ON THE TWO-POINT BOUNDARY VALUE PROBLEM FOR QUADRATIC SECOND-ORDER DIFFERENTIAL EQUATIONS AND INCLUSIONS ON MANIFOLDS

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The two-point boundary value problem for second-order differential inclusions of the form $(D/dt)m(t) \in F(t,m(t),\dot{m}(t))$ on complete Riemannian manifolds is investigated for a couple of points, nonconjugate along at least one geodesic of Levi-Civita connection, where $D/dt$ is the covariant derivative of Levi-Civita connection and $F(t,m,X)$ is a set-valued vector with quadratic or less than quadratic growth in the third argument. Some interrelations between certain geometric characteristics, the distance between points, and the norm of right-hand side are found that guarantee solvability of the above problem for $F$ with quadratic growth in $X$. It is shown that this interrelation holds for all inclusions with $F$ having less than quadratic growth in $X$, and so for them the problem is solvable.

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1. Introduction and discussion of the problem

Let $M$ be a finite-dimensional manifold and $TM$ be its tangent bundle with the natural projection $\pi : TM \to M$. Consider a set-valued map $F : R \times TM \to TM$ such that for any point $(m,X) \in TM$ (this means that $X \in T_mM$, i.e., $X$ is a tangent vector to $M$ at the point $m \in M$) the relation $\pi F(t,m,X) = \pi(m,X) = m$ holds.

The main aim of this paper is investigation of two-point boundary value problem for second-order differential inclusions of the form

$$\frac{D}{dt}m(t) \in F(t,m(t),\dot{m}(t)) \quad (1.1)$$

with $F$ having quadratic or less than quadratic growth in the third argument where $D/dt$ is the covariant derivative of a certain connection.

Such inclusions arise in description of complicated mechanical systems on nonlinear configuration spaces where the set-valued right-hand side $F$ is generated by an essentially discontinuous force field or by a force with control (see, e.g., [8, 10]). That is why everywhere below we call $F$ a set-valued force field.
2 Two-point boundary value problem

Besides its mechanical meaning this problem with $F$ quadratic in $X$ is important since it is a generalization of the well-known classical problem on the possibility to join two given points in a manifold by a geodesic curve of a certain connection (see, e.g., [17]). Recall that if $\nabla$ and $\bar{\nabla}$ are covariant derivatives of two different connections on a manifold $M$, there exists a $(1, 2)$-tensor field $S(\cdot, \cdot)$ on $M$ such that for any two vector fields $X$ and $Y$ on $M$ the equality $\bar{\nabla}_X Y = \nabla_X Y + S(X, Y)$ holds (see, e.g., [17, Statement 7.10]). From this it follows that in terms of covariant derivative $\nabla$ the geodesics of another connection $\bar{\nabla}$ are always described by an equation of the form

$$\frac{D}{dt} \dot{m}(t) = a(m(t), \dot{m}(t)), \quad (1.2)$$

where $a(m, X) = S_m(X, X)$ is a vector filed on $M$ that is quadratic in $X \in T_mM$ at any point $m \in M$.

For the Levi-Civita connection on a complete Riemannian manifold the solvability of two-point boundary value problem for $(1.2)$ for any points $m_0, m_1$ follows from Hopf-Rinow theorem (see, e.g., [2, 17]). But it is not the case even for a Riemannian connection with nonzero torsion: in [1, 6, 14] examples of Riemannian connections (in particular, on a compact manifold, two-dimensional torus) are presented for which this problem may not be solvable.

Consider two elementary and nevertheless characteristic examples where the two-point boundary value problem for $(1.2)$ (and so for $(1.1)$) may not be solvable in spite of the fact that $(1.1)$ is given in terms of Levi-Civita connection of a complete Riemannian metric.

**Example 1.1.** Consider a mechanical system on the unit sphere $S^2$, embedded into $\mathbb{R}^3$, with the force field $\alpha(\bar{r}, \dot{\bar{r}}) = [\bar{r}, \dot{\bar{r}}]||\dot{\bar{r}}||$ where the square brackets denote vector product. Taking into account the fact that $S^2$ is embedded into $\mathbb{R}^3$, we can apply d’Alembert principle and reduce $(1.2)$ to the equation of motion with a constraint in the form: $\ddot{\bar{r}} = [\bar{r}, \dot{\bar{r}}]||\dot{\bar{r}}|| - 2T\bar{r}$ where the kinetic energy $T = (1/2)\dot{\bar{r}}^2$. Since the acceleration is everywhere orthogonal to the velocity, it is obvious that $\dot{T} = 0$. Consider the vector $\dot{\bar{b}} = [\ddot{\bar{r}}, \dot{\bar{r}}]$. Direct calculations yield $\dot{\bar{b}} = 0$. This means that any trajectory satisfies the relation $(\dot{\bar{b}}, \ddot{\bar{r}}) = \text{const}$ (the parentheses denote scalar product in $\mathbb{R}^3$), that is, it is a circle on the sphere that also lies in a plane orthogonal to the constant vector $\dot{\bar{b}}$. Antipodal points are joint by a great circle, that is, $(\dot{\bar{b}}, \ddot{\bar{r}}) = 0$. From this we get the equality for mixed product $(\bar{r}, \dot{\bar{r}}, \ddot{\bar{r}}) = 0$ that is impossible. Thus the antipodal points on the sphere cannot be connected with a trajectory.

**Example 1.2.** Let $X = (x, y)$ be a vector from $\mathbb{R}^2$ and let $a > 0$ be a real number; by $|| \cdot ||$ denote the norm in $\mathbb{R}^2$. In $\mathbb{R}^2$ consider the following system of $(1.2)$ type:

$$\dot{x}(t) = -a||\dot{X}||y, \quad \dot{y}(t) = a||\dot{X}||x \quad (1.3)$$

with initial condition $X(0) = 0, X(0) = X_0$. Since here the vectors $\dot{X}$ and $\dot{X}$ are orthogonal to each other along the solution, $||\dot{X}||$ is constant. Let $||X_0|| = C$, represent the vector $X_0$ in
the form \( X_0 = C(-\sin \varphi_0, \cos \varphi_0) \). Then the solution of above-mentioned Cauchy problem takes the form 
\[
x(t) = \left( \frac{1}{a} \right) \cos(Cat + \varphi_0) - \left( \frac{1}{a} \right) \cos \varphi_0, \quad y(t) = \left( \frac{1}{a} \right) \sin(Cat + \varphi_0) - \left( \frac{1}{a} \right) \sin \varphi_0.
\]
Hence any solution is a circle with the radius \( 1/a \) and it does not leave the disc of radius \( 2/a \) with the center at the initial point. We would like to emphasize that the radius is being reduced as \( a \) is increasing.

If the points are conjugate along all geodesics of Levi-Civita connection joining them (like antipodal points in Example 1.1), the problem may not be solvable even for uniformly bounded \( \alpha(m,X) \) and for \( \alpha(m,X) \) having linear growth in velocities (see [8, 10]). Example 1.2 is representative specially for quadratic right-hand sides.

The two-point boundary value problem for (1.1) and (1.2) with nonconjugate points has been investigated under various conditions, more restrictive than ours in this paper. For (1.2) (i.e., for single-valued force fields) its solvability was shown by Gliklikh for continuous force fields in [7] (bounded case) and in [9] (linear growth in \( X \)), by Yakovlev, for example, in [18] for smooth force fields under some complicated conditions and by Ginzburg in [6] for smooth force fields with less than quadratic growth in \( X \). The solvability of this problem for inclusion (1.1) was shown for set-valued force fields of several types (Gel’man and Gliklikh [5], Gliklikh and Obukhovskii [12, 13], Kisielewicz [16], etc.) but only in uniformly bounded case.

In this paper, we consider the above-mentioned problem for (1.1) with force fields having quadratic or less than quadratic growth in \( X \). We deal with \( F(t,m,X) \) either almost lower semicontinuous or satisfying upper Carathéodory condition (in the latter case \( F(t,m,X) \) has convex images). We suppose that \( m_0 \) and \( m_1 \) are not conjugate along at least one Levi-Civita geodesic and show that if \( F(t,m,X) \) has less than quadratic growth in \( X \) (see Definition 3.1 below), there exists a solution of (1.1) that joins those points. For the case of \( F \) having quadratic bound in \( X \) (see Definition 3.2 below, it is a natural generalization of quadratic growth property for a right-hand side of (1.2)) we find a certain condition on geometric properties of \( M \), Riemannian distance between \( m_0 \) and \( m_1 \) and the norm of operator \( F \) that guarantees the solvability of the problem (see Remark 3.9 below). The former result is a generalization of that from [6] for second-order differential equations with smooth force fields having less that quadratic growth in velocities. Notice that in [6] the arguments based on uniqueness of solution to Cauchy problem for (1.2) are used that are not applicable to the case of inclusion (1.1).

Preliminary material from set-valued analysis can be found in [3, 4, 15], from geometry of manifolds, in [2, 14, 17].

2. Mathematical machinery

In this section, we modify some constructions from [8, 10] for the problem under consideration.

Let \( M \) be a complete Riemannian manifold. Consider \( m_0 \in M, \, [0,1] \subset R \) and let \( v: [0,1] \to T_{m_0}M \) be a continuous curve. It is shown that there exists unique \( C^1 \)-curve \( m: [0,1] \to M \) such that \( m(0) = m_0 \) and the vector \( \dot{m}(t) \) is parallel along \( m(\cdot) \) to the vector \( v(t) \in T_{m_0}M \) at any \( t \in [0,1] \).

Denote the curve \( m(t) \), constructed above from the curve \( v(t) \), by the symbol \( Fv(t) \). Thus, we have defined a continuous operator \( F: C^0([0,1], T_{m_0}M) \to C^1([0,1],M) \) that
Lemma 2.1. There exists a ball \( U_\varepsilon \subset C^0([0,1], T_{m_0}M) \) with a radius \( \varepsilon > 0 \) such that for any curve \( \hat{u}(t) \in U_\varepsilon \subset C^0([0,1], T_{m_0}M) \) there exists a unique vector \( C_u \), belonging to a certain bounded neighbourhood \( V \) of the vector \( \hat{\gamma}(0) \) in \( T_{m_0}M \), that is continuous in \( \hat{u} \) and such that \( \mathcal{F}(\hat{u} + C_u)(1) = m_1 \).

\[ \text{Proof.} \text{ By the construction of operator } \mathcal{F} \text{ its value } \mathcal{F}_\gamma(1) \text{ on the constant curve } v_\gamma(t) = \hat{\gamma}(0) \text{ coincides with } \exp_{m_0} \hat{\gamma}(0) = m_1. \text{ Since } m_0 \text{ and } m_1 \text{ are not conjugate along } \gamma, \exp_{m_0} \text{ is a diffeomorphism of a certain neighbourhood } \hat{\gamma}(0) \in T_{m_0}M \text{ onto a neighbourhood of the point } m_1 \text{ in } M. \text{ Applying the implicit function theorem, one can easily show that the perturbation of exponential map, that sends } X \in T_{m_0}M \text{ to } \mathcal{F}(X + \hat{u})(1), \text{ is also a diffeomorphism of a certain neighbourhood } V \text{ of } \hat{\gamma}(0) \text{ onto a neighbourhood of } m_1 \text{ in } M \text{ for any curve } \hat{u}(t) \text{ from a small enough } \varepsilon\text{-neighbourhood of the origin in } C^0([0,1], T_{m_0}M). \]

Introduce the notation \( \sup_{C \in V} \|C\| = C \) where \( V \) is from Lemma 2.1.

Lemma 2.2. In conditions and notations of Lemma 2.1 let \( K > 0 \) and \( t_1 > 0 \) be such that \( t_1^{-1} \varepsilon > K. \) Then for any curve \( u(t) \in U_K \subset C^0([0,t_1], T_{m_0}M) \) there exists a unique vector \( C_u \) in a neighbourhood \( t_1^{-1}V \) of the vector \( t_1^{-1}\hat{\gamma}(0) \) in \( T_{m_0}M \), continuously depending on \( u \) and such that \( S(u + C_u)(t_1) = m_1. \)

\[ \text{Proof.} \text{ For } u(t) \in U_K \subset C^0([0,t_1], T_{m_0}M) \text{ introduce } \hat{u}(t) = t_1 u(t_1 \cdot t) \in U_\varepsilon \subset C^0([0,1], T_{m_0}M) \text{ and } C_u = t_1^{-1}C_{\hat{u}}. \text{ From Lemma 2.1 we get } \mathcal{F}_\gamma(u + C_u)(1) = m_1 \text{ and } (d/dt)\mathcal{F}_\gamma(\hat{u}(t) + C_u) \text{ is parallel to } \hat{u}(t) + C_u. \text{ For the curve } \gamma(t) = \mathcal{F}_\gamma(u + C_u)(t \cdot t_1) \text{ we have } (d/dt)\gamma(t) = t_1^{-1}(d/dt)\mathcal{F}_\gamma(\hat{u}(t) + C_u)(t \cdot t_1) \text{ and this vector is parallel along the same curve to the vector } t_1^{-1}(\hat{u}(t) + C_u) = u(t) + C_u. \text{ Thus } \gamma(t) = \mathcal{F}_\gamma(u + C_u)(t) = \mathcal{F}_\gamma(\hat{u}(t) + C_u)(t \cdot t_1^{-1}) \text{ for } t \in [0,t_1]. \text{ Hence } \mathcal{F}_\gamma(u + C_u)(t_1) = \mathcal{F}_\gamma(\hat{u}(t) + C_u)(1) = m_1. \]

Lemmas 2.1 and 2.2 form a modification of [10, Theorem 3.3].

Lemma 2.3. For specified \( t_1 > 0 \) and \( K > 0 \) all curves \( S(v(t) + C_v)(t) \) with \( v \in U_K \subset C^0([0,t_1], T_{m_0}M) \) lie in a compact set \( \Xi \subset M \) where \( \Xi \) depends on \( \varepsilon \) and \( C \) introduced above.

Indeed, since the parallel translation preserves the norm of a vector, for any \( v(t) \) as above the length of \( S(v(t) + C_v)(t) \) is not greater than \( \int_0^1 (K + \|C_v\|)dt \leq \int_0^1 t_1^{-1}(\varepsilon + C)dt = \int_0^1 (\varepsilon + C)dt = \varepsilon + C. \) Since \( M \) is complete, by Hopf-Rinow theorem any metric ball of finite radius \( \varepsilon + C \) is compact.

Lemma 2.4. Let a real number \( \delta \) satisfy the inequality \( 0 < \delta < \varepsilon/(\varepsilon + C)^2. \) Then there exists a small enough positive number \( \varphi \) such that \( (\varepsilon t_1^{-1} - \varphi) > 0 \) and the inequality \( \delta ((\varepsilon t_1^{-1} - \varphi) + C t_1^{-1})^2 < \varepsilon t_1^{-2} - \varphi t_1^{-1} \) holds.
Proof. For δ satisfying the hypothesis of the lemma we get δ(εt₁⁻¹ + Ct₁⁻¹)² < εt₁⁻². From continuity of both sides of this inequality it follows that there exists a small enough number φ > 0 such that (εt₁⁻¹ − φ) > 0 and the inequality δ((εt₁⁻¹ − φ) + Ct₁⁻¹)² < εt₁⁻² − φt₁⁻¹ holds.

3. The main statements

Everywhere below M is a complete Riemannian manifold, by ∥ · ∥ we denote the norm in a tangent space generated by the Riemannian metric. Introduce the norm of the set ∥F(t, m, X)∥ ∈ TmM by usual formula ∥F(t, m, X)∥ = supy∈F(t,m,X)∥y∥.

Definition 3.1. We say that F(t, m, X) has less than quadratic growth in X if for any compact Θ ⊂ M and any finite interval [0, l] the relation

\[ \lim_{∥X∥→∞} \frac{∥F(t, m, X)∥}{∥X∥^2} = 0 \]  

(3.1)

holds uniformly in t ∈ [0, l] and m ∈ Θ.

Definition 3.2. We say that F(t, m, X) has quadratic bound in X if for any compact Θ ⊂ M and any finite interval [0, l] the relation

\[ \lim_{∥X∥→∞} \frac{∥F(t, m, X)∥}{∥X∥^2} = a(t, m) \]  

(3.2)

holds uniformly in t ∈ [0, l] and m ∈ Θ where a(t, m) ≥ 0 is a real bounded function on [0, l] × Θ that is not identical zero.

Definition 3.3. We say that F(t, m, X) satisfies upper Carathéodory conditions if:

1. for every (m, X) ∈ TM the map F(·, m, X) : I → TmM is measurable,
2. for almost all t ∈ I the map F(t, ·, ·) : TM → TM is upper semicontinuous.

Definition 3.4. Let I = [0, l] ⊂ R. The set-valued force field F : I × TM → TM is called almost lower semicontinuous if there exists a countable sequence of disjoint compact sets \{In\}, In ⊂ I such that: (i) the measure of I \ ∪n In is equal to zero; (ii) the restriction of F on each In × TM is lower semicontinuous.

Theorem 3.5. Let F(t, m, X) satisfy the upper Carathéodory condition, has convex closed bounded images and has less than quadratic growth in X. Let the points m₁ and m₀ be nonconjugate along a certain geodesic g of the Levi-Civita connection. Then there exists a positive number L(m₀, m₁, g) such that if 0 < t₁ < L(m₀, m₁, g) there exists a solution m(t) of (1.1), for which m(0) = m₀ and m(t₁) = m₁.

Proof. For a C¹-curve γ(t) = Fv(t), v(·) ∈ C⁰(I, Tm₀M), consider the set-valued vector field F(t, γ(t), ˙γ(t)). Denote by Γ the operator of parallel translation of vectors along γ(·) at the point γ(0) = m₀. Apply operator Γ to all sets F(t, γ(t), ˙γ(t)) along γ(·). As a result for any v ∈ C⁰(I, Tm₀M) we obtain a set-valued map ΓFv : [0, l] → Tm₀M that has convex images. It is shown in [13] that the map ΓFv ∈ C⁰([0, l], Tm₀M) × [0, l] → Tm₀M satisfies upper Carathéodory conditions. Denote by PTFv the set of all measurable selections
of $\Gamma F \mathcal{F} v : [0, l] \to T_{m_0} M$ (such selections exist by [3]). Define on $C^0([0, t_1], T_{m_0} M)$ the set-valued operator $\mathcal{P}T \mathcal{F} \mathcal{F}$ by the formula

$$ \mathcal{P}T \mathcal{F} \mathcal{F} v = \left\{ \int_0^t f(\tau)d\tau \mid f(\cdot) \in \mathcal{P}T \mathcal{F} \mathcal{F} v \right\}. $$

(3.3)

It is shown in [13] that $\mathcal{P}T \mathcal{F} \mathcal{F}$ is upper semicontinuous, has convex images and sends bounded sets from $C^0([0, t_1], T_{m_0} M)$ into compacts.

Consider the numbers $\varepsilon$ and $C$ constructed for the points $m_0$ and $m_1$ and geodesic $g$. Let $\Xi$ be a compact from Lemma 2.3, and let $[0, l]$ be a certain interval. Choose a positive number $\delta < \varepsilon/(\varepsilon + C)^2$. Since $F$ satisfies Definition 3.1, one can easily see that there exists a number $Q > 0$ such that for $\|X\| \geq Q$ the inequality

$$ \max_{(t, m) \in T \times \Xi} \|F(t, m, Y)\| < \delta \|X\|^2 $$

(3.4)

holds for all $\|Y\| < \|X\|$. For $t_1 > 0$ small enough we get $t_1 \in [0, l]$ and $(t_1^{-1} \varepsilon - \varphi > Q$ where $\varphi$ is from Lemma 2.4. Determine $L(m_0, m_1, g)$ as the upper bound of $t_1$ such that the above relations hold. Let $0 < t_1 < L(m_0, m_1, g)$. For this $t_1$ denote by $K$ the corresponding number $t_1^{-1} \varepsilon - \varphi$.

By the construction $t_1^{-1} \varepsilon > K$ and so by Lemma 2.2 the operator $\mathfrak{L} : U_K \to C^0([0, t_1], T_{m_0} M)$:

$$ \mathfrak{L}(v) = \int \mathcal{P}T \mathcal{F} \mathcal{F}(v + C_v) $$

(3.5)

is well posed. As well as $\mathcal{P}T \mathcal{F} \mathcal{F}$ this operator is upper semicontinuous, has convex images and sends bounded sets from $C^0([0, t_1], T_{m_0} M)$ into compacts.

For $v \in U_K \subset C^0([0, t_1], T_{m_0} M)$, since the parallel translation preserves the norm of a vector, from the construction of operator $\mathcal{F}$, from (3.4) and from Lemma 2.4 it follows that

$$ \left\| F \left( t, \mathcal{F} \left( v(t) + C_v \right), \frac{d}{dt} \mathcal{F} \left( v(t) + C_v \right) \right) \right\| < \delta (t_1^{-1} \varepsilon - \varphi + C_t^{-1} \varepsilon)^2 < (t_1^{-2} \varepsilon - t_1^{-1} \varphi). $$

(3.6)

Since parallel translation preserves the norm of a vector, from the last inequality it follows that

$$ \| \mathfrak{L}(v + C_v) \| = \left\| \int \mathcal{P}T \mathcal{F} \mathcal{F}(v(\tau) + C_v) \right\|_{C^0([0, t_1], T_{m_0} M)} < (t_1^{-1} \varepsilon - \varphi) = K. $$

(3.7)

Thus $\mathfrak{L}$ sends the ball $U_K$ into itself and from Schauder’s principle for upper semicontinuous set-valued maps (see, e.g., [3]) it follows that it has a fixed point $u^* \in U_K$, that is, $u^* \in \mathfrak{L}u^*$. Let us show that $m(t) = \mathcal{F}(u^*(t) + C_{u^*})$ is the desired solution. By the construction we have $m(0) = m_0$ and $m(t_1) = m_1$, $m(t)$ is a $C^1$-curve and $m(t)$ is absolutely continuous. Note that $u^*$ is a selection of $\Gamma F(t, \mathcal{F}(u^* + C_{u^*}),(d/dt)\mathcal{F}(u^* + C_{u^*}))$ because $u^*$ is a fixed point of $\mathfrak{L}$. In other words, the inclusion $u^*(t) \in \Gamma F(t, \mathcal{F}(u^* + C_{u^*}),(d/dt)\mathcal{F}(u^* + C_{u^*}))$ holds for all points $t$ at which the derivative exists. Using the properties of the covariant derivative and the definition of $u^*$, one can show that $u^*(t)$ is
parallel to \((D/dt)m(t)\) along \(m(\cdot)\) and \(\Gamma F(t, \mathcal{F}(u^* + C_{u^*}), (d/dt)\mathcal{F}(u^* + C_{u^*}))\) is parallel to \(F(t, m(t), \dot{m}(t))\). Hence, \((D/dt)m(t) \in F(t, m(t), \dot{m}(t))\).

**Theorem 3.6.** Let \(F(t, m, X)\) satisfy the upper Carathéodory condition, has convex closed bounded images and has quadratic bound in \(X\). Let the points \(m_1\) and \(m_0\) be nonconjugate along a certain geodesic \(g\) of the Levi-civitá connection. Let in addition for \(t \in [0, 1]\) and \(m \in \Xi\), where \([0, 1]\) is a certain interval and \(\Xi\) is the compact from Lemma 2.3, for the function \(a(t, m)\) from Definition 3.2 there exists a real number \(\delta\) such that the estimate \(a(t, m) < \delta < \varepsilon/(\varepsilon + C)^2\) holds. Then there exists a positive number \(L(m_0, m_1, g)\) such that if \(0 < t_1 < L(m_0, m_1, g)\) there exists a solution \(m(t)\) of (1.1), for which \(m(0) = m_0\) and \(m(t_1) = m_1\).

The proof of Theorem 3.6 follows the same scheme of arguments as that for Theorem 3.5. The only modification is that here for \(F\) with quadratic bound in \(X\) we assume the existence of \(\delta\) such that \(a(t, m) < \delta < \varepsilon/(\varepsilon + C)^2\) while in the proof of Theorem 3.5 analogous \(\delta\) is shown to exist for any \(F\) with less than quadratic growth in \(X\).

**Theorem 3.7.** Let \(F(t, m, X)\) be almost lower semicontinuous, has closed bounded images and has less than quadratic growth in \(X\). Let the points \(m_1\) and \(m_0\) be nonconjugate along a certain geodesic \(g\) of the Levi-civitá connection. Then there exists a positive number \(L(m_0, m_1, g)\) such that if \(0 < t_1 < L(m_0, m_1, g)\) there exists a solution \(m(t)\) of (1.1), for which \(m(0) = m_0\) and \(m(t_1) = m_1\).

**Proof.** Here we use the same notations as in the proof of Theorem 3.5. Notice that from the condition of less than quadratic growth for \(F\) it follows that for all \(v \in C^0([0, 1], T_{m_0}M)\) the curves from \(\mathcal{P}TF\mathcal{F}v\) are integrable. Hence the set-valued map \(\mathcal{P}TF\mathcal{F}\) sends \(C^0([0, 1], T_{m_0}M)\) into \(L^1(([0, 1], \mathcal{A}, \mu), T_{m_0}M)\), where \(\mathcal{A}\) is the Borel \(\sigma\)-algebra and \(\mu\) is the normalized Lebesgue’s measure. Since \(F\) is almost lower semicontinuous, in complete analogy with [15] one can easily show that \(\mathcal{P}TF\mathcal{F} : C^0([0, 1], T_{m_0}M) \rightarrow L^1(([0, 1], \mathcal{A}, \mu), T_{m_0}M)\) is lower semicontinuous and has decomposable images (see the definition of decomposable image, e.g., in [4]). Then by Bressan-Kolombo theorem (see, e.g., [4]) it has a continuous selection that we denote by \(p\mathcal{P}TF\mathcal{F}\).

Choose the numbers \(Q, L(m_0, m_1, g), 0 < t_1 < L(m_0, m_1, g)\) and \(K\) as in the proof of Theorem 3.5. Then on the ball \(U_K \subset C^0([0, t_1], T_{m_0}M)\) the operator

\[
\mathcal{G}v = \int_0^t p\Gamma F\mathcal{F} \left( (v(s) + C_v), \frac{d}{dt} \mathcal{F}(v(s) + C_v) \right) ds : U_K \rightarrow C^0([0, t_1], T_{m_0}M) \tag{3.8}
\]

is well posed. As a corollary to [11, Lemma 19], we get that \(\mathcal{G}\) is completely continuous. Since parallel translation preserves the norm of a vector, from the construction of \(\mathcal{F}\) for any \(u \in U_K\) with given \(F\) we get

\[
\|\mathcal{G}v\| = \left\| \int_0^t p\Gamma F\mathcal{F} \left( (s, \mathcal{F}(v(s) + C_v), \frac{d}{dt} \mathcal{F}(v(s) + C_v) \right) ds \right\|_{C^0([0, t_1], T_{m_0}M)} \tag{3.9}
\]

\[
\leq (t_1^{-1} \varepsilon - \varphi) = K.
\]
Hence $G$ sends $U_K$ into itself and by classical Schauder’s principle it has a fixed point $u^* \in U_K$. Using the same arguments, as in the proof of Theorem 3.5, one can easily prove that $m(t) = G(u^* + C_u^*)(t)$ is a solution of (1.1) such that $m(0) = m_0$ and $m(t_1) = m_1$. □

Theorem 3.8. Let $F(t, m, X)$ be almost lower semicontinuous, has closed bounded images and quadratic bound in $X$. Let the points $m_1$ and $m_0$ be nonconjugate along a certain geodesic $g$ of the Levi-civita connection. Let in addition for $t \in [0, l]$ and $m \in \Xi$, where $[0, l]$ is a certain interval and $\Xi$ is the compact from Lemma 2.3, for the function $a(t, m)$ from Definition 3.2 there exists a real number $\delta$ such that the estimate $a(t, m) < \delta < \varepsilon/(\varepsilon + C)^2$ holds. Then there exists a positive number $L(m_0, m_1, g)$ such that if $0 < t_1 < L(m_0, m_1, g)$ there exists a solution $m(t)$ of (1.1), for which $m(0) = m_0$ and $m(t_1) = m_1$.

As well as in the case of Theorems 3.5 and 3.6, Theorem 3.8 is proved in complete analogy with Theorem 3.7 with the following minor modification: in Theorem 3.8 for $F$ with quadratic bound in $X$ we assume the existence of $\delta$ such that $a(t, m) < \delta < \varepsilon/(\varepsilon + C)^2$ while in the proof of Theorem 3.7 we use the fact that analogous $\delta$ does exist for any $F$ with less than quadratic growth in $X$ (see the proof of Theorem 3.5).

Remark 3.9. Notice that if a geodesic, along which $m_0$ and $m_1$ are not conjugate, is a length minimizing one, the number $C$ characterizes the Riemannian distance between these points. The numbers $C$ and $\varepsilon$ together provide a certain characteristics of the Riemannian geometry on $M$ in a neighbourhood of $m_0$. Theorems 3.6 and 3.8 establishes an interrelation between $C$, $\varepsilon$ and the quadratic bounds of (1.1), under which the two-point boundary value problem for nonconjugate points $m_0$ and $m_1$ is solvable for sure.

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