EXISTENCE OF POSITIVE SOLUTIONS FOR NONLINEAR BOUNDARY VALUE PROBLEMS IN BOUNDED DOMAINS OF $\mathbb{R}^n$

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Let D be a bounded domain in $\mathbb{R}^n$ ($n \geq 2$). We consider the following nonlinear elliptic problem: $\Delta u = f(\cdot, u)$ in D (in the sense of distributions), $u|_{\partial D} = \varphi$, where φ is a non-negative continuous function on ∂D and f is a nonnegative function satisfying some appropriate conditions related to some Kato class of functions $K(D)$. Our aim is to prove that the above problem has a continuous positive solution bounded below by a fixed harmonic function, which is continuous on $\overline{D}$. Next, we will be interested in the Dirichlet problem $\Delta u = -\rho(\cdot, u)$ in D (in the sense of distributions), $u|_{\partial D} = 0$, where ρ is a nonnegative function satisfying some assumptions detailed below. Our approach is based on the Schauder fixed-point theorem.

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1. Introduction

Let D be a bounded $C^{1,1}$ -domain in $\mathbb{R}^n$ ($n \geq 2$), and let G be the Green function for the Laplace operator with zero Dirichlet boundary condition on ∂D . In [4], Chung and Zhao have established interesting inequalities for the Green function G . In particular, they showed that there exists a constant $C > 0$ such that for each x, y in D ,

$$\frac{1}{C}H(x, y) \leq G(x, y) \leq CH(x, y), \quad (1.1)$$

where

$$H(x, y) := \begin{cases} \frac{1}{|x - y|^{n-2}} \min \left(1, \frac{\delta(x)\delta(y)}{|x - y|^2} \right), & \text{if } n \geq 3, \\ \text{Log} \left(1 + \frac{\delta(x)\delta(y)}{|x - y|^2} \right), & \text{if } n = 2, \end{cases} \quad (1.2)$$

and $\delta(x)$ denotes the Euclidean distance between x and ∂D .

2 Nonlinear elliptic problems

Another crucial inequality for the Green function G called 3G-theorem is given by Kalton and Verbitsky [7] for $n \geq 3$ and by Selmi [12] for $n = 2$, namely, there exists a constant $C_0 > 0$ depending only on D such that for all x, y , and z in D ,

$$\frac{G(x, z)G(z, y)}{G(x, y)} \leq C_0 \left(\frac{\delta(z)}{\delta(x)} G(x, z) + \frac{\delta(z)}{\delta(y)} G(y, z) \right). \quad (1.3)$$

This 3G-theorem was investigated by Mâagli and Zribi [10], Zeddini [13], and Mâagli and Mâatoug [9] to introduce a new class of functions denoted by $K(D)$, (see Definition 1.1 below), which contains properly the classical Kato class introduced by Aizenman and Simon [1]. Moreover, they used the properties of functions belonging to this class $K(D)$ to study some nonlinear differential equations.

Definition 1.1. A Borel measurable function q in D belongs to the class $K(D)$ if q satisfies

$$\lim_{\alpha \rightarrow 0} \left(\sup_{x \in D} \int_{D \cap B(x, \alpha)} \frac{\delta(y)}{\delta(x)} G(x, y) |q(y)| dy \right) = 0. \quad (1.4)$$

In this paper, we will exploit the properties pertaining to $K(D)$ to give some results about the existence of positive solutions of nonlinear elliptic problems. Our plan is as follows.

In Section 2, we establish some estimates on the Green function G and some properties of functions belonging to the Kato class $K(D)$.

In Section 3, we are concerned with the existence of positive continuous solutions of the nonlinear elliptic problem

$$\begin{aligned} \Delta u &= f(\cdot, u) \quad (\text{in the sense of distributions}), \\ u &> 0 \quad \text{in } D, \quad u|_{\partial D} = \varphi, \end{aligned} \quad (1.5)$$

where φ is a nontrivial nonnegative continuous function on ∂D . Then, we fix a nontrivial nonnegative harmonic function h_0 in D , which is continuous in $\bar{D}$, and we suppose that f satisfies the following hypotheses.

(H₁) $f : D \times (0, +\infty) \rightarrow [0, +\infty)$ is measurable, continuous with respect to the second variable and satisfies

$$f(x, t) \leq \theta(x, t), \quad \text{for } (x, t) \in D \times (0, +\infty), \quad (1.6)$$

where θ is a nonnegative measurable function on $D \times (0, +\infty)$ such that the function $t \rightarrow \theta(x, t)$ is nonincreasing on $(0, +\infty)$.

(H₂) The function ψ defined on D by $\psi(x) = \theta(x, h_0(x))/h_0(x)$ belongs to the class $K(D)$.

Remark 1.2. Note that the condition “ $\forall c > 0$, $\theta(\cdot, c\delta(\cdot))/\delta(\cdot) \in K(D)$ ” implies the hypothesis (H₂). Indeed, from [14], there exists $c > 0$ such that for each $x \in D$, $h_0(x) \geq c\delta(x)$. So, using the fact that $t \rightarrow \theta(x, t)/t$ is nonincreasing function on $(0, +\infty)$, we obtain (H₂).

Under the assumptions (H_1) – (H_2) , we aim at proving the following result: there exists a constant $c > 1$ such that if $\varphi \geq ch_0$ on ∂D , then problem (1.5) has a positive continuous solution u satisfying for each $x \in D$,

$$h_0(x) \leq u(x) \leq H_D \varphi(x), \quad (1.7)$$

where $H_D \varphi$ is the harmonic continuous function having boundary value φ on ∂D .

This result improves the one of Atherya [2], who considered the following problem:

$$\Delta u = g(u) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = \varphi, \quad (P)$$

where Ω is a simply connected bounded C^2 -domain in $\mathbb{R}^n$ ($n \geq 3$) and $g(u) \leq \max(1, u^{-\alpha})$, for $0 < \alpha < 1$. He proved the existence of a positive continuous solution bounded below by a fixed positive harmonic function h_0 provided that there exists a positive constant $c > 1$ such that $\varphi \geq ch_0$ on ∂D .

In the last section, we will study the following nonlinear problem:

$$\Delta u = -\rho(\cdot, u) \quad \text{in } D \text{ (in the sense of distributions),} \quad u|_{\partial D} = 0, \quad (1.8)$$

where ρ is required to verify the following hypotheses.

(H₃) ρ is nonnegative Borel measurable function on $D \times (0, \infty)$, continuous with respect to the second variable.

(H₄) There exist $p, q : D \rightarrow (0, \infty)$ nontrivial Borel measurable functions and $h, k : (0, \infty) \rightarrow [0, \infty)$ nontrivial and nondecreasing Borel measurable functions satisfying

$$p(x)h(t) \leq \rho(x, t) \leq q(x)k(t), \quad \text{for } (x, t) \in D \times (0, \infty), \quad (1.9)$$

such that

$$(A_1) \quad p \in L^1_{\text{loc}}(D),$$

$$(A_2) \quad q \in K(D),$$

$$(A_3) \quad \lim_{t \rightarrow 0^+} (h(t)/t) = +\infty,$$

$$(A_4) \quad \lim_{t \rightarrow +\infty} (k(t)/t) = 0.$$

Under these hypotheses, we will prove that (1.8) has a positive continuous solution u satisfying on D ,

$$a\delta(x) \leq u(x) \leq b, \quad (1.10)$$

where a, b are positive constants.

Problem (1.8) has been studied by Dalmasso [5] on the unit ball with more restrictive conditions on ρ . Indeed, Dalmasso proved the existence of positive solutions provided that ρ is nondecreasing with respect to the second variable and satisfies

$$\lim_{t \rightarrow 0^+} \left(\min_{x \in \bar{B}} \frac{\rho(x, t)}{t} \right) = +\infty, \quad \lim_{t \rightarrow +\infty} \left(\max_{x \in \bar{B}} \frac{\rho(x, t)}{t} \right) = 0. \quad (1.11)$$

When $\rho(x, t) = \rho(|x|, t)$, he showed the uniqueness of positive radial solution of (1.8).

4 Nonlinear elliptic problems

On the other hand, problem (1.8) has been studied on the entire space $\mathbb{R}^n$ by Brezis and Kamin [3] for the special nonlinearity $\rho(x, t) = v(x)t^\alpha$, $0 < \alpha < 1$. More precisely they proved the existence and the uniqueness of positive solution for the problem below:

$$\Delta u = -v(x)u^\alpha \quad \text{in } \mathbb{R}^n \quad \lim_{|x| \rightarrow \infty} \inf u = 0. \quad (1.12)$$

Notations and preliminaries. In order to simplify our statement, we adopt the following notations.

(i) $C_0(D) := \{f \in C(D) : \lim_{x \rightarrow \partial D} f(x) = 0\}$.

We note that $C_0(D)$ is a Banach space endowed with the uniform norm

$$\|f\|_\infty = \sup_{x \in D} |f(x)|. \quad (1.13)$$

(ii) Let f and g be two nonnegative functions on a set S .

We call $f \sim g$, if there exists a constant $c > 0$ such that

$$\frac{1}{c}g(x) \leq f(x) \leq cg(x) \quad \forall x \in S. \quad (1.14)$$

We call $f \leq g$, if there exists a constant $c > 0$ such that

$$f(x) \leq cg(x) \quad \forall x \in S. \quad (1.15)$$

(iii) Let f be a nonnegative function in D , then we denote by Vf the potential of f defined on D by

$$Vf(x) = \int_D G(x, y)f(y)dy. \quad (1.16)$$

We recall that if $f \in L^1_{\text{loc}}(D)$ and $Vf \in L^1_{\text{loc}}(D)$, then we have $\Delta(Vf) = -f$ in D (in the sense of distributions) (see [4, page 52]).

(iv) We denote by d the diameter of D .

(v) For $x, y \in D$, we denote $[x, y]^2 = |x - y|^2 + \delta(x)\delta(y)$.

2. Properties of the Green function and the class $K(D)$

In this section, we establish some results concerning the Green function $G(x, y)$ and the Kato class $K(D)$.

PROPOSITION 2.1 (see [9, 10]). *Let q be a nonnegative function in $K(D)$. Then*

(i) *the potential $Vq \in C_0(D)$,*

(ii) *the function $x \rightarrow \delta(x)q(x)$ is in $L^1(D)$.*

In the sequel, we put

$$\|q\|_D = \sup_{x \in D} \int_D \frac{\delta(y)}{\delta(x)} G(x, y) |q(y)| dy, \quad (2.1)$$

$$\alpha_q = \sup_{x, y \in D} \int_D \frac{G(x, z)G(z, y)}{G(x, y)} |q(z)| dz \quad (2.2)$$

We recall that if $q \in K(D)$, then $\|q\|_D < \infty$.

Now, it is obvious to see that by (1.3), we have

$$\alpha_q \leq 2C_0 \|q\|_D, \quad (2.3)$$

where C_0 is the constant given by (1.3).

Next, we will prove that $\alpha_q \sim \|q\|_D$.

PROPOSITION 2.2. *Let q be a function in $K(D)$. Then*

(i) *for any nonnegative superharmonic function h in D , we have*

$$\int_D G(x, y) |q(y)| h(y) dy \leq \alpha_q h(x), \quad \forall x \in D, \quad (2.4)$$

(ii) *there exists a constant $C > 0$ such that*

$$C \|q\|_D \leq \alpha_q. \quad (2.5)$$

Proof. (i) Let h be a nonnegative superharmonic function in D , then from [11, Theorem 2.1, page 164], there exists a sequence (f_k) of nonnegative measurable functions on D such that for all $y \in D$,

$$h_k(y) = \int_D G(x, z) f_k(z) dz \quad (2.6)$$

increases to $h(y)$.

Since for each $x, y \in D$, we have

$$\int_D G(x, y) |q(y)| h_k(y) dy \leq \alpha_q h_k(x). \quad (2.7)$$

Thus, from the monotone convergence theorem, we deduce the result.

(ii) Let φ_1 be a positive eigenfunction corresponding to the first eigenvalue of the Dirichlet problem $\Delta u + \lambda u = 0$, $u|_{\partial D} = 0$. Then, from [8, Proposition 2.6] we have for $x \in D$

$$\varphi_1(x) \sim \delta(x). \quad (2.8)$$

Since, φ_1 is a superharmonic function in D , then by applying (i) to φ_1 , we deduce (ii). $\square$

PROPOSITION 2.3. *Let $p > n/2$. Then for each $\lambda < 2 - n/p$, we have*

$$\frac{1}{(\delta(\cdot))^\lambda} L^p(D) \subset K(D). \quad (2.9)$$

6 Nonlinear elliptic problems

To prove Proposition 2.3, we need the two next lemmas.

LEMMA 2.4. *On D^2 , we have*

- (i) *for $n \geq 3$, $G(x, y) \sim \delta(x)\delta(y)/|x - y|^{n-2}[x, y]^2$,*
- (ii) *for $n = 2$, $G(x, y) \sim (\delta(x)\delta(y)/[x, y]^2) \text{Log}(1 + [x, y]^2/|x - y|^2)$.*

Proof. (i) For each $a, b \geq 0$, we have

$$\min(a, b) \sim \frac{ab}{a + b}. \quad (2.10)$$

So, by (1.1) we deduce (i).

(ii) Using (1.1), the fact that for each $t \geq 0$, $\text{Log}(1 + t) \sim \min(1, t) \text{Log}(2 + t)$, and (2.10) we obtain that

$$G(x, y) \sim \min\left(1, \frac{\delta(x)\delta(y)}{|x - y|^2}\right) \text{Log}\left(2 + \frac{\delta(x)\delta(y)}{|x - y|^2}\right) \sim \frac{\delta(x)\delta(y)}{[x, y]^2} \text{Log}\left(1 + \frac{[x, y]^2}{|x - y|^2}\right). \quad (2.11)$$

□

LEMMA 2.5. *Let $\lambda \in \mathbb{R}$. Then on D^2 , we have*

$$\frac{1}{(\delta(y))^\lambda} \frac{\delta(y)}{\delta(x)} G(x, y) \leq \begin{cases} \frac{1}{|x - y|^{n-2+\lambda^+}}, & \text{if } n \geq 3, \\ \frac{1}{|x - y|^{\lambda^+}} \text{Log}\left(\frac{2d}{|x - y|}\right), & \text{if } n = 2, \end{cases} \quad (2.12)$$

where $\lambda^+ = \max(0, \lambda)$.

Proof. By Lemma 2.4, we have on D^2

$$\frac{1}{(\delta(y))^\lambda} \frac{\delta(y)}{\delta(x)} G(x, y) \leq \begin{cases} \frac{1}{|x - y|^{n-2}} \frac{(\delta(y))^{2-\lambda}}{[x, y]^2}, & \text{if } n \geq 3, \\ \frac{(\delta(y))^{2-\lambda}}{[x, y]^2} \text{Log}\left(1 + \frac{[x, y]^2}{|x - y|^2}\right), & \text{if } n = 2. \end{cases} \quad (2.13)$$

Now, we remark that

$$[x, y]^2 \sim |x - y|^2 + 4\delta(x)\delta(y). \quad (2.14)$$

So, we have

$$\begin{aligned} [x, y]^2 &\geq \max(|\delta(x) - \delta(y)|^2 + 4\delta(x)\delta(y), |x - y|^2) \\ &\geq \max((\delta(y))^2, |x - y|^2). \end{aligned} \quad (2.15)$$

Therefore by (2.15) we have

$$\frac{1}{[x, y]^2} \leq \frac{1}{|x - y|^{\lambda^+} (\delta(y))^{2-\lambda^+}}. \quad (2.16)$$

Hence, it follows that

$$\frac{(\delta(y))^{2-\lambda}}{[x, y]^2} \leq \frac{1}{|x - y|^{\lambda^+}}. \quad (2.17)$$

Thus, for $n \geq 3$, we obtain

$$\frac{1}{(\delta(y))^\lambda} \frac{\delta(y)}{\delta(x)} G(x, y) \leq \frac{1}{|x - y|^{n-2+\lambda^+}}. \quad (2.18)$$

Next, it is obvious to see that

$$\text{Log} \left(1 + \frac{[x, y]^2}{|x - y|^2} \right) \leq \text{Log} \left(2 \frac{[x, y]^2}{|x - y|^2} \right) \leq \text{Log} \left(\frac{4d^2}{|x - y|^2} \right). \quad (2.19)$$

Then, for $n = 2$, we obtain by (2.17) and (2.19) that

$$\frac{1}{(\delta(y))^\lambda} \frac{\delta(y)}{\delta(x)} G(x, y) \leq \frac{1}{|x - y|^{\lambda^+}} \text{Log} \left(\frac{2d}{|x - y|} \right). \quad (2.20)$$

This completes the proof. $\square$

Proof of Proposition 2.3. Let $\alpha > 0$, $p > n/2$ and $q \geq 1$ such that $(1/p) + (1/q) = 1$. To show the claim, we use Lemma 2.5 and the Hölder inequality. We distinguish two cases.

Case 1 ($n \geq 3$). Let $f \in L^p(D)$ and $\lambda < 2 - n/p$. Then, for $x \in D$, we have

$$\begin{aligned} & \int_{D \cap B(x, \alpha)} \frac{\delta(y)}{\delta(x)} G(x, y) \frac{|f(y)|}{(\delta(y))^\lambda} dy \\ & \leq \int_{D \cap B(x, \alpha)} \frac{|f(y)|}{|x - y|^{n-2+\lambda^+}} dy \leq \|f\|_p \left(\int_0^\alpha r^{n(1-q)+(2-\lambda^+)q-1} dr \right)^{1/q} \leq \|f\|_p \alpha^{2-n/p-\lambda^+}, \end{aligned} \quad (2.21)$$

which tends to zero as $\alpha \rightarrow 0$.

Case 2 ($n = 2$). Let $f \in L^p(D)$ and $\lambda < 2/q$. Then, for $x \in D$, we have

$$\begin{aligned} & \int_{D \cap B(x, \alpha)} \frac{\delta(y)}{\delta(x)} G(x, y) \frac{|f(y)|}{(\delta(y))^\lambda} dy \\ & \leq \int_{D \cap B(x, \alpha)} \frac{|f(y)|}{|x - y|^{\lambda^+}} \text{Log} \left(\frac{2d}{|x - y|} \right) dy \leq \|f\|_p \left(\int_0^\alpha r^{n-1-\lambda^+q} \left(\text{Log} \frac{2d}{r} \right)^q dr \right)^{1/q}, \end{aligned} \quad (2.22)$$

which tends to zero as $\alpha \rightarrow 0$. This completes the proof. $\square$

8 Nonlinear elliptic problems

In the sequel, we put for $f \in \mathcal{B}(D)$ and $x \in D$,

$$v(x) = \int_D G(x, y) \frac{|f(y)|}{(\delta(y))^\lambda} dy. \quad (2.23)$$

Remark 2.6. From (1.1), we remark that for $x, y \in D$, we have $\delta(x)\delta(y) \leq G(x, y)$. This implies that there exists a constant $C > 0$ such that for each $f \in \mathcal{B}(D)$ and $x \in D$,

$$C\delta(x) \int_D (\delta(y))^{1-\lambda} |f(y)| dy \leq v(x). \quad (2.24)$$

In the next proposition, we will give upper estimates on the function v .

PROPOSITION 2.7. *Let $p > n/2$ and $\lambda < 2 - n/p$. Then there exists a constant $c > 0$, such that for each $f \in L^p(D)$ and $x \in D$,*

$$v(x) \leq \begin{cases} c\|f\|_p (\delta(x))^{2-n/p-\lambda}, & \text{if } 1 - \frac{n}{p} < \lambda < 2 - \frac{n}{p}, \\ c\|f\|_p \delta(x) \left(\text{Log } \frac{2d}{\delta(x)} \right)^{1/q}, & \text{if } \lambda = 1 - \frac{n}{p}, \\ c\|f\|_p \delta(x), & \text{if } \lambda < 1 - \frac{n}{p}. \end{cases} \quad (2.25)$$

To prove Proposition 2.7, we need the following lemma.

LEMMA 2.8 (see [8]). *Let $x, y \in D$. Then we have the following properties:*

- (i) *if $\delta(x)\delta(y) \leq |x - y|^2$ then $\min(\delta(x), \delta(y)) \leq ((\sqrt{5} + 1)/2)|x - y|$,*
- (ii) *if $|x - y|^2 \leq \delta(x)\delta(y)$ then $((3 - \sqrt{5})/2)\delta(x) \leq \delta(y) \leq ((3 + \sqrt{5})/2)\delta(x)$.*

Proof of Proposition 2.7. Let $p > n/2$, $q \geq 1$ such that $(1/p) + (1/q) = 1$ and $\lambda < 2 - n/p$. Let $f \in L^p(D)$, then for each $x \in D$, we have

$$v(x) = \int_{D_1} G(x, y) \frac{|f(y)|}{(\delta(y))^\lambda} dy + \int_{D_2} G(x, y) \frac{|f(y)|}{(\delta(y))^\lambda} dy = I_1 + I_2, \quad (2.26)$$

where

$$\begin{aligned} D_1 &= \{y \in D : \delta(x)\delta(y) \geq |x - y|^2\}, \\ D_2 &= \{y \in D : \delta(x)\delta(y) \leq |x - y|^2\}. \end{aligned} \quad (2.27)$$

Now, we remark that for each $x \in D$ and $y \in D_1$, we have by (1.1) and Lemma 2.8

$$\frac{1}{(\delta(y))^\lambda} G(x, y) \leq \begin{cases} \frac{1}{(\delta(x))^\lambda} \frac{1}{|x - y|^{n-2}}, & \text{for } n \geq 3, \\ \frac{1}{(\delta(x))^\lambda} \text{Log} \left(1 + \left(\frac{2\delta(x)}{|x - y|} \right)^2 \right), & \text{for } n = 2. \end{cases} \quad (2.28)$$

Then, by the Hölder inequality and Lemma 2.8, we obtain for $n \geq 3$

$$\begin{aligned}
 I_1 &\leq \|f\|_p (\delta(x))^{-\lambda} \left(\int_{D_1} \frac{1}{|x-y|^{(n-2)q}} dy \right)^{1/q} \\
 &\leq \|f\|_p (\delta(x))^{-\lambda} \left(\int_0^{((\sqrt{5}+1)/2)\delta(x)} r^{n-1-(n-2)q} dr \right)^{1/q} \\
 &\leq \|f\|_p (\delta(x))^{2-\lambda-n/p}.
 \end{aligned} \tag{2.29}$$

Now, assume that $n = 2$, then since $q > 1$ and $\text{Log}(1+t) \leq t^{1/2}q$, for each $t \geq 1$, we obtain

$$\frac{1}{(\delta(y))^\lambda} G(x, y) \leq \frac{(\delta(x))^{1/q-\lambda}}{|x-y|^{1/q}}. \tag{2.30}$$

So, by the Hölder inequality and Lemma 2.8, it follows that

$$\begin{aligned}
 I_1 &\leq \|f\|_p (\delta(x))^{1/q-\lambda} \left(\int_{D_1} \frac{1}{|x-y|} dy \right)^{1/q} \\
 &\leq \|f\|_p (\delta(x))^{1/q-\lambda} \left(\int_0^{((\sqrt{5}+1)/2)\delta(x)} dr \right)^{1/q} \\
 &\leq \|f\|_p (\delta(x))^{2/q-\lambda} = \|f\|_p (\delta(x))^{2-\lambda-2/p}.
 \end{aligned} \tag{2.31}$$

Next, by (1.1), we have for each $x \in D$ and $y \in D_2$

$$\frac{1}{(\delta(y))^\lambda} G(x, y) \sim \frac{\delta(x)(\delta(y))^{1-\lambda}}{|x-y|^n}. \tag{2.32}$$

Then, using the Hölder inequality and Lemma 2.8, we obtain

$$I_2 \leq \|f\|_p \left(\int_{D_2} \left(\frac{\delta(x)(\delta(y))^{1-\lambda}}{|x-y|^n} \right)^q dy \right)^{1/q}. \tag{2.33}$$

For each $y \in D_2$, it follows from Lemma 2.8 that $\delta(y) \leq |x-y|$. So, we will discuss two cases.

Case 3. If $\lambda \leq 1$, it follows that

$$I_2 \leq \|f\|_p \delta(x) \left(\int_{D_2} \frac{1}{|x-y|^{(n-1+\lambda)q}} dy \right)^{1/q} \tag{2.34}$$

$$\leq \|f\|_p \delta(x) \left(\int_{((\sqrt{5}-1)/2)\delta(x)}^d r^{n-1-(n-1+\lambda)q} dr \right)^{1/q}. \tag{2.35}$$

Thus, we distinguish the following two subcases.

(a) If $\lambda \leq 1 - n/p$, then from (2.35) it follows that

$$\begin{aligned} I_2 &\leq \|f\|_p \delta(x) \left(\int_{((\sqrt{5}-1)/2)\delta(x)}^d r^{(1-n-\lambda p)/(p-1)} dr \right)^{1/q} \\ &\leq \|f\|_p \delta(x) \begin{cases} \left(\text{Log} \frac{2d}{\delta(x)} \right)^{1/q} & \text{if } \lambda = 1 - \frac{n}{p}; \\ 1 & \text{if } \lambda < 1 - \frac{n}{p}. \end{cases} \end{aligned} \quad (2.36)$$

(b) If $1 - n/p < \lambda \leq 1$, then by (2.34) we obtain

$$\begin{aligned} I_2 &\leq \|f\|_p (\delta(x))^{2-\lambda-n/p} \left(\int_{D_2} \frac{(\delta(x))^{(\lambda+n/p-1)q}}{|x-y|^{(n-1+\lambda)q}} dy \right)^{1/q} \\ &= \|f\|_p (\delta(x))^{2-\lambda-n/p} \left(\int_{((\sqrt{5}-1)/2)\delta(x) \leq |x-y| \leq d} \frac{1}{|x-y|^n} dy \right)^{1/q} \\ &\leq \|f\|_p (\delta(x))^{2-\lambda-n/p}. \end{aligned} \quad (2.37)$$

Case 4. If $\lambda > 1$, then from (2.33) it follows that

$$\begin{aligned} I_2 &\leq \|f\|_p (\delta(x))^{2-\lambda-n/p} \left(\int_{D_2} \left(\frac{\delta(x)}{\delta(y)} \right)^{(\lambda-1)q} \frac{(\delta(x))^{n/(p-1)}}{|x-y|^{n+n/p-1}} dy \right)^{1/q} \\ &\leq \|f\|_p (\delta(x))^{2-\lambda-n/p} \left(\int_{D_2} \left(\frac{\delta(x)}{\delta(y)} \right)^{(\lambda-1)q} \frac{1}{|x-y|^n} dy \right)^{1/q}. \end{aligned} \quad (2.38)$$

Since $(\lambda-1)q \in]0, 1[$, it follows from [8, Corollary 2.8] that

$$I_2 \leq \|f\|_p (\delta(x))^{2-\lambda-n/p}. \quad (2.39)$$

This completes the proof. $\square$

Remark 2.9. By taking $p = +\infty$ (i.e., $q = 1$), in Propositions 2.3 and 2.7, we find again the results of Maaagli in [8].

3. First existence result

In this section, we are interested in the existence of positive solutions for problem (1.5). We recall that h_0 is a fixed nontrivial nonnegative harmonic function in D , which is continuous in $\bar{D}$. Let φ be a nontrivial nonnegative continuous function on ∂D .

We denote by $H_D \varphi$ the solution of the Dirichlet problem

$$\Delta w = 0 \quad \text{in } D, \quad w|_{\partial D} = \varphi. \quad (3.1)$$

The main result of this section is the following.

THEOREM 3.1. *Assume (H_1) – (H_2) . Then there exists a constant $c > 1$ such that if $\varphi \geq ch_0$ on ∂D , then problem (1.5) has a positive continuous solution satisfying for each $x \in D$*

$$h_0(x) \leq u(x) \leq H_D \varphi(x). \quad (3.2)$$

To prove Theorem 3.1, we need the following lemma.

For a fixed $q \in K^+(D)$, put

$$\Gamma_q = \{v \in K(D) : |v| \leq q\}, \quad (3.3)$$

then, we have

LEMMA 3.2. *Let q be a nonnegative function belonging to $K(D)$, the family of functions*

$$\mathfrak{F}_q = \left\{ \int_D G(\cdot, y) v(y) dy : v \in \Gamma_q \right\} \quad (3.4)$$

is uniformly bounded and equicontinuous in $\overline{D}$, and consequently, it is relatively compact in $C_0(D)$.

Proof. Let $q \in K(D)$ and T be the operator defined on $\mathfrak{F}_q$ by

$$Tv(x) = \int_D G(x, y) v(y) dy. \quad (3.5)$$

By Proposition 2.1(i), we obtain

$$\sup_{x \in D} |Tv(x)| \leq \sup_{x \in D} \int_D G(x, y) q(y) dy < \infty. \quad (3.6)$$

Then the family $T(\mathfrak{F}_q)$ is uniformly bounded.

Next, we propose to prove the equicontinuity of $T(\mathfrak{F}_q)$ in $\overline{D}$.

Let $v \in \mathfrak{F}_q$, $x_0 \in D$, and $\alpha > 0$. Let $x, x' \in B(x_0, \alpha) \cap D$.

Then

$$|Tv(x) - Tv(x')| \leq |Vq(x) - Vq(x')|. \quad (3.7)$$

Since, by Proposition 2.1(i), $Vq \in C_0(D)$, it follows that

$$|Tv(x) - Tv(x')| \longrightarrow 0 \quad \text{as } |x - x'| \longrightarrow 0. \quad (3.8)$$

Similarly, we have $\lim_{x \rightarrow \partial D} Tv(x) = 0$. Which implies that the family $T(\mathfrak{F}_q)$ is equicontinuous in $\overline{D}$.

Finally, by Ascoli's theorem, the family $T(\mathfrak{F}_q)$ is relatively compact in $C_0(D)$. Which completes the proof. $\square$

Proof of Theorem 3.1. We will use a fixed-point argument.

Let $c = 1 + \alpha_\psi$, where α_ψ is the constant defined by (2.2) associated to the function ψ given in (H_2) . Let $\varphi \in C^+(\partial D)$ such that $\varphi \geq ch_0$ on ∂D .

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We consider the set Λ given by

$$\Lambda = \{u \in C(\overline{D}) : h_0 \leq u \leq H_D \varphi\}. \quad (3.9)$$

Since $\varphi \geq ch_0$ on ∂D , we obtain

$$H_D \varphi \geq ch_0 \quad \text{on } D. \quad (3.10)$$

So Λ is a nonempty closed bounded and convex set in $C(\overline{D})$.

For each $u \in \Lambda$, define

$$Tu(x) = H_D \varphi(x) - \int_D G(x, y) f(y, u(y)) dy, \quad \forall x \in D. \quad (3.11)$$

Now, we will prove that the family $T\Lambda$ is relatively compact in $C(\overline{D})$.

For each $y \in D$ and $u \in \Lambda$, we have by (H_2)

$$0 \leq f(y, u(y)) \leq \frac{\theta(y, h_0(y))}{h_0(y)} h_0(y) \leq c\psi(y). \quad (3.12)$$

with $c = \sup_{y \in D} h_0(y)$. Then, the function $y \rightarrow f(y, u(y)) \in \Gamma_{c\psi}$.

Hence the family

$$\left\{ \int_D G(\cdot, y) f(y, u(y)) dy : u \in \Lambda \right\} \subseteq \mathfrak{F}_{c\psi}. \quad (3.13)$$

So, using Lemma 3.2 and the fact that $H_D \varphi$ is continuous in $\overline{D}$, we conclude that $T\Lambda$ is a relatively compact set in $C(\overline{D})$.

Next, we intend to show that T maps Λ to itself.

It's obvious to see that

$$Tu(x) \leq H_D \varphi(x), \quad \forall x \in D. \quad (3.14)$$

Moreover, from (H_1) , and by using (3.11), (2.4), and (3.10), we obtain that for each $x \in D$

$$Tu(x) \geq H_D \varphi(x) - \alpha_\psi h_0(x) \geq h_0(x), \quad (3.15)$$

which proves that $T\Lambda \subset \Lambda$.

Now, let us prove the continuity of the operator T in Λ in the supremum norm. Let $(u_k)_k$ be a sequence in Λ which converges uniformly to a function u in Λ . Then, for each $x \in D$, we have

$$|Tu_k(x) - Tu(x)| \leq \int_D G(x, y) |f(y, u_k(y)) - f(y, u(y))| dy. \quad (3.16)$$

On the other hand, by hypothesis (H_1) , we have

$$|f(y, u_k(y)) - f(y, u(y))| \leq 2h_0(y)\psi(y) \leq \psi(y). \quad (3.17)$$

Since $V\psi \in C_0(D)$, we conclude by the continuity of f with respect to the second variable and the dominated convergence theorem that

$$\forall x \in \overline{D}, \quad Tu_k(x) \longrightarrow Tu(x) \quad \text{as } k \longrightarrow +\infty. \quad (3.18)$$

Since $T\Lambda$ is a relatively compact family in $C(\overline{D})$, therefore the pointwise convergence implies the uniform convergence, namely,

$$\|Tu_k - Tu\|_\infty \longrightarrow 0 \quad \text{as } k \longrightarrow +\infty. \quad (3.19)$$

Thus, T is a compact mapping on Λ .

Finally the Schauder fixed-point theorem implies the existence of $u \in \Lambda$ such that $Tu = u$, that is, for each $x \in D$

$$u(x) = H_D\varphi(x) - \int_D G(x, y)f(y, u(y))dy. \quad (3.20)$$

Now, let us verify that u is a solution of problem (1.5).

Since $\psi \in K(D)$, it follows from Proposition 2.1(ii), that $\psi \in L^1_{\text{loc}}(D)$.

Furthermore, we have $f(\cdot, u) \leq c\psi$, then $f(\cdot, u) \in L^1_{\text{loc}}(D)$ and $V(f(\cdot, u)) \in \mathfrak{F}_{c\psi}$. So by Lemma 3.2, we have

$$V(f(\cdot, u)) \in C_0(D) \subset L^1_{\text{loc}}(D). \quad (3.21)$$

Thus, applying Δ to both sides of (3.20) and using the fact that $\Delta(Vf) = -f$, we obtain, that u satisfies the elliptic differential equation

$$\Delta u = f(\cdot, u) \quad \text{in } D \text{ (in the sense of distributions)}. \quad (3.22)$$

Moreover, since $H_D\varphi = \varphi$ in ∂D and $V(f(\cdot, u)) \in C_0(D)$, we conclude that $u|_{\partial D} = \varphi$. So u is a positive continuous solution of problem (1.5). $\square$

Now, let us state another comparison result for the solution u of problem (1.5), in the case of the special nonlinearity $f(x, t) = q(x)\Phi(t)$.

For this aim, suppose that the following hypotheses on q and Φ are adopted.

- (i) $\Phi : (0, \infty) \rightarrow (0, \infty)$ is continuously differentiable nonincreasing function.
- (ii) q is a nontrivial nonnegative function on D such that

$$q \in C^\alpha_{\text{loc}}(D), \quad 0 < \alpha < 1, \quad \forall c > 0, \quad x \longrightarrow \frac{q(x)}{\delta(x)}\Phi(c\delta(x)) \in K(D). \quad (3.23)$$

Moreover, let F be the function defined on $[0, \infty)$ by

$$F(t) = \int_0^t \frac{1}{\Phi(s)} ds. \quad (3.24)$$

It is obviously seen, from hypotheses adopted on Φ , that the function F is a bijection from $[0, \infty)$ to itself. Then, we have the following.

THEOREM 3.3. *Let u be the solution given by (3.20) of the following problem:*

$$\Delta u + q\Phi(u) = 0, \quad \text{in } D, \quad u|_{\partial D} = \varphi. \quad (3.25)$$

Then, we have $u \in C^{2+\alpha}(D) \cap C(\overline{D})$. Further, u satisfies on D

$$u(x) \leq \min(H_D\varphi(x), F^{-1}(H_D(F \circ \varphi)(x) - Vq(x))). \quad (3.26)$$

Proof. Let v be the function defined on D by

$$v = F(u) - H_D(F \circ \varphi) + Vq. \quad (3.27)$$

Then $v \in C^2(D)$ and we have

$$\Delta v = \frac{1}{\Phi(u)} \Delta u - \frac{\Phi'(u)}{(\Phi(u))^2} |\nabla u|^2 - q = -\frac{\Phi'(u)}{(\Phi(u))^2} |\nabla u|^2. \quad (3.28)$$

Thus, $\Delta v \geq 0$. In addition, since $Vq \in C_0(D)$, it follows that $v \in C_0(D)$. Then, the maximum principle (see [6, pages 465-466]) implies that $v \leq 0$, in D . This completes the proof. $\square$

Remark 3.4. (1) Let $\lambda > 0$ and $\varphi(x) = \lambda$, $\forall x \in \partial D$. Then, we have for each $x \in D$,

$$H_D(F \circ \varphi)(x) - Vq(x) = F(\lambda)(x) - Vq(x) \leq F(\lambda). \quad (3.29)$$

Thus for each $x \in D$,

$$F^{-1}(H_D(F \circ \varphi)(x) - Vq(x)) \leq \lambda = H_D\varphi(x). \quad (3.30)$$

Therefore, from (3.26) we have for each $x \in D$,

$$h_0(x) \leq u(x) \leq F^{-1}(H_D(F \circ \varphi)(x) - Vq(x)). \quad (3.31)$$

(2) By hypothesis (i), we have

$$\Phi(\|u\|_\infty) \geq \Phi(\|\varphi\|_\infty). \quad (3.32)$$

Therefore,

$$h_0(x) \leq u(x) \leq H_D\varphi(x) - \Phi(\|\varphi\|_\infty)Vq(x). \quad (3.33)$$

Then we have

$$h_0 \leq u \leq \min(H_D\varphi - \Phi(\|\varphi\|_\infty), F^{-1}(H_D(F \circ \varphi) - Vq)). \quad (3.34)$$

Example 3.5. Let h_0 be a nontrivial nonnegative harmonic function, which is continuous on $\overline{D}$. Then, from [14], there exists c_1 such that for each $x \in D$

$$h_0(x) \geq c_1\delta(x). \quad (3.35)$$

Let $\alpha > 0$, and f be a nonnegative measurable function on $D \times (0, \infty)$, continuous with respect to the second variable satisfying

$$f(x, t) \leq t^{-\alpha} (\delta(x))^{\alpha+1} q(x), \quad (3.36)$$

where the function q belongs to $K^+(D)$.

Then, there exists $c > 0$ such that if $\varphi \geq (1 + c)h_0$ on ∂D , the problem

$$\begin{aligned} \Delta u &= f(\cdot, u) \quad (\text{in the sense of distributions}) \\ u &> 0 \quad \text{in } D, \quad u|_{\partial D} = \varphi, \end{aligned} \quad (3.37)$$

has a positive continuous solution in $\overline{D}$ satisfying

$$h_0(x) \leq u(x) \leq H_D \varphi(x). \quad (3.38)$$

4. Second existence result

In this section, we prove the following result for problem (1.8).

THEOREM 4.1. *Assume (H_3) – (H_4) . Then problem (1.8) has a positive solution $u \in C_0(D)$. Moreover there exist positive constants a and b , such that*

$$a\delta(x) \leq u(x) \leq b. \quad (4.1)$$

Proof. By (A_2) and (H_4) , the function $q \in K^+(D)$. Then, from Proposition 2.1(i), we have $Vq \in C_0(D)$. So, $M := \sup_{x \in D} (Vq(x)) < \infty$.

From (A_4) , there exists $b > 0$ such that $Mk(b) \leq b$.

On the other hand, by (A_1) , there exists a compact $K \subset D$ such that

$$0 < \int_K \delta(y) p(y) dy < \infty. \quad (4.2)$$

Furthermore, by (1.1), there exists $\alpha > 0$ such that for each x, y in D

$$G(x, y) \geq \alpha \delta(x) \delta(y). \quad (4.3)$$

Next, let r be the constant given by

$$r := \inf_{y \in K} \delta(y). \quad (4.4)$$

Then, from (H_4) , there exists $a > 0$ such that

$$\alpha h(ar) \int_K \delta(y) p(y) dy \geq a. \quad (4.5)$$

Now, let Ω be the convex set

$$\Omega := \{u \in C_0(D) : a\delta(x) \leq u(x) \leq b\} \quad (4.6)$$

and S be the operator defined on Ω by

$$Su(x) = \int_D G(x, y) \rho(y, u(y)) dy. \quad (4.7)$$

We will prove that S is a compact mapping on Ω .

By (H_4) , we have for each $u \in \Omega$

$$\rho(\cdot, u) \leq k(b)q = \tilde{q}. \quad (4.8)$$

Since $q \in K^+(D)$, it follows that the function $y \rightarrow \rho(y, u(y)) \in \Gamma_{\tilde{q}}$.

Hence, the family

$$\left\{ \int_{\Omega} G(\cdot, y) \rho(y, u(y)) dy : u \in \Omega \right\} \subseteq \mathfrak{F}_{\tilde{q}}. \quad (4.9)$$

Consequently, by Lemma 3.2, the family $S(\Omega)$ is relatively compact in $C_0(D)$. Next, we need to verify that for $u \in \Omega$ and $x \in D$, we have

$$a\delta(x) \leq Su(x) \leq b. \quad (4.10)$$

Let $u \in \Omega$ and $x \in D$, then by (H_4) , we have

$$\begin{aligned} Su(x) &\leq \int_D G(x, y) q(y) k(u(y)) dy \\ &\leq k(b) \int_D G(x, y) q(y) dy \\ &\leq Mk(b) \leq b. \end{aligned} \quad (4.11)$$

On the other hand, from (H_4) and using (1.1) and (4.5), we have

$$\begin{aligned} Su(x) &\geq \alpha\delta(x) \int_D \delta(y) p(y) h(u(y)) dy \\ &\geq \alpha\delta(x) \int_K \delta(y) p(y) h(a\delta(y)) dy \\ &\geq \delta(x) \left[\alpha h(ar) \int_K \delta(y) p(y) dy \right] \geq a\delta(x). \end{aligned} \quad (4.12)$$

Thus, it follows that $S(\Omega) \subset \Omega$.

Now, we consider a sequence $(u_k)_k$ in Ω which converges uniformly to u in Ω . Since ρ is continuous with respect to the second variable, we deduce by the dominated convergence theorem that for all $x \in D$,

$$Su_k(x) \longrightarrow Su(x) \quad \text{as } k \longrightarrow +\infty. \quad (4.13)$$

Therefore, using the fact that $S(\Omega)$ is relatively compact in $C_0(D)$, we conclude that $\|Su_k - Su\|_{\infty} \rightarrow 0$ as $k \rightarrow +\infty$. Hence S is a compact mapping from Ω to itself. Then by the

Schauder fixed-point theorem, there exists a function $u \in \Omega$ such that

$$u(x) = \int_D G(x, y) \rho(y, u(y)) dy. \quad (4.14)$$

Now, since $q \in K^+(D)$ then by Proposition 2.1(ii), we have $\rho(\cdot, u) \in L^1_{\text{loc}}(D)$ and $V(\rho(\cdot, u)) \in C_0(D) \subset L^1_{\text{loc}}(D)$.

So, u satisfies (in the sense of distributions) $\Delta u = -\rho(\cdot, u)$ in D . Moreover, $\lim_{x \rightarrow \partial D} u(x) = \lim_{x \rightarrow \partial D} V(\rho(\cdot, u(\cdot)))(x) = 0$. So u is a solution of problem (1.8). $\square$

Example 4.2. Let $p > n/2$ and $f \in L^p_+(D)$. Assume that the function $g : (0, \infty) \rightarrow [0, \infty)$ is a nontrivial continuous and nondecreasing function satisfying

$$\lim_{t \rightarrow 0^+} \frac{g(t)}{t} = +\infty, \quad \lim_{t \rightarrow +\infty} \frac{g(t)}{t} = 0. \quad (4.15)$$

Then for each $\lambda < 2 - n/p$ the problem

$$\Delta u = -(\delta(x))^{-\lambda} f(x) g(u) \quad \text{in } D, \quad u|_{\partial D} = 0, \quad (4.16)$$

has a positive solution $u \in C_0(D)$. Moreover, from Proposition 2.7, we have for each $x \in D$,

$$u(x) \leq \begin{cases} c \|f\|_p (\delta(x))^{2-n/p-\lambda}, & \text{if } 1 - \frac{n}{p} < \lambda < 2 - \frac{n}{p}, \\ c \|f\|_p \delta(x) \left(\text{Log} \frac{2d}{\delta(x)} \right)^{(p-1)/p}, & \text{if } \lambda = 1 - \frac{n}{p}, \\ c \|f\|_p \delta(x), & \text{if } \lambda < 1 - \frac{n}{p}. \end{cases} \quad (4.17)$$

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