epsilon}^2 |\Pi v_{\epsilon}^0| \right).
\]

(2.57)

Thus, using the evenness of \(\delta_{\epsilon}\), the oddness of \(\Pi v_{\epsilon}^0\) and Holder inequality, we obtain

\[
\int_{B_\epsilon} P \delta^{9-\epsilon}_{\epsilon} \Pi v_{\epsilon}^0 = O\left(\frac{\|\Pi v_{\epsilon}^0\|}{(\lambda_{\epsilon} d_{\epsilon})^2}\right).
\]

(2.58)

Using (2.54), (2.56), (2.58) and Propositions 2.2 and 2.4, we easily derive our lemma. □

**Lemma 2.11.** For \(\epsilon\) small, we have

\[
\|\Pi v_{\epsilon}^0\| = O\left(\epsilon^{3/2} + \frac{1}{(\lambda_{\epsilon} d_{\epsilon})^{3/2}}\right).
\]

(2.59)

**Proof.** We write

\[
\Pi v_{\epsilon}^0 = \tilde{\Pi} v_{\epsilon}^0 + \alpha \Pi \delta_{\epsilon} + \beta \lambda_{\epsilon} \frac{\partial \Pi \delta_{\epsilon}}{\partial \lambda} + \sum_{r=1}^{5} \gamma_r \frac{1}{\lambda_{\epsilon}} \frac{\partial \Pi \delta_{\epsilon}}{\partial a_r}
\]

(2.60)

with

\[
\langle \tilde{\Pi} v_{\epsilon}^0, \Pi \delta_{\epsilon} \rangle = \left\langle \tilde{\Pi} v_{\epsilon}^0, \frac{\partial \Pi \delta_{\epsilon}}{\partial \lambda} \right\rangle = \left\langle \tilde{\Pi} v_{\epsilon}^0, \frac{\partial \Pi \delta_{\epsilon}}{\partial a_r} \right\rangle = 0 \quad \text{for each } r \in \{1,2,3,4,5\}.
\]

(2.61)
Taking the scalar product in $H^2 \cap H^1_0(B_\varepsilon)$ of (2.60) with $\Pi \delta_\varepsilon, \lambda_\varepsilon \partial \Pi \delta_\varepsilon/\partial \lambda, \lambda_\varepsilon^{-1} \partial \Pi \delta_\varepsilon/\partial a_r, 1 \leq r \leq 5$, provides us with the following invertible linear system in $\alpha, \beta, y_r$ (with $1 \leq r \leq 5$)

\[
\langle \Pi \delta_\varepsilon, \Pi \nu^\varepsilon \rangle = \alpha(C' + o(1)) + \beta \left\langle \Pi \delta_\varepsilon, \lambda_\varepsilon \frac{\partial \Pi \delta_\varepsilon}{\partial \lambda} \right\rangle + \sum_{r=1}^5 y_r \left\langle \Pi \delta_\varepsilon, \frac{1}{\lambda_\varepsilon} \frac{\partial \Pi \delta_\varepsilon}{\partial a_r} \right\rangle
\]

\[
\left\langle \lambda_\varepsilon \frac{\partial \Pi \delta_\varepsilon}{\partial \lambda}, \Pi \nu^\varepsilon \right\rangle = \alpha \left\langle \Pi \delta_\varepsilon, \lambda_\varepsilon \frac{\partial \Pi \delta_\varepsilon}{\partial \lambda} \right\rangle + \beta \left(C'' + o(1)\right) + \sum_{r=1}^5 y_r \left\langle \Pi \delta_\varepsilon, \frac{1}{\lambda_\varepsilon} \frac{\partial \Pi \delta_\varepsilon}{\partial a_r} \right\rangle \tag{S}
\]

\[
\left\langle \frac{1}{\lambda_\varepsilon} \frac{\partial \Pi \delta_\varepsilon}{\partial a_k}, \Pi \nu^\varepsilon \right\rangle = \alpha \left\langle \Pi \delta_\varepsilon, \frac{1}{\lambda_\varepsilon} \frac{\partial \Pi \delta_\varepsilon}{\partial a_k} \right\rangle + \beta \left(\frac{\lambda_\varepsilon \partial \Pi \delta_\varepsilon}{\lambda_\varepsilon \partial a_k} \right) + \sum_{r=1}^5 y_r \left\langle \frac{1}{\lambda_\varepsilon} \frac{\partial \Pi \delta_\varepsilon}{\partial a_k}, \frac{1}{\lambda_\varepsilon} \frac{\partial \Pi \delta_\varepsilon}{\partial a_r} \right\rangle.
\]

Observe that

\[
\left\langle \Pi \delta_\varepsilon, \lambda_\varepsilon \frac{\partial \Pi \delta_\varepsilon}{\partial \lambda} \right\rangle = O\left(\frac{1}{\lambda_\varepsilon d_\varepsilon}\right);
\]

\[
\left\langle \lambda_\varepsilon \frac{\partial \Pi \delta_\varepsilon}{\partial \lambda}, \lambda_\varepsilon \frac{\partial \Pi \delta_\varepsilon}{\partial a_r} \right\rangle = O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^2}\right);
\]

\[
\left\langle \Pi \delta_\varepsilon, \frac{\partial \Pi \delta_\varepsilon}{\partial a_r} \right\rangle = O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^2}\right);
\]

\[
\left\langle \frac{1}{\lambda_\varepsilon} \frac{\partial \Pi \delta_\varepsilon}{\partial a_k}, \frac{1}{\lambda_\varepsilon} \frac{\partial \Pi \delta_\varepsilon}{\partial a_r} \right\rangle = (C'' + o(1)) \delta_{kr} + O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^2}\right),
\]

where $\delta_{kr}$ denotes the Kronecker symbol.

Now, because of the evenness of $\delta_\varepsilon$ and the oddness of $\Pi \nu^\varepsilon$ with respect to $(x - a_\varepsilon)_k$ we obtain

\[
\langle \Pi \delta_\varepsilon, \Pi \nu^\varepsilon \rangle = \int_{B_\varepsilon} \Delta \Pi \delta_\varepsilon \cdot \Delta \Pi \nu^\varepsilon = \int_{B_\varepsilon} \Pi \nu^\varepsilon \Delta \Pi \delta_\varepsilon = 0. \tag{2.63}
\]

In the same way we have

\[
\left\langle \lambda_\varepsilon \frac{\partial \Pi \delta_\varepsilon}{\partial \lambda}, \Pi \nu^\varepsilon \right\rangle = \left\langle \frac{1}{\lambda_\varepsilon} \frac{\partial \Pi \delta_\varepsilon}{\partial a_k}, \Pi \nu^\varepsilon \right\rangle = 0 \quad \text{for each } r \neq k. \tag{2.64}
\]

We also have

\[
\left\langle \frac{1}{\lambda_\varepsilon} \frac{\partial \Pi \delta_\varepsilon}{\partial a_k}, \Pi \nu^\varepsilon \right\rangle = \int_{B_\varepsilon} \Delta \left(\frac{1}{\lambda_\varepsilon} \frac{\partial \Pi \delta_\varepsilon}{\partial a_k}\right) \cdot \Delta (\nu^\varepsilon - \Pi \nu^\varepsilon - w_\varepsilon)
\]

\[
= \int_{B_\varepsilon} \Delta \left(\frac{1}{\lambda_\varepsilon} \frac{\partial \Pi \delta_\varepsilon}{\partial a_k}\right) \cdot \Delta \nu^\varepsilon - \int_{B_\varepsilon} \Delta \left(\frac{1}{\lambda_\varepsilon} \frac{\partial \Pi \delta_\varepsilon}{\partial a_k}\right) \cdot \Delta w_\varepsilon. \tag{2.65}
\]
where we have used in the last equality the fact that $\Pi v_\varepsilon^o$ is even with respect to $(x - a_\varepsilon)_k$. Using (2.37) and Holder inequality, we obtain

$$
\int_{B_\varepsilon} \Delta \left( \frac{1}{\lambda_\varepsilon} \frac{\partial \Pi \delta_\varepsilon}{\partial a_k} \right) \cdot \Delta v_\varepsilon \leq c \|v_\varepsilon\| \left( \int_{\Omega \setminus B_\varepsilon} \left| \Delta \frac{1}{\lambda_\varepsilon} \frac{\partial \delta_\varepsilon}{\partial a_k} \right|^2 \right)^{1/2} = O \left( \frac{\|v_\varepsilon\|}{(\lambda_\varepsilon d_\varepsilon)^{3/2}} \right). \quad (2.66)
$$

Equation (2.66) and Lemma 2.7 imply that

$$
\left\langle \frac{1}{\lambda_\varepsilon} \frac{\partial \Pi \delta_\varepsilon}{\partial a_k}, \Pi v_\varepsilon^o \right\rangle = O \left( \frac{\|v_\varepsilon\|}{(\lambda_\varepsilon d_\varepsilon)^{3/2}} \right). \quad (2.67)
$$

Inverting the linear system (S), we deduce from the above estimates

$$\alpha = O \left( \frac{\|v_\varepsilon\|}{(\lambda_\varepsilon d_\varepsilon)^{3/2}} \right), \quad \beta = O \left( \frac{\|v_\varepsilon\|}{(\lambda_\varepsilon d_\varepsilon)^{3/2}} \right),
$$

$$y_k = O \left( \frac{\|v_\varepsilon\|}{(\lambda_\varepsilon d_\varepsilon)^{3/2}} \right), \quad y_r = O \left( \frac{\|v_\varepsilon\|}{(\lambda_\varepsilon d_\varepsilon)^{3/2}} \right), \quad r \neq k. \quad (2.68)
$$

This implies through (2.60)

$$\|\Pi v_\varepsilon^o - \tilde{\Pi} v_\varepsilon^o\| = O \left( \frac{\|v_\varepsilon\|}{(\lambda_\varepsilon d_\varepsilon)^{3/2}} \right), \quad \|\Pi v_\varepsilon^o\|^2 = \|\tilde{\Pi} v_\varepsilon^o\|^2 + O \left( \frac{\|v_\varepsilon\|^2}{(\lambda_\varepsilon d_\varepsilon)^{3}} \right). \quad (2.69)
$$

We now turn to the last step, which consists in estimating $\|\tilde{\Pi} v_\varepsilon^o\|$. Since $u_\varepsilon$ is a solution of $(P_\varepsilon)$, we have

$$
\int_{B_\varepsilon} \Delta^2 u_\varepsilon \Pi v_\varepsilon^o = \int_{B_\varepsilon} u_\varepsilon^{9-\varepsilon} \Pi v_\varepsilon^o. \quad (2.70)
$$

Because of the evenness of $\delta_\varepsilon$ and the oddness of $\Pi v_\varepsilon^o$ with respect to $(x - a_\varepsilon)_k$, (2.70) becomes

$$
\|\Pi v_\varepsilon^o\|^2 = \int_{B_\varepsilon} u_\varepsilon^{9-\varepsilon} \Pi v_\varepsilon^o. \quad (2.71)
$$

By (2.69), (2.71) and Lemma 2.10, we obtain

$$
\|\tilde{\Pi} v_\varepsilon^o\|^2 - 9 \int_{B_\varepsilon} \delta_\varepsilon^8 (\Pi v_\varepsilon^o)^2 + o(\|\tilde{\Pi} v_\varepsilon^o\|^2) = O \left( \varepsilon^3 + \frac{1}{(\lambda_\varepsilon d_\varepsilon)^{3}} \right). \quad (2.72)
$$

Using now (2.72) and the fact that the quadratic form

$$
u \longmapsto \int_{B_\varepsilon} |\Delta \nu|^2 - 9 \int_{B_\varepsilon} \delta_\varepsilon^8 \nu^2 \quad (2.73)$$
is positive definite (see [6]) on the subset \([\text{Span}(\Pi\delta_\epsilon, \partial\Pi\delta_\epsilon/\partial\lambda, \partial\Pi\delta_\epsilon/\partial a_k \ 1 \leq k \leq 5)]_{H^2 \cap H^1_0(B_\epsilon)},\) we obtain

\[
\|\widetilde{\Pi v}_\epsilon\| \leq C \left( \frac{1}{(\lambda_\epsilon d_\epsilon)^{3/2}} + \epsilon^{3/2} \right).
\] (2.74)

Our lemma follows from (2.69) and (2.74). \(\square\)

Before ending this section, let us prove the following estimate which will be needed later.

**Lemma 2.12.** For \(\epsilon\) small, we have

\[
\left\langle \frac{\partial^2 P\delta_\epsilon}{\partial\lambda \partial a_k}, v_\epsilon \right\rangle = O \left( \frac{1}{(\lambda_\epsilon d_\epsilon)^{3/2}} + \epsilon^{3/2} \right).
\] (2.75)

**Proof.** We have

\[
\int_{\Omega} \Delta \left( \frac{\partial^2 P\delta_\epsilon}{\partial\lambda \partial a_k} \right) \Delta v_\epsilon = \int_{B_\epsilon} \Delta^2 \left( \frac{\partial^2 P\delta_\epsilon}{\partial\lambda \partial a_k} \right) v_\epsilon + O \left( \frac{\|v_\epsilon\|}{(\lambda_\epsilon d_\epsilon)^{9/2}} \right) = \int_{B_\epsilon} \Delta^2 \left( \frac{\partial^2 P\delta_\epsilon}{\partial\lambda \partial a_k} \right) \Pi v_\epsilon^0 + \int_{B_\epsilon} \Delta^2 \left( \frac{\partial^2 P\delta_\epsilon}{\partial\lambda \partial a_k} \right) w_\epsilon + O \left( \frac{\|v_\epsilon\|}{(\lambda_\epsilon d_\epsilon)^{9/2}} \right). \tag{2.76}
\]

For the first integral on the right-hand side in (2.76), we have

\[
\int_{B_\epsilon} \Delta^2 \left( \frac{\partial^2 P\delta_\epsilon}{\partial\lambda \partial a_k} \right) \Pi v_\epsilon^0 = O(\|\Pi v_\epsilon^0\|) = O \left( \frac{1}{(\lambda_\epsilon d_\epsilon)^{3/2}} + \epsilon^{3/2} \right), \tag{2.77}
\]

where we have used in the last equality Lemma 2.11.

Now let \(\psi_4\) be the solution of

\[
\Delta^2 \psi_4 = \Delta^2 \left( \frac{\partial^2 P\delta_\epsilon}{\partial\lambda \partial a_k} \right) \text{ in } B_\epsilon, \quad \Delta \psi_4 = \psi_4 = 0 \text{ on } \partial B_\epsilon. \tag{2.78}
\]

Thus, as in the proof of Lemma 2.6, we obtain for \(y \in \partial B_\epsilon\)

\[
\frac{\partial \psi_4}{\partial \nu}(y) = O \left( \frac{1}{\lambda_\epsilon^{1/2} d_\epsilon^2} \right), \quad \frac{\partial \Delta \psi_4}{\partial \nu}(y) = O \left( \frac{1}{\lambda_\epsilon^{1/2} d_\epsilon^4} \right) \tag{2.79}
\]

and therefore

\[
\int_{B_\epsilon} \Delta^2 \left( \frac{\partial^2 P\delta_\epsilon}{\partial\lambda \partial a_k} \right) w_\epsilon = O \left( \frac{\|v_\epsilon\|}{(\lambda_\epsilon d_\epsilon)^{1/2}} \right). \tag{2.80}
\]

From (2.76), (2.77), (2.80) and Proposition 2.4, we easily deduce our lemma. \(\square\)
3. Proof of theorems

Let us start by proving the following crucial result.

**Proposition 3.1.** For \( u_\varepsilon = \alpha_\varepsilon P_{a_\varepsilon} + v_\varepsilon \) solution of \( (P_\varepsilon) \) with \( \lambda_\varepsilon = 1 + o(1) \) as \( \varepsilon \) goes to zero, we have the following estimates

- (a) \( c_2 \varepsilon + O(\varepsilon^2) - c_1 (H(a_\varepsilon, a_\varepsilon)/\lambda_\varepsilon) + o(1/\lambda_\varepsilon) = 0 \),
- (b) \( (c_3/\lambda_\varepsilon^2) (\partial H(a_\varepsilon, a_\varepsilon)/\partial a) + o(1/\lambda_\varepsilon^2) + O(1/\lambda_\varepsilon^2 + 1/(\lambda_\varepsilon d_\varepsilon)^{1/2}) = 0 \),

where \( c_1, c_2 \) are the constants defined in Theorem 1.1, and where \( c_3 > 0 \).

**Proof.** Since claim (a) was proved in [5], we only need to prove claim (b). Multiplying \( (P_\varepsilon) \) by \( (1/\lambda_\varepsilon) (\partial P_\varepsilon/\partial a_k) \) and integrating on \( \Omega \), we obtain for \( 1 \leq k \leq 5 \)

\[
0 = \int_\Omega \Delta^2 u_\varepsilon \frac{1}{\lambda_\varepsilon} \frac{\partial P_\varepsilon}{\partial a_k} - \int_\Omega u_\varepsilon^{9-\varepsilon} \frac{1}{\lambda_\varepsilon} \frac{\partial P_\varepsilon}{\partial a_k}
= \alpha_\varepsilon \int_\Omega \delta_\varepsilon^{7-\varepsilon} v_\varepsilon \frac{1}{\lambda_\varepsilon} \frac{\partial P_\varepsilon}{\partial a_k}
- \int_\Omega \left[ (\alpha_\varepsilon P_\varepsilon)^{9-\varepsilon} + (9-\varepsilon)(\alpha_\varepsilon P_\varepsilon)^{8-\varepsilon} v_\varepsilon \right] \frac{1}{\lambda_\varepsilon} \frac{\partial P_\varepsilon}{\partial a_k} + O\left( ||v_\varepsilon||^3 \right). \tag{3.1}
\]

We estimate each term on the right-hand side in (3.1). First, by Proposition 2.1 and the Holder inequality, we have

\[
\int_\Omega P_\varepsilon^{7-\varepsilon} v_\varepsilon \frac{1}{\lambda_\varepsilon} \frac{\partial P_\varepsilon}{\partial a_k} = \int_\Omega \delta_\varepsilon^{7-\varepsilon} v_\varepsilon \frac{1}{\lambda_\varepsilon} \frac{\partial \theta_\varepsilon}{\partial a_k} + O\left( ||v_\varepsilon||^2 \right). \tag{3.2}
\]

Secondly, we compute

\[
\begin{align*}
\int_\Omega P_\varepsilon^{8-\varepsilon} v_\varepsilon \frac{1}{\lambda_\varepsilon} \frac{\partial P_\varepsilon}{\partial a_k} &= \int_\Omega \delta_\varepsilon^{8-\varepsilon} v_\varepsilon \frac{1}{\lambda_\varepsilon} \frac{\partial \theta_\varepsilon}{\partial a_k} + (8-\varepsilon) \int_\Omega \delta_\varepsilon^{7-\varepsilon} v_\varepsilon \frac{1}{\lambda_\varepsilon} \frac{\partial \theta_\varepsilon}{\partial a_k} + O\left( \int_\Omega \delta_\varepsilon^{7-\varepsilon} \theta_\varepsilon^2 \right) v_\varepsilon |v_\varepsilon|.
\end{align*}
\]

By Proposition 2.1 and the Holder inequality, we obtain

\[
\begin{align*}
\int_\Omega \delta_\varepsilon^{7-\varepsilon} \theta_\varepsilon^2 |v_\varepsilon| &= O\left( \frac{||v_\varepsilon||}{(\lambda_\varepsilon d_\varepsilon)^2} \right), \\
\int_\Omega \delta_\varepsilon^{7-\varepsilon} \theta_\varepsilon |v_\varepsilon| \left| \frac{1}{\lambda_\varepsilon} \frac{\partial \theta_\varepsilon}{\partial a_k} \right| &= O\left( \frac{||v_\varepsilon||}{(\lambda_\varepsilon d_\varepsilon)^3} \right), \\
\int_\Omega \delta_\varepsilon^{8-\varepsilon} v_\varepsilon \left| \frac{1}{\lambda_\varepsilon} \frac{\partial \theta_\varepsilon}{\partial a_k} \right| &= O\left( \frac{||v_\varepsilon||}{(\lambda_\varepsilon d_\varepsilon)^2} \right). \tag{3.4}
\end{align*}
\]
We also have by Proposition 2.2
\[
\int_{\Omega} \delta_{\epsilon}^{8-\epsilon} v_{\epsilon} \frac{1}{\lambda_{\epsilon}} \frac{\partial \delta_{\epsilon}}{\partial a_k} = \int_{\Omega} \delta_{\epsilon}^{8} \left( \delta_{\epsilon}^{-\epsilon} \frac{c_{0}^{-\epsilon}}{\lambda_{\epsilon}^{\epsilon/2}} \right) v_{\epsilon} \frac{1}{\lambda_{\epsilon}} \frac{\partial \delta_{\epsilon}}{\partial a_k} + O \left( \frac{\|v_{\epsilon}\|}{(\lambda_{\epsilon} d_{\epsilon})^{9/2}} \right).
\]
Using (2.17), Lemma 2.3 and the Hölder inequality, we derive that
\[
\int_{\Omega} \delta_{\epsilon}^{8-\epsilon} v_{\epsilon} \frac{1}{\lambda_{\epsilon}} \frac{\partial \delta_{\epsilon}}{\partial a_k} = \int_{B_{\epsilon}} \delta_{\epsilon}^{8} \left( \delta_{\epsilon}^{-\epsilon} \frac{c_{0}^{-\epsilon}}{\lambda_{\epsilon}^{\epsilon/2}} \right) v_{\epsilon} \frac{1}{\lambda_{\epsilon}} \frac{\partial \delta_{\epsilon}}{\partial a_k} + O \left( \frac{\|v_{\epsilon}\|}{(\lambda_{\epsilon} d_{\epsilon})^{9/2}} \right)
\]
where we have used Lemma 2.6 in the last equality.

Using (3.2)–(3.6), Lemmas 2.5, 2.8, 2.9, Proposition 2.2 and the fact that $\lambda_{\epsilon} = 1 + O(\epsilon \log \lambda_{\epsilon})$, we easily derive our result.

We are now able to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let $(u_{\epsilon})$ be a solution of $(P_{\epsilon})$ which satisfies (H). Then, using Proposition 2.2, $u_{\epsilon} = \alpha_{\epsilon} P\delta_{a_{\epsilon},\lambda_{\epsilon}} + v_{\epsilon}$ with $\alpha_{\epsilon} \to 1, \lambda_{\epsilon} \to 1, \lambda_{\epsilon} d(a_{\epsilon}, \partial \Omega) \to \infty, v_{\epsilon}$ satisfies $(V_{0})$ and $\|v_{\epsilon}\| \to 0$. Now, using claim (a) of Proposition 3.1, we derive that
\[
\epsilon = \frac{c_{1}}{c_{2}} \frac{H(a_{\epsilon}, a_{\epsilon})}{\lambda_{\epsilon}} + o \left( \frac{1}{\lambda_{\epsilon} d_{\epsilon}} \right) = O \left( \frac{1}{\lambda_{\epsilon} d_{\epsilon}} \right).
\]
Therefore, it follows from claim (b) of Proposition 3.1 that
\[
\frac{\partial H(a_{\epsilon}, a_{\epsilon})}{\partial a} = o \left( \frac{1}{d_{\epsilon}^{2}} \right).
\]
Using (3.8) and the fact that for $a$ near the boundary $(\partial H/\partial a)(a_{\epsilon}, a_{\epsilon}) \sim c d(a_{\epsilon}, \partial \Omega)^{-2}$, we derive that $a_{\epsilon}$ is away from the boundary and it converges to a critical point $x_{0}$ of $\varphi$.

Finally, using (3.7), we obtain
\[
\epsilon \lambda_{\epsilon} \to \frac{c_{1}}{c_{2}} \varphi(x_{0}) \quad \text{as} \quad \epsilon \to 0.
\]
By Proposition 2.2, we have
\[
\|u_{\epsilon}\|_{L^{\infty}}^{2} \sim c_{2}^{2} \epsilon \lambda_{\epsilon} \quad \text{as} \quad \epsilon \to 0.
\]
This concludes the proof of Theorem 1.1.

The sequel of this section is devoted to the proof of Theorem 1.2.
Proof of Theorem 1.2. Let \( x_0 \) be a nondegenerate critical point of \( \varphi \). It is easy to see that \( d(a, \partial \Omega) > d_0 > 0 \) for \( a \) near \( x_0 \). We will take a function \( u = \alpha P \delta_{(a, \lambda)} + \nu \) where \( (\alpha - \alpha_0) \) is very small, \( \lambda \) is large enough, \( \| \nu \| \) is very small, \( a \) is close to \( x_0 \) and \( \alpha_0 = S^{-5/8} \) and we will prove that we can choose the variables \((\alpha, \lambda, a, \nu)\) so that \( u \) is a critical point of \( J_\epsilon \) with \( \| u \| = 1 \). Here \( J_\epsilon \) denotes the functional corresponding to problem \((P_\epsilon)\) defined by

\[
J_\epsilon(u) = \left( \int_{\Omega} |\Delta u|^2 \right)^{1/2} \left( \int_{\Omega} |u|^{10-\epsilon} \right)^{-1}. 
\tag{3.11}
\]

Let

\[
M_\epsilon = \{ (\alpha, \lambda, a, \nu) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \times \Omega \times \mathcal{H}(\Omega) : |\alpha - \alpha_0| < \nu_0, \\
d_a > d_0, \lambda > \nu_0^{-1}, \epsilon \log \lambda < \nu_0, \| \nu \| < \nu_0 \text{ and } \nu \in E_{(a, \lambda)} \},
\tag{3.12}
\]

where \( \nu_0 \) and \( d_0 \) are two suitable positive constants and where \( d_a = d(a, \partial \Omega) \).

Let us define the functional

\[
K_\epsilon : M_\epsilon \rightarrow \mathbb{R}, \quad K_\epsilon(\alpha, a, \lambda, \nu) = J_\epsilon(\alpha P \delta_{(a, \lambda)} + \nu).
\tag{3.13}
\]

It is known that \((\alpha, \lambda, a, \nu)\) is a critical point of \( K_\epsilon \) if and only if \( u = \alpha P \delta_{(a, \lambda)} + \nu \) is a critical point of \( J_\epsilon \) on \( \mathcal{H}(\Omega) \). So this fact allows us to look for critical points of \( J_\epsilon \) by successive optimizations with respect to the different parameters on \( M_\epsilon \).

First, arguing as in \([25, \text{Proposition 4}]\) we see that the following problem

\[
\min \{ J_\epsilon(\alpha P \delta_{(a, \lambda)} + \nu), \nu \text{ satisfying } (V_0) \text{ and } \| \nu \| < \nu_0 \}
\tag{3.14}
\]

is achieved by a unique function \( \overline{\nu} \) which satisfies the estimate of Proposition 2.4. This implies that there exist \( A, B \) and \( C_i \)'s such that

\[
\frac{\partial K_\epsilon}{\partial \nu}(\alpha, \lambda, a, \overline{\nu}) = \nabla J_\epsilon(\alpha P \delta_{(a, \lambda)} + \overline{\nu}) = AP \delta_{(a, \lambda)} + B \frac{\partial}{\partial \lambda} P \delta_{(a, \lambda)} + \sum_{i=1}^{5} C_i \frac{\partial}{\partial a_i} P \delta_{(a, \lambda)}, \tag{3.15}
\]

where \( a_i \) is the \( i \)-th component of \( a \).

According to \([5]\), we have that

\[
A = O\left( \epsilon \log \lambda + |\beta| + \frac{1}{\lambda} \right), \quad B = O(\lambda \epsilon + 1), \quad C_j = O\left( \frac{\epsilon^2}{\lambda} + \frac{1}{\lambda^3} \right). \tag{3.16}
\]

To find critical points of \( K_\epsilon \), we have to solve the following system

\[
\frac{\partial K_\epsilon}{\partial \alpha} = 0 \]

\[
\frac{\partial K_\epsilon}{\partial \lambda} = B \left( \frac{\partial^2 P \delta}{\partial \lambda^2}, \overline{\nu} \right) + \sum_{i=1}^{5} C_i \left( \frac{\partial^2 P \delta}{\partial \lambda \partial a_i}, \overline{\nu} \right) \tag{E_1}
\]

\[
\frac{\partial K_\epsilon}{\partial a_j} = B \left( \frac{\partial^2 P \delta}{\partial \lambda \partial a_j}, \overline{\nu} \right) + \sum_{i=1}^{5} C_i \left( \frac{\partial^2 P \delta}{\partial a_i \partial a_j}, \overline{\nu} \right), \quad \text{for each } j = 1, \ldots, 5.
\]
Observe that for $\psi = P(\lambda, \lambda) + \bar{v}$, we have

$$\mathcal{J}_\varepsilon(\alpha P(\lambda, \lambda) + \bar{v}) = S + O\left(\frac{1}{\lambda^2}\right),$$

(3.18)

$$\frac{\partial K_\varepsilon}{\partial \alpha} = \left\langle \nabla \mathcal{J}_\varepsilon(\alpha P(\lambda, \lambda) + \bar{v}), \alpha \frac{H(a, a)}{\lambda} \right\rangle = \mathcal{J}_\varepsilon(u) \left(\frac{c}{\lambda^2} \frac{1}{\sqrt{H(a, a)}} + \rho \right) \varepsilon + O\left(\frac{1}{\lambda^5}\right).$$

(3.19)

Following the proof of claim (b) of Proposition 3.1, we obtain, for each $j = 1, \ldots, 5$,

$$\frac{1}{\lambda} \frac{\partial K_\varepsilon}{\partial a_j} = \left\langle \nabla \mathcal{J}_\varepsilon(\alpha P(\lambda, \lambda) + \bar{v}), \frac{1}{\lambda} \frac{\partial P(\lambda, \lambda)}{\partial a_j} \right\rangle = -\frac{c}{\lambda^2} \frac{H(a, a)}{\lambda} (1 - 2\alpha S^5) + O\left(\frac{1}{\lambda^5}\right).$$

(3.20)

On the other hand, one can easily verify that

$$\left\| \frac{\partial^2 P(\lambda, \lambda)}{\partial \lambda^2} \right\| = O\left(\frac{1}{\lambda^2}\right), \quad \left\| \frac{\partial^2 P(\lambda, \lambda)}{\partial a \partial a_j} \right\| = O(\lambda^2).$$

(3.21)

Now, we take the following change of variables:

$$\alpha = \alpha_0 + \beta, \quad a = x_0 + \xi, \quad \frac{1}{\lambda^{1/2}} = \sqrt{\frac{c_2}{c_1}} \left(\frac{1}{\sqrt{H(x_0, x_0)}} + \rho \right) \sqrt{\varepsilon}. \quad (3.23)$$

Then, using estimates (3.18)–(3.22), Lemma 2.12, Proposition 2.4 and the fact that $x_0$ is a nondegenerate critical point of $\phi$, the system $(E_1)$ becomes

$$\beta = O\left(\frac{\varepsilon}{\log \varepsilon} \right) + |\beta|^2 \frac{1}{\lambda^{1/2}}, \quad \rho = O\left(\frac{\varepsilon}{\log \varepsilon} \right) + |\beta|^2 + |\xi|^2 + \rho^2 \frac{1}{\lambda^{1/2}},$$

(3.24)

Thus Brower’s fixed point theorem shows that the system $(E_2)$ has a solution $(\beta_\varepsilon, \rho_\varepsilon, \xi_\varepsilon)$ for $\varepsilon$ small enough such that

$$\beta_\varepsilon = O(\varepsilon), \quad \rho_\varepsilon = O(\varepsilon), \quad \xi_\varepsilon = O(\varepsilon).$$
By construction, the corresponding \(u_\varepsilon\) is a critical point of \(J_\varepsilon\) that is \(w_\varepsilon = J_\varepsilon(u_\varepsilon)^{(\varepsilon - 5)/2}/(8 - \varepsilon)\) satisfies

\[
\Delta^2 w_\varepsilon = |w_\varepsilon|^{8 - \varepsilon} w_\varepsilon \quad \text{in } \Omega, \quad w_\varepsilon = \Delta w_\varepsilon = 0 \text{ on } \partial \Omega \tag{3.25}
\]

with \(|w_\varepsilon^-|_{L^{10}(\Omega)}\) very small, where \(w_\varepsilon^- = \max(0, -w_\varepsilon)\).

As in [7, Proposition 4.1], we prove that \(w_\varepsilon^- = 0\). Thus, since \(w_\varepsilon\) is a non-negative function which satisfies (3.25), the strong maximum principle ensures that \(w_\varepsilon > 0\) on \(\Omega\) and then \(w_\varepsilon\) is a solution of \((P_\varepsilon)\), which blows up at \(x_0\) as \(\varepsilon\) goes to zero. This ends the proof of Theorem 1.2. \(\square\)

References

20 Single blow-up solutions for a biharmonic equation


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