EXISTENCE OF POSITIVE SOLUTIONS FOR SOME POLYHARMONIC NONLINEAR EQUATIONS IN $\mathbb{R}^n$

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We will study the following polyharmonic nonlinear elliptic equation $(-\Delta)^m u + f(\cdot, u) = 0$ in $\mathbb{R}^n$, $n > 2m$. Under appropriate conditions on the nonlinearity $f(x, t)$, related to a class of functions called m -Green-tight functions, we give some existence results for the above equation.

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1. Introduction

In this paper, we deal with the higher order elliptic equation

$$(-\Delta)^m u = f(\cdot, u), \quad \text{in } \mathbb{R}^n, \quad (1.1)$$

where m is a positive integer such that $n > 2m$.

In the case $m = 1$, (1.1) contains several well-known types which have been studied extensively by many authors (see for example [1–3, 8, 9, 11, 12, 14] and the references therein). Their basic tools are essentially some properties of functions belonging to the classical Kato class $K_n(\mathbb{R}^n)$ and the subclass of Green-tight functions $K_n^\infty(\mathbb{R}^n)$ (some properties pertaining to these classes can be found in [1, 4, 14]).

In this paper, we are concerned with the high order. Our purpose is two folded. One is to extend the Kato class $K_n(\mathbb{R}^n)$ and the subclass $K_n^\infty(\mathbb{R}^n)$ to the order $m \geq 2$. The second purpose is to investigate the existence of positive solutions for (1.1). The outline of the paper is as follows. The existence results are given in Sections 3, 4 and 5. In Section 2, we give the explicit formula of the Green function $G_{m,n}(x, y)$ of $(-\Delta)^m$ in $\mathbb{R}^n$. Namely, for each x, y in $\mathbb{R}^n$

$$G_{m,n}(x, y) = k_{m,n} \frac{1}{|x - y|^{n-2m}}, \quad (1.2)$$

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where $k_{m,n}$ is a positive constant which will be precised later. The 3G-Theorem proved in [13] for the case $m = 1$, is also valid for every m . Indeed, for each x, z in $\mathbb{R}^n$, we have

$$\frac{G_{m,n}(x, z)G_{m,n}(z, y)}{G_{m,n}(x, y)} \leq 2^{n-2m-1} [G_{m,n}(x, z) + G_{m,n}(z, y)]. \quad (1.3)$$

This 3G-Theorem will be useful to state our existence results.

Next, we study the Kato class $K_{m,n}(\mathbb{R}^n)$ defined as follows.

Definition 1.1. A Borel measurable function φ in $\mathbb{R}^n$ ($n > 2m$), belongs to the Kato class $K_{m,n}(\mathbb{R}^n)$ if

$$\lim_{\alpha \rightarrow 0} \left(\sup_{x \in \mathbb{R}^n} \int_{|x-y| \leq \alpha} \frac{|\varphi(y)|}{|x-y|^{n-2m}} dy \right) = 0. \quad (1.4)$$

Indeed, first we prove some properties of functions belonging to this class similar to those established in [1, 4]. In particular, we have the following characterization

$$\varphi \in K_{m,n}(\mathbb{R}^n) \iff \lim_{t \rightarrow 0} \left(\sup_{x \in \mathbb{R}^n} \int_0^t s^{m-1} \int_{\mathbb{R}^n} p(s, x, y) |\varphi(y)| dy ds \right) = 0, \quad (1.5)$$

where $p(t, x, y) = (1/(4\pi t)^{n/2}) \exp(-|x-y|^2/4t)$, for $t \in (0, \infty)$ and $x, y \in \mathbb{R}^n$, is the density of the Gauss semi-group on $\mathbb{R}^n$.

Secondly, we study a subclass of $K_{m,n}(\mathbb{R}^n)$ denoted by $K_{m,n}^\infty(\mathbb{R}^n)$ and defined by the following.

Definition 1.2. A Borel measurable function φ belongs to the class $K_{m,n}^\infty(\mathbb{R}^n)$ and it is called m -Green-tight function if $\varphi \in K_{m,n}(\mathbb{R}^n)$ and satisfies

$$\lim_{M \rightarrow \infty} \left(\sup_{x \in \mathbb{R}^n} \int_{|y| \geq M} \frac{|\varphi(y)|}{|x-y|^{n-2m}} dy \right) = 0. \quad (1.6)$$

In particular, we characterize the class $K_{m,n}^\infty(\mathbb{R}^n)$ as follows.

THEOREM 1.3. Let $\varphi \in \mathcal{B}^+(\mathbb{R}^n)$, ($n > 2m$). Then the following assertions are equivalent

- (1) $\varphi \in K_{m,n}^\infty(\mathbb{R}^n)$.
- (2) The m -potential of φ , $V\varphi(x) := \int_{\mathbb{R}^n} G_{m,n}(x, y)\varphi(y)dy$ is in $C_0^+(\mathbb{R}^n)$.

This Theorem improves the result of Zhao in [14], for the case $m = 1$. A more fine characterization will be given in the radial case.

One can easily check that $L^1(\mathbb{R}^n) \cap K_{m,n}(\mathbb{R}^n) \subset K_{m,n}^\infty(\mathbb{R}^n)$. Also we show that for $p > n/2m$ and $\lambda < 2m - n/p < \mu$, we have

$$\frac{L^p(\mathbb{R}^n)}{(1 + |\cdot|)^{\mu-\lambda} |\cdot|^\lambda} \subset K_{m,n}^\infty(\mathbb{R}^n), \quad (1.7)$$

and we precise the behaviour of the m -potential of functions in this class.

In Section 3, we are interested in the following polyharmonic problem

$$\begin{aligned} (-\Delta)^m u + u\varphi(\cdot, u) &= 0, \quad \text{in } \mathbb{R}^n \text{ (in the sense of distributions)} \\ \lim_{|x| \rightarrow \infty} u(x) &= c > 0. \end{aligned} \quad (1.8)$$

The function φ is required to verify the following assumptions.

(H₁) φ is a nonnegative measurable function on $\mathbb{R}^n \times (0, \infty)$.

(H₂) For each $\lambda > 0$, there exists a nonnegative function $q_\lambda \in K_{m,n}^\infty(\mathbb{R}^n)$ with $\alpha_{q_\lambda} \leq 1/2$ (see (1.24)) and such that for each $x \in \mathbb{R}^n$, the mapping $t \rightarrow t(q_\lambda(x) - \varphi(x, t))$ is continuous and nondecreasing on $[0, \lambda]$.

Under these hypotheses, we give an existence result for the problem (1.8). In fact, we will prove the following theorem.

THEOREM 1.4. *Assume (H₁) and (H₂). Then the problem (1.8) has a positive continuous solution u in $\mathbb{R}^n$ satisfying for each $x \in \mathbb{R}^n$, $c/2 \leq u(x) \leq c$.*

To establish this result, we use a potential theory approach. In particular, we prove that if the function $q \in K_{m,n}^\infty(\mathbb{R}^n)$ is sufficiently small and f is a nonnegative function on $\mathbb{R}^n$, then the equation

$$(-\Delta)^m u + qu = f, \quad (1.9)$$

has a positive solution on $\mathbb{R}^n$. In [6], Grunau and Sweers gave a similar result in the unit ball of $\mathbb{R}^n$, with operators perturbed by small lower order terms:

$$(-\Delta)^m u + \sum_{|k| < 2m} a_k(u) D^k u = f. \quad (1.10)$$

In the case $m = 1$, the problem (1.8) has been studied by Mâagli and Masmoudi in [7, 8], where they gave an existence and an uniqueness result in both bounded and unbounded domain Ω .

In Section 4, we are concerned with the following polyharmonic problem

$$\begin{aligned} (-\Delta)^m u &= f(\cdot, u), \quad \text{in } \mathbb{R}^n \text{ (in the sense of distributions)} \\ \lim_{|x| \rightarrow \infty} u(x) &= 0. \end{aligned} \quad (1.11)$$

Here f is required to satisfy the following assumptions.

(H₃) f is a nonnegative measurable function on $\mathbb{R}^n \times (0, \infty)$, continuous with respect to the second variable.

(H₄) There exist a nonnegative function p in $\mathbb{R}^n$ such that

$$0 < \alpha_0 := \int_{\mathbb{R}^n} \frac{p(y)}{(|y| + 1)^{2(n-2m)}} dy < \infty \quad (1.12)$$

and a nonnegative function $q \in K_{m,n}^\infty(\mathbb{R}^n)$ such that for $x \in \mathbb{R}^n$ and $t > 0$

$$p(x)h(t) \leq f(x, t) \leq q(x)g(t), \quad (1.13)$$

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where h is a nonnegative nondecreasing measurable function on $[0, \infty)$ satisfying

$$m_0 := \frac{1}{k_{m,n}\alpha_0} < h_0 := \liminf_{t \rightarrow 0^+} \frac{h(t)}{t} \leq \infty \quad (1.14)$$

and g is a nonnegative measurable function locally bounded on $[0, \infty)$ satisfying

$$0 \leq g^\infty := \limsup_{t \rightarrow \infty} \frac{g(t)}{t} < M_0 := \frac{1}{\|Vq\|_\infty}. \quad (1.15)$$

By using a fixed point argument, we will state the following existence result.

THEOREM 1.5. *Assume (H_3) and (H_4) . Then the problem (1.11) has a positive continuous solution u in $\mathbb{R}^n$ satisfying for each $x \in \mathbb{R}^n$,*

$$\frac{a}{(|x| + 1)^{n-2m}} \leq u(x) \leq bVq(x), \quad (1.16)$$

where a, b are positive constants.

This result follows up the one of Dalmasso (see [5]), who studied the problem (1.11) in the unit ball B , with more restrictive conditions on the function f . Indeed, he assumed that f is nondecreasing with respect to the second variable and satisfies

$$\lim_{t \rightarrow 0^+} \min_{x \in \bar{B}} \frac{f(x, t)}{t} = +\infty, \quad \lim_{t \rightarrow +\infty} \max_{x \in \bar{B}} \frac{f(x, t)}{t} = 0. \quad (1.17)$$

He proved the existence of a positive solution and he gave also an uniqueness result for positive radial solution when $f(x, t) = f(|x|, t)$.

When $m = 1$, similar conditions, but more restrictive, on the nonlinearity f have been adopted by Mâagli and Masmoudi in [8]. In fact in [8], the authors studied (1.11) in an unbounded domain D of $\mathbb{R}^n$, $n \geq 3$, with compact nonempty boundary ∂D and gave an existence result as Theorem 1.5.

On the other hand, Brezis and Kamin proved in [3], the existence and the uniqueness of a positive solution for the problem

$$\begin{aligned} -\Delta u &= \rho(x)u^\alpha \quad \text{in } \mathbb{R}^n, \\ \liminf_{|x| \rightarrow \infty} u(x) &= 0, \end{aligned} \quad (1.18)$$

with $0 < \alpha < 1$ and ρ is a nonnegative measurable function satisfying some appropriate conditions. We improve in this section the result of Brezis and Kamin in [3] and the one of Mâagli and Masmoudi in [8].

In Section 5, we will study the existence of solutions to the following polyharmonic problem

$$\begin{aligned} (-\Delta)^m u &= f(\cdot, u), \quad \text{in } \mathbb{R}^n \text{ (in the sense of distributions)} \\ u(x) &> 0, \quad \text{in } \mathbb{R}^n, \end{aligned} \quad (1.19)$$

under the following assumptions on the nonlinearity f .

(H₅) f is a nonnegative measurable function on $\mathbb{R}^n \times (0, \infty)$, continuous with respect to the second variable on $(0, \infty)$.

(H₆) $f(x, t) \leq q(x, t)$, where q is a nonnegative measurable function on $\mathbb{R}^n \times (0, \infty)$ such that the function $t \rightarrow q(x, t)$ is nondecreasing on $(0, \infty)$.

(H₇) There exists a constant $c > 0$ such that $q(\cdot, c) \in K_{m,n}^\infty(\mathbb{R}^n)$ and

$$\|V(q(\cdot, c))\|_\infty < c. \quad (1.20)$$

Put $c^* = c - \|V(q(\cdot, c))\|_\infty$. We give in this section the following existence result.

THEOREM 1.6. *Assume (H₅), (H₆), and (H₇). Then for each $\delta \in (0, c^*]$, the problem (1.19) has a positive continuous solution u in $\mathbb{R}^n$ satisfying for each $x \in \mathbb{R}^n$*

$$\begin{aligned} \delta &\leq u(x) \leq c, \\ \lim_{|x| \rightarrow \infty} u(x) &= \delta. \end{aligned} \quad (1.21)$$

If $m = 1$, Yin gave in [11] an existence result of the following problem

$$\begin{aligned} \Delta u + f(x, u) &= 0, \quad \text{in } G_B, \\ u(x) &> 0, \end{aligned} \quad (1.22)$$

where $G_B = \{x \in \mathbb{R}^n, |x| > B\}$, for some $B \geq 0$. His method relies on the technique of radial super/subsolutions. Our approach is different, in fact we will use a fixed point argument. We improve the result of Yin under more general assumptions (see Remark 5.3).

In order to simplify our statements, we define some convenient notations.

Notations.

- (i) $\mathcal{B}(\mathbb{R}^n)$ denotes the set of Borel measurable functions in $\mathbb{R}^n$ and $\mathcal{B}^+(\mathbb{R}^n)$ the set of nonnegative ones.
- (ii) $C_0(\mathbb{R}^n) := \{w \text{ continuous on } \mathbb{R}^n \text{ and } \lim_{|x| \rightarrow \infty} w(x) = 0\}$ and $C_0^+(\mathbb{R}^n)$ the set of nonnegative ones.
- (iii) For $\varphi \in \mathcal{B}^+(\mathbb{R}^n)$, we put the m -potential of φ on $\mathbb{R}^n$ by

$$V\varphi(x) := V_{m,n}\varphi(x) = \int_{\mathbb{R}^n} G_{m,n}(x, y)\varphi(y)dy = k_{m,n} \int_{\mathbb{R}^n} \frac{\varphi(y)}{|x - y|^{n-2m}} dy. \quad (1.23)$$

- (iv) For $\varphi \in \mathcal{B}^+(\mathbb{R}^n)$, we put

$$\alpha_\varphi = \sup_{x, y \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{G_{m,n}(x, z)G_{m,n}(z, y)}{G_{m,n}(x, y)} |\varphi(z)| dz. \quad (1.24)$$

- (v) Let $\lambda \in \mathbb{R}$, we denote by $\lambda^+ = \max(\lambda, 0)$.

- (vi) Let f and g be two positive functions on a set S .

We call $f \sim g$, if there is $c > 0$ such that

$$\frac{1}{c}g(x) \leq f(x) \leq cg(x) \quad \forall x \in S. \quad (1.25)$$

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We call $f \preceq g$, if there is $c > 0$ such that

$$f(x) \leq cg(x) \quad \forall x \in S. \quad (1.26)$$

The following properties will be used several times: for $s, t \geq 0$, we have

$$\begin{aligned} \min(s, t) &= s \wedge t \sim \frac{st}{s+t}, \\ (s+t)^p &\sim s^p + t^p, \quad p \in \mathbb{R}^+. \end{aligned} \quad (1.27)$$

2. Properties of the Kato class

In this section, we characterize functions belonging to the Kato class $K_{m,n}(\mathbb{R}^n)$ and the subclass $K_{m,n}^\infty(\mathbb{R}^n)$ of m -Green-tight functions and we prove Theorem 1.3. We recall that throughout this paper, we are concerned with $n > 2m$.

We set $p(t, x, y) = (1/(4\pi t)^{n/2}) \exp(-|x - y|^2/4t)$, for $t \in (0, \infty)$ and $x, y \in \mathbb{R}^n$, the density of the Gauss semi-group on $\mathbb{R}^n$. By a simple computation, we obtain that the Green function of $(-\Delta)^m$ in $\mathbb{R}^n$, for each $m \geq 1$, is given by

$$G_{m,n}(x, y) = \frac{1}{(m-1)!} \int_0^\infty s^{m-1} p(s, x, y) ds, \quad \text{for } x, y \text{ in } \mathbb{R}^n. \quad (2.1)$$

Then we have the following explicit expression

$$G_{m,n}(x, y) = k_{m,n} \frac{1}{|x - y|^{n-2m}}, \quad \text{for } x, y \text{ in } \mathbb{R}^n, \quad (2.2)$$

where $k_{m,n} = \Gamma(n/2 - m)/4^m \pi^{n/2} (m-1)!$.

2.1. The class $K_{m,n}(\mathbb{R}^n)$. We will study properties of functions belonging to $K_{m,n}(\mathbb{R}^n)$. First we remark the following comparison on the classes $K_{j,n}(\mathbb{R}^n)$, for $j \geq 1$.

Remark 2.1. Let $j, m \in \mathbb{N}$ such that $1 \leq j \leq m$, then we have for each $n > 2m$

$$K_n(\mathbb{R}^n) := K_{1,n}(\mathbb{R}^n) \subseteq K_{j,n}(\mathbb{R}^n) \subseteq K_{m,n}(\mathbb{R}^n), \quad (2.3)$$

where $K_n(\mathbb{R}^n)$ is the classical Kato class introduced in [1].

Example 2.2. Let $\varphi \in \mathcal{B}(\mathbb{R}^n)$. Suppose that for $p > n/2m$, we have

$$\sup_{x \in \mathbb{R}^n} \int_{|x-y| \leq 1} |\varphi(y)|^p dy < \infty. \quad (2.4)$$

Then by the Hölder inequality, we conclude that $\varphi \in K_{m,n}(\mathbb{R}^n)$.

In particular, we have that for $p > n/2m$, $L^p(\mathbb{R}^n) \subset K_{m,n}(\mathbb{R}^n)$.

To establish the characterization (1.5) of the Kato class $K_{m,n}(\mathbb{R}^n)$, we need the following lemmas.

LEMMA 2.3. For each $t > 0$ and $x, y \in \mathbb{R}^n$, we have

$$\int_0^t s^{m-1} p(s, x, y) ds \leq G_{m,n}(x, y). \quad (2.5)$$

Moreover, for $|x - y| \leq 2\sqrt{t}$, we have that

$$G_{m,n}(x, y) \leq \int_0^t s^{m-1} p(s, x, y) ds. \quad (2.6)$$

Proof. Let $t > 0$ and $x, y \in \mathbb{R}^n$. Then (2.5) follows immediately from (2.1).

If we suppose further that $|x - y| \leq 2\sqrt{t}$, then we have

$$\begin{aligned} \int_0^t s^{m-1} p(s, x, y) ds &= c \int_0^t s^{m-n/2-1} \exp\left(-\frac{|x-y|^2}{4s}\right) ds \\ &= \frac{c}{|x-y|^{n-2m}} \int_{|x-y|^2/4t}^{\infty} r^{n/2-m-1} e^{-r} dr \\ &\geq \frac{c}{|x-y|^{n-2m}} \int_1^{\infty} r^{n/2-m-1} e^{-r} dr \\ &= c G_{m,n}(x, y), \end{aligned} \quad (2.7)$$

where the letter c is a positive constant which may vary from line to line. □

LEMMA 2.4. Let $\varphi \in K_{m,n}(\mathbb{R}^n)$. Then for each compact $L \subset \mathbb{R}^n$, we have

$$\sup_{x \in \mathbb{R}^n} \int_{x+L} |\varphi(y)| dy < \infty. \quad (2.8)$$

In particular, we have $K_{m,n}(\mathbb{R}^n) \subset L^1_{\text{loc}}(\mathbb{R}^n)$.

Proof. Let $\varphi \in K_{m,n}(\mathbb{R}^n)$, then by (1.4) there exists $\alpha > 0$ such that

$$\sup_{x \in \mathbb{R}^n} \int_{|x-y| \leq \alpha} \frac{|\varphi(y)|}{|x-y|^{n-2m}} dy \leq 1. \quad (2.9)$$

Let $a_1, \dots, a_p \in L$ such that $L \subseteq \bigcup_{i=1}^p B(a_i, \alpha)$. Hence for each $x \in \mathbb{R}^n$, we have

$$\begin{aligned} \int_{x+L} |\varphi(y)| dy &\leq \sum_{i=1}^p \int_{B(x+a_i, \alpha)} |\varphi(y)| dy \\ &\leq \sum_{i=1}^p \alpha^{n-2m} \int_{B(x+a_i, \alpha)} \frac{|\varphi(y)|}{|x+a_i-y|^{n-2m}} dy \\ &\leq p \alpha^{n-2m}. \end{aligned} \quad (2.10)$$

So, $\sup_{x \in \mathbb{R}^n} \int_{x+L} |\varphi(y)| dy < \infty$. □

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PROPOSITION 2.5. *Let $\varphi \in K_{m,n}(\mathbb{R}^n)$. Then for each fixed $\alpha > 0$, we have*

$$\sup_{0 \leq t \leq 1} \left(\sup_{x \in \mathbb{R}^n} \int_{|x-y| \geq \alpha} t^{m-1} p(t, x, y) |\varphi(y)| dy \right) := M(\alpha) < \infty. \quad (2.11)$$

Proof. Let $\varphi \in K_{m,n}(\mathbb{R}^n)$, $0 < t \leq 1$. Let $\alpha > 0$, then we have that

$$\begin{aligned} & \sup_{x \in \mathbb{R}^n} \int_{|x-y| \geq \alpha} t^{m-1} p(t, x, y) |\varphi(y)| dy \\ & \leq \frac{\exp(-\alpha^2/8t)}{t^{n/2-m+1}} \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{8}\right) |\varphi(y)| dy. \end{aligned} \quad (2.12)$$

So to prove (2.11), we need to show that

$$\sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{8}\right) |\varphi(y)| dy < \infty. \quad (2.13)$$

Indeed, using Lemma 2.4, we denote by

$$c := \sup_{x \in \mathbb{R}^n} \int_{x+B(0,1)} |\varphi(y)| dy < \infty. \quad (2.14)$$

On the other hand, since any ball $B(0, k)$ of radius $k \geq 1$ in $\mathbb{R}^n$ can be covered by $\alpha(n) := A_n k^n$ balls of radius 1, where A_n is a constant depending only on n (see [4, page 67]), then there exist $a_1, a_2, \dots, a_{\alpha(n)} \in B(0, k)$ such that

$$B(0, k) \subset \bigcup_{1 \leq i \leq \alpha(n)} B(a_i, 1). \quad (2.15)$$

Hence for each $x \in \mathbb{R}^n$, we have

$$\int_{x+B(0,k)} |\varphi(y)| dy \leq \sum_{i=1}^{\alpha(n)} \int_{B(x+a_i, 1)} |\varphi(y)| dy \leq c A_n k^n, \quad (2.16)$$

which implies that for each $x \in \mathbb{R}^n$,

$$\begin{aligned} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{8}\right) |\varphi(y)| dy & \leq \sum_{k=0}^{\infty} \exp\left(-\frac{k^2}{8}\right) \int_{k \leq |x-y| \leq k+1} |\varphi(y)| dy \\ & \leq c A_n \sum_{k=0}^{\infty} \exp\left(-\frac{k^2}{8}\right) (k+1)^n \\ & < \infty. \end{aligned} \quad (2.17)$$

Thus (2.13) holds. This ends the proof. $\square$

PROPOSITION 2.6. *Let $\varphi \in B(\mathbb{R}^n)$. Then $\varphi \in K_{m,n}(\mathbb{R}^n)$ if and only if*

$$\lim_{t \rightarrow 0} \left(\sup_{x \in \mathbb{R}^n} \int_0^t s^{m-1} \int_{\mathbb{R}^n} p(s, x, y) |\varphi(y)| dy ds \right) = 0. \quad (2.18)$$

Proof. Suppose φ verifies (2.18), then from (2.6) we have that

$$\int_{|x-y| \leq \alpha} \frac{|\varphi(y)|}{|x-y|^{n-2m}} dy \leq \int_{\mathbb{R}^n} \int_0^{\alpha^{2/4}} s^{m-1} p(s, x, y) |\varphi(y)| ds dy, \quad (2.19)$$

which implies that the function φ satisfies (1.4).

Conversely, suppose that $\varphi \in K_{m,n}(\mathbb{R}^n)$. Let $\varepsilon > 0$, then by (1.4), there exists $\alpha > 0$ such that

$$\sup_{x \in \mathbb{R}^n} \int_{|x-y| \leq \alpha} \frac{|\varphi(y)|}{|x-y|^{n-2m}} dy \leq \varepsilon. \quad (2.20)$$

Thus from (2.5) and (2.11), we deduce that for each $x \in \mathbb{R}^n$ and $t \leq 1$, we have

$$\begin{aligned} & \int_0^t s^{m-1} \int_{\mathbb{R}^n} p(s, x, y) |\varphi(y)| dy ds \\ & \leq \int_{|x-y| \leq \alpha} \int_0^t s^{m-1} p(s, x, y) |\varphi(y)| dy ds \\ & \quad + \int_0^t \int_{|x-y| \geq \alpha} s^{m-1} p(s, x, y) |\varphi(y)| dy ds \\ & \leq \int_{|x-y| \leq \alpha} \frac{|\varphi(y)|}{|x-y|^{n-2m}} dy + tM(\alpha) \\ & \leq \varepsilon + tM(\alpha). \end{aligned} \quad (2.21)$$

This implies (2.18) and completes the proof. $\square$

2.2. The class $K_{m,n}^\infty(\mathbb{R}^n)$. We will characterize the subclass of m -Green-tight functions $K_{m,n}^\infty(\mathbb{R}^n)$. In fact, we will prove Theorem 1.3 and we give in particular a more precise characterization in the radial case.

Example 2.7. Let $p > n/2m$. Then $L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \subset K_{m,n}^\infty(\mathbb{R}^n)$.

Proof of Theorem 1.3. Let $\varphi \in \mathcal{B}^+(\mathbb{R}^n)$. First we suppose that $\varphi \in K_{m,n}^\infty(\mathbb{R}^n)$, then using similar arguments as in the proof [9, Proposition 6], we obtain easily that $V\varphi \in C_0^+(\mathbb{R}^n)$.

Conversely we suppose that $V\varphi \in C_0^+(\mathbb{R}^n)$. Then, we aim at proving that $\varphi \in K_{m,n}^\infty(\mathbb{R}^n)$. So we divide the proof into two steps.

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Step 1. We will prove that φ satisfies (2.18). Indeed it is clear from (2.1), that for each $x \in \mathbb{R}^n$, we have that

$$\begin{aligned} V\varphi(x) &= \frac{1}{(m-1)!} \int_0^t s^{m-1} \int_{\mathbb{R}^n} p(s, x, y) \varphi(y) dy ds \\ &\quad + \frac{1}{(m-1)!} \int_t^\infty s^{m-1} \int_{\mathbb{R}^n} p(s, x, y) \varphi(y) dy ds \\ &= I_1(x) + I_2(x). \end{aligned} \quad (2.22)$$

From the properties of the density $p(s, x, y)$, we deduce that $x \rightarrow I_1(x)$ and $x \rightarrow I_2(x)$ are nonnegative lower semi-continuous functions in $\mathbb{R}^n$. Then using the fact that $V\varphi \in C_0^+(\mathbb{R}^n)$, we get that the function $x \rightarrow I_1(x)$ is also in $C_0^+(\mathbb{R}^n)$. So, for each $x \in \mathbb{R}^n$, the family $\{\int_0^t s^{m-1} \int_{\mathbb{R}^n} p(s, x, y) \varphi(y) dy ds, t > 0\}$ is decreasing in $C_0^+(\mathbb{R}^n)$, which together with the fact that for each $x \in \mathbb{R}^n$,

$$\lim_{t \rightarrow 0} \int_0^t s^{m-1} \int_{\mathbb{R}^n} p(s, x, y) \varphi(y) dy ds = 0 \quad (2.23)$$

imply by Dini Lemma, that (2.18) is satisfied.

Step 2. We will prove that φ satisfies (1.6). Let $\varepsilon > 0$, then since $V\varphi \in C_0^+(\mathbb{R}^n)$, there exists $a > 0$ such that for $|x| \geq a$, we have that $V\varphi(x) \leq \varepsilon$.

Let $M \geq 2a$, then

$$\begin{aligned} \sup_{x \in \mathbb{R}^n} \int_{|y| \geq M} \frac{\varphi(y)}{|x - y|^{n-2m}} dy &\leq \sup_{|x| \geq a} \int_{\mathbb{R}^n} \frac{\varphi(y)}{|x - y|^{n-2m}} dy + \sup_{|x| \leq a} \int_{|y| \geq M} \frac{\varphi(y)}{|x - y|^{n-2m}} dy \\ &\leq \varepsilon + \int_{|y| \geq M} \frac{\varphi(y)}{|y|^{n-2m}} dy. \end{aligned} \quad (2.24)$$

Now, since $V\varphi(0) < \infty$, we deduce that

$$\lim_{M \rightarrow \infty} \int_{|y| \geq M} \frac{\varphi(y)}{|y|^{n-2m}} dy = 0. \quad (2.25)$$

Then (1.6) holds and this ends the proof. $\square$

For a nonnegative function ρ in $K_{m,n}^\infty(\mathbb{R}^n)$, we denote by

$$M_\rho := \{\varphi \in \mathcal{B}(\mathbb{R}^n), |\varphi| \leq \rho\}. \quad (2.26)$$

PROPOSITION 2.8. *For a nonnegative function ρ in $K_{m,n}^\infty(\mathbb{R}^n)$, the family of functions*

$$V(M_\rho) := \{V\varphi, \varphi \in M_\rho\} \quad (2.27)$$

is uniformly bounded and equicontinuous in $C_0(\mathbb{R}^n)$ and consequently it is relatively compact in $C_0(\mathbb{R}^n)$.

Proof. Let $\rho \in K_{m,n}^\infty(\mathbb{R}^n)$. Obviously, since each function φ in M_ρ is in $K_{m,n}^\infty(\mathbb{R}^n)$, we obtain by Theorem 1.3 that the family $V(M_\rho) \subset C_0(\mathbb{R}^n)$ and is uniformly bounded. Next, we prove the equicontinuity of functions in $V(M_\rho)$ on $\mathbb{R}^n \cup \{\infty\}$ by same arguments as in the proof of [9, Proposition 6]. Thus by Ascoli's Theorem the family $V(M_\rho)$ is relatively compact in $C_0(\mathbb{R}^n)$. This ends the proof. $\square$

Remark 2.9. We recall (see [12, 14]) that for $m = 1$ and $n \geq 3$, a radial function is in $K_n^\infty(\mathbb{R}^n)$ if and only if $\int_0^\infty r |\varphi(r)| dr < \infty$.

Similarly, we will give in the sequel a characterization of radial functions belonging to $K_{m,n}^\infty(\mathbb{R}^n)$.

PROPOSITION 2.10. *Let φ be a radial function in $\mathbb{R}^n$, then $\varphi \in K_{m,n}^\infty(\mathbb{R}^n)$ if and only if*

$$\int_0^\infty r^{2m-1} |\varphi(r)| dr < \infty. \quad (2.28)$$

In order to prove Proposition 2.10, we will use the following behaviour of the m -potential of radial functions on $\mathbb{R}^n$.

PROPOSITION 2.11. *Let $\varphi \in \mathcal{B}^+(\mathbb{R}^n)$ be a radial function on $\mathbb{R}^n$, then for $x \in \mathbb{R}^n$, we have*

$$V\varphi(x) \sim \int_0^\infty \frac{r^{n-1}}{(|x| \vee r)^{n-2m}} \varphi(r) dr. \quad (2.29)$$

Proof. Let $\varphi \in \mathcal{B}^+(\mathbb{R}^n)$. First, we recall the well known results for $x, y \in \mathbb{R}^n$,

$$\begin{aligned} (n-2)k_{1,n} \int_{\mathbb{R}^n} \frac{\varphi(y)}{|x-y|^{n-2}} dy &= \int_0^\infty \frac{r^{n-1}}{(|x| \vee r)^{n-2}} \varphi(r) dr, \\ \int_{\mathbb{R}^n} \frac{dz}{|x-z|^{n-2} |y-z|^{n-2}} &= \frac{c_n}{|x-y|^{n-4}}. \end{aligned} \quad (2.30)$$

This implies that there exists a constant $c > 0$ such that

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{\varphi(y)}{|x-y|^{n-4}} dy &= c \int_0^\infty r^{n-1} \varphi(r) \int_0^\infty \frac{t^{n-1}}{(|x| \vee t)^{n-2} (t \vee r)^{n-2}} dt dr \\ &\geq c \int_0^\infty r^{n-1} \varphi(r) \int_{|x| \vee r}^\infty \frac{1}{t^{n-3}} dt dr \\ &\geq \frac{c}{n-4} \int_0^\infty \frac{r^{n-1} \varphi(r)}{(|x| \vee r)^{n-4}} dr. \end{aligned} \quad (2.31)$$

Hence, we obtain by recurrence that

$$\int_0^\infty \frac{r^{n-1}}{(|x| \vee r)^{n-2m}} \varphi(r) dr \leq \int_{\mathbb{R}^n} \frac{\varphi(y)}{|x-y|^{n-2m}} dy. \quad (2.32)$$

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On the other hand, there exists a constant $\tilde{c} > 0$ such that for each $x \in \mathbb{R}^n$,

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{\varphi(y)}{|x-y|^{n-2m}} dy &= \tilde{c} \int_0^\infty \int_0^\pi \frac{r^{n-1} \varphi(r) (\sin \theta)^{n-2}}{(|x|^2 + r^2 - 2r|x| \cos \theta)^{(n-2m)/2}} d\theta dr \\ &\leq \tilde{c} \int_0^\infty \int_0^\pi \frac{r^{n-1} \varphi(r) (\sin \theta)^{n-2}}{(|x| \vee r)^{n-2m} (\sin \theta)^{n-2m}} d\theta dr \\ &= \tilde{c} \left(\int_0^\pi (\sin \theta)^{2m-2} d\theta \right) \left(\int_0^\infty \frac{r^{n-1} \varphi(r)}{(|x| \vee r)^{n-2m}} dr \right). \end{aligned} \quad (2.33)$$

Thus (2.29) holds. $\square$

Proof of Proposition 2.10. Suppose that φ is a radial function in $K_{m,n}^\infty(\mathbb{R}^n)$, then by Theorem 1.3, $V\varphi(0) < \infty$ and so we deduce (2.28) from (2.29).

Conversely, suppose that φ satisfies (2.28). Let $\alpha > 0$ and $t = |x|$, then by (2.29), we have

$$\begin{aligned} \int_{|x-y| \leq \alpha} \frac{|\varphi(y)|}{|x-y|^{n-2m}} dy &\leq \int_{(t-\alpha)^+}^{t+\alpha} \frac{r^{n-1}}{(t \vee r)^{n-2m}} |\varphi(r)| dr \\ &\leq \int_{(t-\alpha)^+}^{t+\alpha} r^{2m-1} |\varphi(r)| dr. \end{aligned} \quad (2.34)$$

Let $\phi(s) = \int_0^s r^{2m-1} |\varphi(r)| dr$, for $s \in [0, \infty]$. Using (2.28), we deduce that ϕ is a continuous function on $[0, \infty]$. This implies that

$$\int_{(t-\alpha)^+}^{t+\alpha} r^{2m-1} |\varphi(r)| dr = \phi(t+\alpha) - \phi((t-\alpha)^+), \quad (2.35)$$

converges to zero as $\alpha \rightarrow 0$ uniformly for $t \in [0, \infty]$. So φ verifies (1.4).

Next, we have by (2.29)

$$\int_{|y| \geq M} \frac{|\varphi(y)|}{|x-y|^{n-2m}} dy \leq \int_M^\infty \frac{r^{n-1}}{(t \vee r)^{n-2m}} |\varphi(r)| dr \leq \int_M^\infty r^{2m-1} |\varphi(r)| dr, \quad (2.36)$$

which, using (2.28), tends to zero as $M \rightarrow \infty$ and so φ verifies (1.6). This completes the proof. $\square$

We close this section by giving a class of functions included in $K_{m,n}^\infty(\mathbb{R}^n)$ and we precise the behaviour of the m -potential of functions in this class. We need the following lemma.

LEMMA 2.12. Let $\alpha > 0$ and $a, b > 0$ such that $a + b < n$. Then

$$\int_{|x-y| \leq \alpha} \frac{dy}{|y|^a |x-y|^b} \leq \alpha^{n-(a+b)}. \quad (2.37)$$

Proof. Let $\alpha > 0$ and a, b be nonnegative real numbers such that $a + b < n$. Then

$$\begin{aligned} \int_{|x-y| \leq \alpha} \frac{dy}{|y|^a |x-y|^b} &\leq \int_{(|x-y| \leq \alpha) \cap (|x-y| \leq |y|)} \frac{dy}{|x-y|^{a+b}} + \int_{(|y| \leq |x-y| \leq \alpha)} \frac{dy}{|y|^{a+b}} \\ &\leq \int_0^\alpha r^{n-1-(a+b)} dr \\ &\leq \alpha^{n-(a+b)}. \end{aligned} \quad (2.38)$$

□

PROPOSITION 2.13. Let $p > n/2m$. Then for $\lambda < 2m - n/p < \mu$, we have

$$\frac{L^p(\mathbb{R}^n)}{(1 + |\cdot|)^{\mu-\lambda} |\cdot|^\lambda} \subset K_{m,n}^\infty(\mathbb{R}^n). \quad (2.39)$$

Proof. Let $p > n/2m$ and $q \geq 1$ such that $1/p + 1/q = 1$. Let a be a function in $L^p(\mathbb{R}^n)$ and $\lambda < 2m - n/p < \mu$. First, we will prove that the function $\varphi(x) := a(x)/(1 + |x|)^{\mu-\lambda} |x|^\lambda$ satisfies (1.4). Let $\alpha > 0$, then by the Hölder inequality and Lemma 2.12, we have for $x \in \mathbb{R}^n$

$$\begin{aligned} \int_{|x-y| \leq \alpha} \frac{|\varphi(y)|}{|x-y|^{n-2m}} dy &\leq \|a\|_p \left(\int_{|x-y| \leq \alpha} \frac{dy}{(1 + |y|)^{(\mu-\lambda)q} |y|^{\lambda q} |x-y|^{(n-2m)q}} \right)^{1/q} \\ &\leq \|a\|_p \left(\int_{|x-y| \leq \alpha} \frac{dy}{|y|^{q\lambda^+} |x-y|^{(n-2m)q}} \right)^{1/q} \\ &\leq \|a\|_p \alpha^{2m-n/p-\lambda^+}, \end{aligned} \quad (2.40)$$

which converges to zero as $\alpha \rightarrow 0$.

Secondly, we claim that φ satisfies (1.6). To show the claim we use the Hölder inequality. Let $M > 1$, then we have

$$\begin{aligned} \int_{|y| \geq M} \frac{|\varphi(y)|}{|x-y|^{n-2m}} dy &\leq \|a\|_p \left(\int_{|y| \geq M} \frac{dy}{(1 + |y|)^{(\mu-\lambda)q} |y|^{\lambda q} |x-y|^{(n-2m)q}} \right)^{1/q} \\ &\sim \|a\|_p \left(\int_{|y| \geq M} \frac{dy}{|y|^{\mu q} |x-y|^{(n-2m)q}} \right)^{1/q} \\ &= \|a\|_p (A(x))^{1/q}. \end{aligned} \quad (2.41)$$

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Furthermore

$$\begin{aligned}
 A(x) &\leq \sup_{|x| \leq M/2} \int_{|y| \geq M} \frac{dy}{|y|^{(n-2m+\mu)q}} \\
 &\quad + \sup_{|x| \geq M/2} \frac{1}{|x|^{\mu q}} \int_{(|y| \geq M) \cap (|x-y| \leq |x|/2)} \frac{dy}{|x-y|^{(n-2m)q}} \\
 &\quad + \sup_{|x| \geq M/2} \frac{1}{|x|^{(n-2m)q}} \int_{(|y| \geq M) \cap (|x|/2 \leq |x-y| \leq 2|x|)} \frac{dy}{|y|^{\mu q}} \\
 &\quad + \sup_{|x| \geq M/2} \int_{(|y| \geq M) \cap (|x-y| \geq 2|x|)} \frac{dy}{|x-y|^{(n-2m+\mu)q}} \\
 &\leq \frac{1}{M^{(n-2m+\mu)q-n}} + \sup_{|z| \geq M/2} \frac{\text{Log}(3|z|/M)}{|z|^{(n-2m)q}},
 \end{aligned} \tag{2.42}$$

which converges to zero as $M \rightarrow \infty$. This ends the proof. $\square$

Remark 2.14. It is obvious to see that for each $\varphi \in \mathcal{B}^+(\mathbb{R}^n)$, we have

$$\frac{k_{m,n}}{(|x|+1)^{n-2m}} \int_{\mathbb{R}^n} \frac{\varphi(y)}{(|y|+1)^{n-2m}} dy \leq V\varphi(x). \tag{2.43}$$

We precise in the following, some upper estimates on the m -potential of functions in the class $L^p(\mathbb{R}^n)/(1+|\cdot|)^{\mu-\lambda}|\cdot|^\lambda$. Indeed, put for a nonnegative function $a \in L^p(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$

$$Wa(x) := V \left(\frac{a}{(1+|\cdot|)^{\mu-\lambda}|\cdot|^\lambda} \right) (x) = \int_{\mathbb{R}^n} G_{m,n}(x,y) \frac{a(y)}{(1+|y|)^{\mu-\lambda}|y|^\lambda} dy. \tag{2.44}$$

Then we have the following.

PROPOSITION 2.15. *Let $p > n/2m$ and $\lambda < 2m - n/p < \mu$. Then there exists $c > 0$ such that for each nonnegative function $a \in L^p(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we have the following estimates*

$$Wa(x) \leq c \|a\|_p \begin{cases} \frac{1}{(1+|x|)^{n-2m}} \text{Log}(|x|+1)^{p/(p-1)}, & \text{if } \mu + \frac{n}{p} = n \\ \frac{1}{(1+|x|)^{(n-2m) \wedge (\mu+n/p-2m)}}, & \text{if } \mu + \frac{n}{p} \neq n. \end{cases} \tag{2.45}$$

Proof. Let $p > n/2m$ and $q \geq 1$ such that $1/p + 1/q = 1$. Let a be a nonnegative function in $L^p(\mathbb{R}^n)$ and $\lambda < 2m - n/p < \mu$. Put $\varphi(x) = a(x)/(1 + |x|)^{\mu-\lambda}|x|^\lambda$, then by the Hölder inequality, we have for each $x \in \mathbb{R}^n$

$$\begin{aligned} V\varphi(x) &\leq \|a\|_p \left(\int_{\mathbb{R}^n} \frac{dy}{|x-y|^{(n-2m)q} (1+|y|)^{(\mu-\lambda)q} |y|^{\lambda q}} \right)^{1/q} \\ &= \|a\|_p (I(x))^{1/q}. \end{aligned} \quad (2.46)$$

Furthermore,

(i) if $|x| \leq 1$, we have by Lemma 2.12, that

$$\begin{aligned} I(x) &\leq \int_{B(x,2)} \frac{dy}{|x-y|^{(n-2m)q} |y|^{q\lambda^+}} + \int_{B^c(x,2)} \frac{dy}{|x-y|^{(n-2m)q} |y|^{\mu q}} \\ &\leq \int_{B(x,2)} \frac{dy}{|x-y|^{(n-2m)q} |y|^{q\lambda^+}} + \int_{B^c(0,2)} \frac{dy}{|x-y|^{(n-2m+\mu)q}} \\ &\leq 1, \end{aligned} \quad (2.47)$$

(ii) if $|x| \geq 1$, we have

$$\begin{aligned} I(x) &\leq \int_{(|y| \leq 1/2)} \frac{dy}{|x-y|^{(n-2m)q} |y|^{\lambda q}} + \int_{(|y| \geq 1/2) \cap (|x-y| \leq |x|/2)} \frac{dy}{|x-y|^{(n-2m)q} |y|^{\mu q}} \\ &\quad + \int_{(|y| \geq 1/2) \cap (|x|/2 \leq |x-y| \leq 2|x|)} \frac{dy}{|x-y|^{(n-2m)q} |y|^{\mu q}} \\ &\quad + \int_{(|y| \geq 1/2) \cap (|x-y| \geq 2|x|)} \frac{dy}{|x-y|^{(n-2m)q} |y|^{\mu q}} \\ &\leq \frac{1}{|x|^{(n-2m)q}} \int_{(|y| \leq 1/2)} \frac{dy}{|y|^{\lambda q}} + \frac{1}{|x|^{\mu q}} \int_{(|x-y| \leq |x|/2)} \frac{dy}{|x-y|^{(n-2m)q}} \\ &\quad + \frac{1}{|x|^{(n-2m)q}} \int_{(1/2 \leq |y| \leq 3|x|)} \frac{dy}{|y|^{\mu q}} + \int_{(|x-y| > 2|x|)} \frac{dy}{|x-y|^{(n-2m+\mu)q}} \\ &\leq \frac{1}{|x|^{(n-2m)q}} \begin{cases} \text{Log}(|x| + 1), & \text{if } \mu + \frac{n}{p} = n \\ |x|^{n-\mu q}, & \text{if } \mu + \frac{n}{p} < n \\ 1, & \text{if } \mu + \frac{n}{p} > n. \end{cases} \end{aligned} \quad (2.48)$$

By combining the above inequalities, we get the result. $\square$

COROLLARY 2.16. *The class of functions $L^\infty(\mathbb{R}^n)/(1 + |\cdot|)^{\mu-\lambda} \cdot |\cdot|^\lambda$ is included in $K_{m,n}^\infty(\mathbb{R}^n)$ if and only if $\lambda < 2m < \mu$.*

Proof. “ $\Leftarrow$ ” follows from Proposition 2.13.

“ $\Rightarrow$ ” Suppose that the function φ defined on $\mathbb{R}^n$ by $\varphi(x) = 1/(1 + |x|)^{\mu-\lambda}|x|^\lambda$ is in $K_{m,n}^\infty(\mathbb{R}^n)$. Then by Proposition 2.10, we have $\int_0^\infty r^{2m-1}\varphi(r)dr < \infty$. This implies that $\lambda < 2m < \mu$. $\square$

Remark 2.17. Let $\lambda < 2m < \mu$ and $\varphi(x) = 1/(1 + |x|)^{\mu-\lambda}|x|^\lambda$, for $x \in \mathbb{R}^n$, then by simple calculus, we obtain the following behaviour on the m -potential

$$V\varphi(x) \sim \begin{cases} \frac{1}{(1 + |x|)^{n-2m}} \text{Log}(|x| + 1), & \text{if } \mu = n \\ \frac{1}{(1 + |x|)^{(n-2m) \wedge (\mu-2m)}}, & \text{if } \mu \neq n. \end{cases} \quad (2.49)$$

3. First existence result

In this section, we aim at proving Theorem 1.4. The following lemmas are useful.

LEMMA 3.1. *Let φ be a nonnegative function in $K_{m,n}^\infty(\mathbb{R}^n)$. Then we have*

$$\|V\varphi\|_\infty \leq \alpha_\varphi \leq 2^{n-2m}\|V\varphi\|_\infty. \quad (3.1)$$

Proof. By (1.3) we obtain easily that $\alpha_\varphi \leq 2^{n-2m}\|V\varphi\|_\infty$. On the other hand, by letting $|y| \rightarrow \infty$ in (1.24), we deduce from Fatou Lemma that $\|V\varphi\|_\infty \leq \alpha_\varphi$. $\square$

LEMMA 3.2. *Let φ be a nonnegative function in $K_{m,n}^\infty(\mathbb{R}^n)$. Then for each $x \in \mathbb{R}^n$, we have*

$$V(\varphi G_{m,n}(\cdot, y))(x) \leq \alpha_\varphi G_{m,n}(x, y). \quad (3.2)$$

Proof. The result holds by (1.24). $\square$

In the sequel, let q be a nonnegative function in $K_{m,n}^\infty(\mathbb{R}^n)$ such that $\alpha_q \leq 1/2$. For $f \in \mathcal{B}^+(\mathbb{R}^n)$, we will define the potential kernel $V_q f := V_{m,n,q} f$ as a solution for the perturbed polyharmonic equation (1.9).

We put for $x, y \in \mathbb{R}^n$,

$$\mathcal{G}_{m,n}(x, y) = \begin{cases} \sum_{k \geq 0} (-1)^k (V(q \cdot))^k (G_{m,n}(\cdot, y))(x), & \text{if } x \neq y \\ \infty, & \text{if } x = y. \end{cases} \quad (3.3)$$

Then we have the following comparison result.

LEMMA 3.3. *Let q be a nonnegative function in $K_{m,n}^\infty(\mathbb{R}^n)$ such that $\alpha_q \leq 1/2$. Then for $x, y \in \mathbb{R}^n$, we have*

$$(1 - \alpha_q)G_{m,n}(x, y) \leq \mathcal{G}_{m,n}(x, y) \leq G_{m,n}(x, y). \quad (3.4)$$

Proof. Since $\alpha_q \leq 1/2$, we deduce from (3.2), that

$$\begin{aligned} |\mathcal{G}_{m,n}(x, y)| &\leq \sum_{k \geq 0} (\alpha_q)^k G_{m,n}(x, y) \\ &= \frac{1}{1 - \alpha_q} G_{m,n}(x, y). \end{aligned} \quad (3.5)$$

Furthermore, we have for $x \neq y$ in $\mathbb{R}^n$

$$\mathcal{G}_{m,n}(x, y) = G_{m,n}(x, y) - V(q\mathcal{G}_{m,n}(\cdot, y))(x), \quad (3.6)$$

which together with (3.2), imply that

$$\begin{aligned} \mathcal{G}_{m,n}(x, y) &\geq G_{m,n}(x, y) - \frac{\alpha_q}{1 - \alpha_q} G_{m,n}(x, y) \\ &= \frac{1 - 2\alpha_q}{1 - \alpha_q} G_{m,n}(x, y) \\ &\geq 0. \end{aligned} \quad (3.7)$$

Hence the result follows from (3.6) and (3.2). $\square$

Let us define the operator V_q on $\mathcal{B}^+(\mathbb{R}^n)$ by

$$V_q f(x) = \int_{\mathcal{B}} \mathcal{G}_{m,n}(x, y) f(y) dy, \quad x \in \mathbb{R}^n. \quad (3.8)$$

Then we obtain the following.

LEMMA 3.4. *Let $f \in \mathcal{B}^+(\mathbb{R}^n)$. Then $V_q f$ satisfies the following resolvent equation*

$$Vf = V_q f + V_q(qVf) = V_q f + V(qV_q f). \quad (3.9)$$

Proof. From the expression of $\mathcal{G}_{m,n}$, we deduce that for $f \in \mathcal{B}^+(\mathbb{R}^n)$ such that $Vf < \infty$,

$$V_q f = \sum_{k \geq 0} (-1)^k (V(q \cdot))^k Vf. \quad (3.10)$$

So we obtain that

$$\begin{aligned} V_q(qVf) &= \sum_{k \geq 0} (-1)^k (V(q \cdot))^k [V(qVf)] \\ &= - \sum_{k \geq 1} (-1)^k (V(q \cdot))^k Vf \\ &= Vf - V_q f. \end{aligned} \quad (3.11)$$

The second equality holds by integrating (3.6). $\square$

PROPOSITION 3.5. *Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that $Vf \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then $V_q f$ is a solution (in the sense of distributions) of the perturbed polyharmonic equation (1.9).*

Proof. Using the resolvent equation (3.9), we have

$$V_q f = V f - V(q V_q f). \quad (3.12)$$

Applying the operator $(-\Delta)^m$ on both sides of the above equality, we obtain that

$$(-\Delta)^m (V_q f) = f - q V_q f \quad (\text{in the sense of distributions}). \quad (3.13)$$

This completes the proof. $\square$

Now, we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let $c > 0$. Then by (H_2) , there exists a nonnegative function $q := q_c \in K_{m,n}^\infty(\mathbb{R}^n)$, such that $\alpha_q \leq 1/2$ and for each $x \in \mathbb{R}^n$, the map

$$t \longrightarrow t(q(x) - \varphi(x, t)) \text{ is continuous and nondecreasing on } [0, c], \quad (3.14)$$

which implies in particular that for each $x \in \mathbb{R}^n$ and $t \in [0, c]$,

$$0 \leq \varphi(x, t) \leq q(x), \quad (3.15)$$

Let

$$\Lambda := \{u \in \mathcal{B}^+(\mathbb{R}^n) : (1 - \alpha_q)c \leq u \leq c\}. \quad (3.16)$$

We define the operator T on Λ by

$$Tu(x) := c(1 - V_q(q)(x)) + V_q[(q - \varphi(\cdot, u))u](x). \quad (3.17)$$

First, we prove that Λ is invariant under T . Indeed, for each $u \in \Lambda$, we have

$$Tu \leq c(1 - V_q(q)(x)) + cV_q(q)(x) \leq c. \quad (3.18)$$

Moreover, from (3.15), (3.4) and Lemma 3.1 we deduce that for each $u \in \Lambda$, we have

$$Tu \geq c(1 - V_q(q)(x)) \geq c(1 - V(q)(x)) \geq c(1 - \alpha_q). \quad (3.19)$$

Next, we prove that the operator T is nondecreasing on Λ . Indeed, let $u, v \in \Lambda$ such that $u \leq v$, then from (3.14) we obtain that

$$Tv - Tu = V_q([(q - \varphi(\cdot, v))v] - [(q - \varphi(\cdot, u))u]) \geq 0. \quad (3.20)$$

Now, consider the sequence (u_k) defined by $u_0 = (1 - \alpha_q)c$ and $u_{k+1} = Tu_k$, for $k \in \mathbb{N}$. Then since Λ is invariant under T , we obtain obviously that $u_1 = Tu_0 \geq u_0$ and so from the monotonicity of T , we have

$$u_0 \leq u_1 \leq \cdots \leq u_k \leq c. \quad (3.21)$$

So from (3.14) and the dominated convergence theorem we deduce that the sequence (u_k) converges to a function $u \in \Lambda$ which satisfies

$$u = c(1 - V_q(q)(x)) + V_q[(q - \varphi(\cdot, u))u](x). \quad (3.22)$$

That is

$$u - V_q(qu) = c(1 - V_q(q)(x)) - V_q(u\varphi(\cdot, u)). \quad (3.23)$$

Applying the operator $(I + V(q \cdot))$ on both sides of the above equality and using (3.9) we deduce that u satisfies

$$u = c - V(u\varphi(\cdot, u)). \quad (3.24)$$

Finally, we claim that u is a positive continuous solution for the Problem (1.6). To prove the claim, we use Lemma 2.4. Indeed, since $u \sim c$ on $\mathbb{R}^n$ and

$$0 \leq u\varphi(\cdot, u) \leq cq, \quad (3.25)$$

we deduce that either u and $u\varphi(\cdot, u)$ are in $L^1_{\text{loc}}(\mathbb{R}^n)$.

Now, from (3.24) we can easily see that $V(u\varphi(\cdot, u)) \in L^1_{\text{loc}}(\mathbb{R}^n)$. Hence u satisfies (in the sense of distributions) the elliptic differential equation

$$(-\Delta)^m u + u\varphi(\cdot, u) = f \quad \text{in } \mathbb{R}^n. \quad (3.26)$$

On the other hand, it follows from (3.25) that $u\varphi(\cdot, u) \in M_q$ and so by Proposition 2.8, we obtain that $V(u\varphi(\cdot, u))$ is in $C_0^+(\mathbb{R}^n)$.

This implies by (3.24) that $\lim_{|x| \rightarrow \infty} u(x) = c$, which completes the proof. $\square$

Remark 3.6. Let $c > 0$ and u be a solution of (1.8). Then we have by Theorem 1.4 that for each $x \in \mathbb{R}^n$, $0 \leq u(x) \leq c$. Let q be the nonnegative function in $K_{m,n}^\infty(\mathbb{R}^n)$ given in the proof of Theorem 1.4. Then we deduce from (3.24) and (3.25), that

$$0 \leq c - u(x) = V(u\varphi(\cdot, u))(x) \leq cV(q)(x). \quad (3.27)$$

Example 3.7. Let $p > n/2m$ and a be a nonnegative function in $L^p(\mathbb{R}^n)$. Let $\lambda < 2m - n/p < \mu$ and α, β be two nonnegative constants.

Put $q(x) = a(x)/(1 + |x|)^{\mu-\lambda}|x|^\lambda$. Then, for each $c > 0$, the following polyharmonic problem

$$\begin{aligned} (-\Delta)^m u + \beta u^{\alpha+1} q &= 0, \quad \text{in } \mathbb{R}^n \text{ (in the sense of distributions)} \\ \lim_{|x| \rightarrow \infty} u(x) &= c, \end{aligned} \quad (3.28)$$

has a positive continuous solution satisfying $c/2 \leq u(x) \leq c$, provided that β is sufficiently small.

Moreover, by Remark 3.6 and Proposition 2.15, we have

$$0 \leq c - u(x) \leq c \|a\|_p \begin{cases} \frac{1}{(1 + |x|)^{n-2m}} \text{Log}(|x| + 1)^{p/(p-1)}, & \text{if } \mu + \frac{n}{p} = n \\ \frac{1}{(1 + |x|)^{(n-2m) \wedge (\mu+n/p-2m)}}, & \text{if } \mu + \frac{n}{p} \neq n. \end{cases} \quad (3.29)$$

Remark 3.8. It is interesting to compare the asymptotics (3.29) with the results of Trubek [10], for the case $m = 1$.

4. Second existence result

In this section, we aim at proving Theorem 1.5.

Proof of Theorem 1.5. Assuming (H_3) and (H_4) , we will use the Schauder fixed point theorem. From (1.14), there exists $\eta > 0$ such that

$$h(t) \geq m_0 t, \quad \text{for each } t \in [0, \eta]. \quad (4.1)$$

On the other hand, let $\alpha \in (g^\infty, M_0)$, then by (1.15), there exists $\rho > 0$ such that for $t \geq \rho$, we have $g(t) \leq \alpha t$. Put $\beta = \sup_{0 \leq t \leq \rho} g(t)$. So we deduce that

$$0 \leq g(t) \leq \alpha t + \beta, \quad \text{for each } t \geq 0. \quad (4.2)$$

By Remark 2.14, we note that there exists a constant $\alpha_1 > 0$ such that

$$\frac{\alpha_1}{(1 + |x|)^{n-2m}} \leq Vq(x). \quad (4.3)$$

Let $a \in (0, \eta)$ and $b = \max\{a/\alpha_1, \beta/(1 - \alpha \|Vq\|_\infty)\}$. So we consider the closed convex set

$$\Lambda = \left\{ u \in C_0(\mathbb{R}^n), \frac{a}{(1 + |x|)^{n-2m}} \leq u(x) \leq bVq(x), \forall x \in \mathbb{R}^n \right\}. \quad (4.4)$$

Obviously by (4.3) we have that the set Λ is nonempty. Next we define the operator T on Λ by

$$Tu(x) = \int_{\mathbb{R}^n} G_{m,n}(x, y) f(y, u(y)) dy. \quad (4.5)$$

Let us prove that $T\Lambda \subset \Lambda$. Let $u \in \Lambda$, then by (4.2) we have

$$\begin{aligned} Tu(x) &\leq \int_{\mathbb{R}^n} G_{m,n}(x, y) q(y) g(u(y)) dy \\ &\leq \int_{\mathbb{R}^n} G_{m,n}(x, y) q(y) [\alpha u(y) + \beta] dy \\ &\leq (\alpha b \|Vq\|_\infty + \beta) Vq(x) \\ &\leq bVq(x). \end{aligned} \quad (4.6)$$

Moreover, since h is nondecreasing, we deduce by (4.1) and (1.14) that

$$\begin{aligned}
 Tu(x) &\geq \int_{\mathbb{R}^n} G_{m,n}(x, y) p(y) h(u(y)) dy \\
 &\geq \int_{\mathbb{R}^n} G_{m,n}(x, y) p(y) h\left(\frac{a}{(1+|y|)^{n-2m}}\right) dy \\
 &\geq m_0 a \int_{\mathbb{R}^n} G_{m,n}(x, y) \frac{p(y)}{(1+|y|)^{n-2m}} dy \\
 &\geq \frac{m_0 a k_{m,n}}{(1+|x|)^{n-2m}} \int_{\mathbb{R}^n} \frac{p(y)}{(1+|y|)^{2(n-2m)}} dy \\
 &= \frac{a}{(1+|x|)^{n-2m}}.
 \end{aligned} \tag{4.7}$$

On the other hand, by (1.13), we have that for each $u \in \Lambda$

$$f(\cdot, u) \leq g(b\|Vq\|_\infty)q. \tag{4.8}$$

This implies by Proposition 2.8 that $Tu \in V(M_q) \subset C_0(\mathbb{R}^n)$. So $T\Lambda \subset \Lambda$.

Next, we prove the continuity of T in Λ . Let (u_k) be a sequence in Λ , which converges uniformly to a function $u \in \Lambda$. Then using (4.8) and (H_3) , we deduce by Theorem 1.3 and the dominated convergence Theorem that for $x \in \mathbb{R}^n$,

$$Tu_k(x) \longrightarrow Tu(x) \quad \text{as } k \longrightarrow \infty. \tag{4.9}$$

Now, since $T\Lambda \subset V(M_q)$, we deduce by Proposition 2.8 that $T\Lambda$ is relatively compact in $C_0(\mathbb{R}^n)$, which implies that

$$\|Tu_k - Tu\|_\infty \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{4.10}$$

Hence T is a compact map from Λ to itself. So the Schauder fixed point theorem leads to the existence of $u \in \Lambda$ such that

$$u = V(f(\cdot, u)). \tag{4.11}$$

Finally by (4.8) and Lemma 2.4, we conclude that $y \rightarrow f(y, u(y))$ is in $L^1_{\text{loc}}(\mathbb{R}^n)$, which together with (4.11) imply that u satisfies (in the sense of distributions) the elliptic differential equation

$$(-\Delta)^m u = f(\cdot, u) \quad \text{in } \mathbb{R}^n. \tag{4.12}$$

This ends the proof. $\square$

Example 4.1. Let p be a nonnegative function in $K^\infty_{m,n}(\mathbb{R}^n)$ and $0 \leq \alpha < 1$. Then the following problem

$$\begin{aligned}
 (-\Delta)^m u + p(x)u^\alpha &= 0, \quad x \in \mathbb{R}^n, \\
 \lim_{|x| \rightarrow \infty} u(x) &= 0,
 \end{aligned} \tag{4.13}$$

has a positive solution $u \in C_0(\mathbb{R}^n)$ satisfying for each $x \in \mathbb{R}^n$

$$\frac{1}{(1 + |x|)^{n-2m}} \leq u(x) \leq Vp(x). \quad (4.14)$$

5. Third existence result

In this section, we aim at proving Theorem 1.6.

Proof of Theorem 1.6. Let $c > 0$ be the constant given by (H_7) and $c^* = c - \|V(q(\cdot, c))\|_\infty$. Let $\delta \in (0, c^*]$. We will use the Schauder fixed point theorem, so we consider the closed convex set

$$\Lambda = \{u \in C(\mathbb{R}^n \cup \{\infty\}) : \delta \leq u(x) \leq c, \forall x \in \mathbb{R}^n\} \quad (5.1)$$

and we define the integral operator T on Λ by

$$Tu(x) = \delta + V(f(\cdot, u))(x). \quad (5.2)$$

First, we prove that $T\Lambda \subset \Lambda$. Let $u \in \Lambda$, then since f is a nonnegative function, we have that $Tu(x) \geq \delta$, for each $x \in \mathbb{R}^n$. Moreover by (H_6) , we have for $x \in \mathbb{R}^n$,

$$Tu(x) \leq \delta + V(q(\cdot, u))(x) \leq c^* + V(q(\cdot, c))(x) \leq c. \quad (5.3)$$

Furthermore by (H_7) , since for all $u \in \Lambda$, $f(\cdot, u) \in M_{q(\cdot, c)}$, then it follows from Proposition 2.8 that $V(f(\cdot, u)) \in C_0(\mathbb{R}^n)$ and more precisely $T\Lambda$ is relatively compact in $C(\mathbb{R}^n \cup \{\infty\})$. Therefore $T\Lambda \subset \Lambda$.

Next, let us prove the continuity of T in Λ . Let (u_k) be a sequence in Λ , which converges uniformly to a function $u \in \Lambda$. Since f is continuous with respect to the second variable, we deduce by the dominated convergence theorem that for each $x \in \mathbb{R}^n \cup \{\infty\}$,

$$Tu_k(x) \longrightarrow Tu(x) \quad \text{as } k \longrightarrow \infty. \quad (5.4)$$

Now, since $T\Lambda$ is relatively compact in $C(\mathbb{R}^n \cup \{\infty\})$, then

$$\|Tu_k - Tu\|_\infty \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \quad (5.5)$$

Finally the Schauder fixed point theorem implies the existence of $u \in \Lambda$ such that

$$u(x) = \delta + V(f(\cdot, u))(x), \quad \forall x \in \mathbb{R}^n. \quad (5.6)$$

Using (H_6) , (H_7) and Lemma 2.4, we deduce that the function $y \rightarrow f(y, u(y))$ is in $L^1_{\text{loc}}(\mathbb{R}^n)$. So u satisfies (in the sense of distributions) the elliptic differential equation

$$(-\Delta)^m u = f(\cdot, u) \quad \text{in } \mathbb{R}^n. \quad (5.7)$$

Moreover since $V(f(\cdot, u)) \in C_0(\mathbb{R}^n)$, then by (5.6) it follows that $\lim_{|x| \rightarrow \infty} u(x) = \delta$. This ends the proof. $\square$

COROLLARY 5.1. *Assume that $q(x, t) = p(x)g(t)$, where g is a nonnegative nondecreasing measurable function and p is a nonnegative function in $K_{m,n}^\infty(\mathbb{R}^n)$. If the function g satisfies either $g(t) = o(t)$ as $t \rightarrow 0$ or $g(t) = o(t)$ as $t \rightarrow \infty$, then the problem (1.19) has a positive solution $u \in C(\mathbb{R}^n \cup \{\infty\})$.*

Example 5.2. Among the equations of form (1.1), we have the Emden-Fowler equation of order m

$$(-\Delta)^m u + p(x)u^\alpha = 0, \quad \alpha > 0, x \in \mathbb{R}^n, n > 2m, \quad (5.8)$$

where $p \in K_{m,n}^\infty(\mathbb{R}^n)$.

(i) For the sublinear ($0 < \alpha < 1$) or the superlinear ($\alpha > 1$) case, let $c > 0$ such that

$$\|Vp\|_\infty c^{\alpha-1} < 1. \quad (5.9)$$

Then applying Theorem 1.6, we deduce that for each $\delta \in (0, c(1 - c^{\alpha-1}\|Vp\|_\infty))$, (5.8) with $\alpha \neq 1$ has a continuous positive solution u in $\mathbb{R}^n$ with $\delta \leq u(x) \leq c$, for all $x \in \mathbb{R}^n$ and $\lim_{|x| \rightarrow \infty} u(x) = \delta$.

(ii) For the linear case ($\alpha = 1$). If $\|Vp\|_\infty < 1$, then applying Theorem 1.6, we deduce that for each $c > 0$ and $\delta \in (0, c(1 - \|Vp\|_\infty))$, (5.8) has a continuous positive solution u in $\mathbb{R}^n$ with $\delta \leq u(x) \leq c$, for all $x \in \mathbb{R}^n$ and $\lim_{|x| \rightarrow \infty} u(x) = \delta$.

Remark 5.3. We improve in this section the Yin's result in [11]. Indeed, Yin proved in particular the existence of bounded positive solutions for the Emden-Fowler equation

$$\Delta u + p(x)u^\alpha = 0, \quad 0 < \alpha \neq 1, x \in \mathbb{R}^n, n \geq 3, \quad (5.10)$$

provided that the function p satisfies

$$\int_0^\infty s \max_{|x|=s} \{p(x)\} ds < \infty. \quad (5.11)$$

However by taking $\lambda > (n-1)/2$ and

$$p(x) = p(x', x_n) = \frac{1}{(1 + x_n^2)(1 + \sum_{i=1}^{n-1} x_i^2)^\lambda}, \quad x \in \mathbb{R}^n, \quad (5.12)$$

then we have

$$\max_{|x|=s} p(x) \geq p(0, s) = \frac{1}{1 + s^2} \quad (5.13)$$

which implies that (5.11) is not satisfied. On the other hand, we have that $p \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \subset K_{m,n}^\infty(\mathbb{R}^n)$. This implies by Corollary 5.1 that the Emden-Fowler equation (5.8) has a positive solution $u \in C(\mathbb{R}^n \cup \{\infty\})$, for each $m \geq 1$.

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