Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2007, Article ID 18187, 21 pages
doi:10.1155/2007/18187

Research Article

Existence and Multiplicity of Positive Solutions for Dirichlet Problems in Unbounded Domains

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Received 30 September 2006; Revised 26 December 2006; Accepted 3 January 2007

Recommended by Vy Khoi Le

We consider the elliptic problem $-\Delta u + u = b(x)|u|^{p-2}u + h(x)$ in Ω, $u \in H_0^1(\Omega)$, where $2 < p < (2N/(N-2))$ $(N \geq 3)$, $2 < p < \infty$ $(N = 2)$, Ω is a smooth unbounded domain in $\mathbb{R}^N$, $b(x) \in C(\Omega)$, and $h(x) \in H^{-1}(\Omega)$. We use the shape of domain Ω to prove that the above elliptic problem has a ground-state solution if the coefficient $b(x)$ satisfies $b(x) \to b^\infty > 0$ as $|x| \to \infty$ and $b(x) \geq c$ for some suitable constants $c \in (0, b^\infty)$, and $h(x) \equiv 0$. Furthermore, we prove that the above elliptic problem has multiple positive solutions if the coefficient $b(x)$ also satisfies the above conditions, $h(x) \geq 0$ and $0 < \|h\|_{H^{-1}} < (p-2)(1/(p-1))^{(p-1)/(p-2)}[b_{\sup} S^p(\Omega)]^{1/(2-p)}$, where $S(\Omega)$ is the best Sobolev constant of subcritical operator in $H_0^1(\Omega)$ and $b_{\sup} = \sup_{x\in\Omega} b(x)$.

1. Introduction

In this paper, we are concerned with the existence and multiplicity of positive solutions of the following elliptic problems:

$$\begin{gathered} -\Delta u + u = b(x)|u|^{p-2}u + h(x) \quad \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{gathered} \tag{1.1}$$

where $2 < p < (2N/(N-2))$ $(N \geq 3)$, $2 < p < \infty$ $(N = 2)$, and Ω is a smooth unbounded domain in $\mathbb{R}^N$. We assume that $b(x) \in C(\Omega) \cap L^\infty(\Omega)$ satisfies

$$b(x) > 0, \quad \forall x \in \Omega, \tag{1.2}$$

and $h(x)$ satisfies

$$h(x) \in H^{-1}(\Omega), \quad h(x) \geq 0. \tag{1.3}$$

Associated with (1.1), we consider the energy functional J_h^b in the Sobolev space $H_0^1(\Omega)$:

$$J_h^b(u) = \frac{1}{2}\|u\|_{H^1}^2 - \frac{1}{p}\int_\Omega b(x)|u|^p - \int_\Omega h(x)u, \tag{1.4}$$

where $\|u\|_{H^1} = (\int_\Omega |\nabla u|^2 + u^2)^{1/2}$. By Rabinowitz [1, Proposition B.10], $J_h^b \in C^1(H_0^1(\Omega), \mathbb{R})$. It is well known that the solutions of (1.1) are the critical points of the energy functional J_h^b in $H_0^1(\Omega)$.

Under the assumption (1.3) and $h(x) \not\equiv 0$, (1.1) can be regarded as a perturbation problem of the following homogeneous elliptic equation:

$$\begin{gathered} -\Delta u + u = b(x)|u|^{p-2}u \quad \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{gathered} \tag{1.5}$$

A typical approach for solving a problem of this kind is to use the minimax method:

$$\alpha_\Gamma^b(\Omega) = \inf_{\gamma \in \Gamma(\Omega)} \max_{t \in [0,1]} J_0^b(\gamma(t)), \tag{1.6}$$

where

$$\Gamma(\Omega) = \{\gamma \in C([0,1], H_0^1(\Omega)) \mid \gamma(0) = 0,\ \gamma(1) = e\}, \tag{1.7}$$

$J_0^b(e) = 0$, and $e \neq 0$. By the mountain pass lemma due to Ambrosetti and Rabinowitz [2], we called the nonzero critical point $u \in H_0^1(\Omega)$ of J_0^b is as ground-state solution of (1.5) in Ω if $J_0^b(u) = \alpha_\Gamma^b(\Omega)$. We note that the ground-state solutions of (1.5) in Ω can also be obtained by the Nehari minimization problem

$$\alpha_0^b(\Omega) = \inf_{v \in \mathbf{M}_0^b(\Omega)} J_0^b(v), \tag{1.8}$$

where $\mathbf{M}_0^b(\Omega) = \{u \in H_0^1(\Omega)\backslash\{0\} \mid \|u\|_{H^1}^2 = \int_\Omega b(x)|u|^p\}$. Note that $\mathbf{M}_0^b(\Omega)$ contains every nonzero solution of (1.5) in Ω, $\alpha_\Gamma^b(\Omega) = \alpha^b(\Omega) > 0$ (see Willem [3] and Wang and Wu [4]), and if $b(x) \equiv b^\infty > 0$ is a constant, then J_0^b and $\alpha_0^b(\Omega)$ are replaced by J_0^∞ and $\alpha_0^\infty(\Omega)$, respectively.

That the existence of ground-state solutions of (1.5) is affected by the shape of the domain Ω and $b(x)$ that satisfies some suitable conditions has been the focus of a great deal of research in recent years. By the Rellich compactness theorem and the minimax method, it is easy to obtain a ground-state solution for (1.5) in bounded domains. When Ω is an unbounded domain and $b(x) \equiv b^\infty$, the existence of ground-state solutions has been established by several authors under various conditions. We mention, in particular, results by Berestycki and Lions [5], Lien et al. [6], Chen and Wang [7], and Del Pino and Felmer [8, 9]. In [5], $\Omega = \mathbb{R}^N$. Actually, Kwong [10] proved that the positive solution of (1.5) in $\mathbb{R}^N$ is unique. In [6], Ω is a periodic domain. In [7, 6], the domain Ω is required

to satisfy that

($\Omega 1$) $\Omega = \Omega_1 \cup \Omega_2$, where Ω_1, Ω_2 are domains in $\mathbb{R}^N$ and $\Omega_1 \cap \Omega_2$ is bounded;

($\Omega 2$) $\alpha_0^\infty(\Omega) < \min\{\alpha_0^\infty(\Omega_1), \alpha_0^\infty(\Omega_2)\}$.

In [8, 9], for $1 \le l \le N-1$, $\mathbb{R}^N = \mathbb{R}^l \times \mathbb{R}^{N-l}$. For a point $x \in \mathbb{R}^N$, we have $x = (y,z)$, where $y \in \mathbb{R}^l$ and $z \in \mathbb{R}^{N-l}$. Let $y \in \mathbb{R}^l$, we denote by $\Omega^y \subset \mathbb{R}^{N-l}$ the projection of Ω onto $\mathbb{R}^{N-l}$, that is,

$$\Omega^y = \{z \in \mathbb{R}^{N-l} \mid (y,z) \in \Omega\}. \tag{1.9}$$

The domain Ω is required to satisfy that

($\Omega 3$) Ω is a smooth subset of $\mathbb{R}^N$ and the projections Ω^y are bounded uniformly in $y \in \mathbb{R}^l$;

($\Omega 4$) there exists a nonempty closed set $F \subset \mathbb{R}^{N-l}$ such that $F \subset \Omega^y$ for all $y \in \mathbb{R}^l$;

($\Omega 5$) for each $\delta > 0$, there exists $K > 0$ such that

$$\Omega^y \subset \{z \in \mathbb{R}^{N-l} \mid \operatorname{dist}(z,F) < \delta\} \tag{1.10}$$

for all $|y| \ge K$.

Moreover, when $\Omega = \mathbb{R}^N \backslash \omega$ is an exterior domain, where ω is a bounded domain. It is well known that (1.5) in $\mathbb{R}^N \backslash \omega$ does not admit any ground-state solution (see Benci and Cerami [12]). However, Bahri and Lions [11] and Benci and Cerami [12] asserted that (1.5) in $\mathbb{R}^N \backslash \omega$ has a higher-energy positive solution. As Ω is an Esteban-Lions domain, (1.5) in Ω does not admit any nontrivial solution (see Esteban and Lions [13]), where the definition of Esteban-Lions domain is as follows: for a proper unbounded domain Ω in $\mathbb{R}^N$, there exists $\chi \in \mathbb{R}^N$, $\|\chi\| = 1$ such that $n(x) \cdot \chi \ge 0$ and $n(x) \cdot \chi \not\equiv 0$ on $\partial\Omega$, where $n(x)$ is the unit outward normal vector to $\partial\Omega$ at the point x.

When $b(x) \not\equiv b^\infty$, which satisfies the condition (1.2), the existence of ground-state solutions of (1.5) has been established by the condition $b(x) \ge b^\infty$ and the existence of ground-state solutions of limit equation

$$\begin{gathered} -\Delta u + u = b^\infty |u|^{p-2} u \quad \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{gathered} \tag{1.11}$$

On the other hand, for $\Omega = \mathbb{R}^N$ and $b(x) \le b^\infty$ on $\mathbb{R}^N$ with a strict inequality on a set of positive measures, (1.5) in $\mathbb{R}^N$ does not admit any ground-state solution. However, Bahri and Lions [11], Cao [14], and Bahri and Li [15] asserted that (1.5) in $\mathbb{R}^N$ has a higher-energy positive solution under the coefficient $b(x)$ which satisfies conditions $b(x) \ge (1/2)^{(p-2)/2} b^\infty$ and $b(x) \to b^\infty$ as $|x| \to \infty$ such that the functional J_0^b in $H_0^1(\Omega)$ satisfies the Palais-Smale condition for energy level β with

$$\alpha_0^\infty(\mathbb{R}^N) < \beta < \alpha_0^\infty(\mathbb{R}^N) + \alpha_0^b(\mathbb{R}^N). \tag{1.12}$$

The first result of our paper is relaxing the condition $b(x) \geq b^\infty$ to show the existence of ground-state solution of (1.5) by the shape of domain Ω. First, we consider the following assumptions:

($\Omega 1'$) given $k \geq 0$ and $1 \leq m \leq k$, the domain $\Omega = \bigcup_{i=1}^{k} \Omega_i$, where $\Omega_i \cap \Omega_j$ is bounded for all $i \neq j$ and Ω_j is unbounded domain for all $j = 1, 2, \ldots, m$;

($\Omega 2'$) the functional J_0^∞ in $H_0^1(\Omega)$ satisfies the Palais-Smale condition for energy level $\alpha_0^\infty(\Omega)$;

($b1$) $b(x) \geq (\alpha_0^\infty(\Omega)/\min\{\alpha_0^\infty(\Omega_1), \alpha_0^\infty(\Omega_2), \ldots, \alpha_0^\infty(\Omega_m)\})^{(p-2)/2} b^\infty$ and $b(x) \to b^\infty$ as $|x| \to \infty$.

Then we have the following result.

Theorem 1.1. *If the domain Ω satisfies the conditions ($\Omega 1'$)-($\Omega 2'$) and $b(x)$ satisfies the condition ($b1$), then (1.5) in Ω has a ground-state solution.*

Remark 1.2. If the domain Ω satisfies the conditions ($\Omega 1$)-($\Omega 2$), then the functional J_0^∞ in $H_0^1(\Omega)$ satisfies the Palais-Smale condition for energy level $\alpha_0^\infty(\Omega)$, and we have

$$0 < \alpha_0^\infty(\Omega) < \min\{\alpha_0^\infty(\Omega_1), \alpha_0^\infty(\Omega_2), \ldots, \alpha_0^\infty(\Omega_m)\} \tag{1.13}$$

(see Lien et al. [6] and Chen and Wang [7]). Thus,

$$0 < \left(\frac{\alpha_0^\infty(\Omega)}{\min\{\alpha_0^\infty(\Omega_1), \alpha_0^\infty(\Omega_2), \ldots, \alpha_0^\infty(\Omega_m)\}}\right)^{(p-2)/2} < 1. \tag{1.14}$$

It is known that the general unbounded domains in $\mathbb{R}^N$ can be classified into three kinds. If Ω is an unbounded domain in $\mathbb{R}^N$, then it satisfies one of the following conditions:

(1) J_0^∞ in $H_0^1(\Omega)$ satisfies the Palais-Smale condition for energy level $\alpha_0^\infty(\Omega)$. In particular, (1.11) in Ω has a ground-state solution u_0 such that $J_0^\infty(u_0) = \alpha_0^\infty(\Omega)$;

(2) J_0^∞ in $H_0^1(\Omega)$ does not satisfy the Palais-Smale condition for energy level $\alpha_0^\infty(\Omega)$, but (1.11) in Ω has a ground-state solution u_0 such that $J_0^\infty(u_0) = \alpha_0^\infty(\Omega)$;

(3) equation (1.11) in Ω does not admit any ground-state solution.

In this motivation, consider a general unbounded domain Ω and its exterior domain $\Omega^c(r) = \Omega \backslash \overline{B^N(0;r)}$, and the following assumptions:

($\Omega 3'$) equation (1.11) in Ω has a ground state solution u_0 such that $J_0^\infty(u_0) = \alpha_0^\infty(\Omega)$.

($b2$) $b(x) \geq (\alpha_0^\infty(\Omega)/\lim_{r\to\infty} \alpha_0^\infty(\Omega^c(r)))^{(p-2)/2} b^\infty$ and $b(x) \to b^\infty$ as $|x| \to \infty$.

Then we have the following result.

Theorem 1.3. *If the unbounded domain Ω satisfies the condition ($\Omega 3'$) and $b(x)$ satisfies the condition ($b2$), then (1.5) in Ω has a ground-state solution.*

Remark 1.4. (1) If the domain Ω satisfies the conditions ($\Omega 3$)–($\Omega 5$), J_0^∞ in $H_0^1(\Omega)$ satisfies the Palais-Smale condition for energy level $\alpha_0^\infty(\Omega)$. Then $\alpha_0^\infty(\Omega) < \alpha_0^\infty(\Omega^c(r))$ for all $r > 0$ (see Del Pino and Felmer [8, 9] or Wu [16]). Since $\alpha_0^\infty(\Omega^c(r))$ is nondecreasing as r is

increasing, we have

$$0 \le \left(\frac{\alpha_0^\infty(\Omega)}{\lim_{r\to\infty} \alpha_0^\infty(\Omega^c(r))} \right)^{(p-2)/2} < 1. \tag{1.15}$$

(2) If Ω is a periodic domain, then J_0^∞ in $H_0^1(\Omega)$ does not satisfy the Palais-Smale condition for energy level $\alpha_0^\infty(\Omega)$, but (1.11) in Ω has a ground-state solution u_0 such that $J_0^\infty(u_0) = \alpha_0^\infty(\Omega)$. Then $\alpha_0^\infty(\Omega) = \alpha_0^\infty(\Omega^c(r))$ for all $r > 0$ (see Lien et al. [6]). Thus,

$$\left(\frac{\alpha_0^\infty(\Omega)}{\lim_{r\to\infty} \alpha_0^\infty(\Omega^c(r))} \right)^{(p-2)/2} \equiv 1. \tag{1.16}$$

Remark 1.5. If the domain $\Omega = \mathbb{R}^N$, coefficient $b(x)$ satisfies the condition (1.2) and $b(x) \le b^\infty$ with a strict inequality on a set of positive measures, then (1.5) in $\mathbb{R}^N$ does not admit any ground-state solution and $\alpha_0^\infty(\mathbb{R}^N) = \alpha_0^b(\mathbb{R}^N)$. However, if the domain Ω satisfies the conditions ($\Omega 1$)-($\Omega 2$) (or ($\Omega 3$)–($\Omega 5$)), $b(x)$ satisfies the condition ($b1$) (or ($b2$)) and $b(x) \le b^\infty$ with a strict inequality on a set of positive measure, then from Theorem 1.1 (or Theorem 1.3), we can conclude that (1.5) has a ground-state solution. Moreover, $\alpha_0^\infty(\Omega) < \alpha_0^b(\Omega)$.

Finally, we consider (1.1). For $\Omega = \mathbb{R}^N$, several authors have shown the existence of at least two positive solutions of (1.1) in $\mathbb{R}^N$ under some suitable conditions. In [17] by Zhu for $b(x) = b^\infty$, $h(x)$ is exponential decay and $\|h\|_{L^2}$ is sufficiently small. By Cao and Zhou in [18] and Jeanjean [19], for $b(x) \ge b^\infty$ and $\|h\|_{H^{-1}}$ sufficiently small. By Adachi and Tanaka in [20], for $b(x) \ge b^\infty - Ce^{-\lambda|x|}$ for some $C, \lambda > 0$ and $\|h\|_{H^{-1}}$ sufficiently small. Moreover, Adachi and Tanaka [21] used that (1.5) in $\mathbb{R}^N$ does not admit any ground-state solution for the condition $b(x) \le b^\infty$ with a strict inequality on a set of positive measures, to show that (1.1) in $\mathbb{R}^N$ has at least four positive solutions for $\|h\|_{H^{-1}}$ sufficiently small. The second aim of our paper is also relaxing the condition $b(x) \ge b^\infty$ to show the existence of at least two positive solutions of (1.1) in Ω. Denote

$$b_{\sup} = \sup_{x\in\Omega} b(x) \tag{1.17}$$

and $S(\Omega) = [(2p/(p-2))\alpha_0^\infty(\Omega)]^{(2-p)/2p}$ is the best Sobolev constant of subcritical operator in $H_0^1(\Omega)$ (see Lin et al. [22] or Willem [3]). Then we have the following results.

Theorem 1.6. *Suppose that the domain Ω satisfies the conditions ($\Omega 1'$)-($\Omega 2'$) and $b(x)$ satisfies the condition ($b1$). If $h \ge 0$ and*

$$0 < \|h\|_{H^{-1}} < (p-2)\left(\frac{1}{p-1}\right)^{(p-1)/(p-2)} [b_{\sup} S^p(\Omega)]^{1/(2-p)}, \tag{1.18}$$

then (1.1) in Ω has at least two positive solutions.

Theorem 1.7. *Suppose that the domain Ω satisfies the condition $(\Omega 3')$ and $b(x)$ satisfies the condition $(b2)$. If $h \geq 0$ and*

$$0 < \|h\|_{H^{-1}} < (p-2)\left(\frac{1}{p-1}\right)^{(p-1)/(p-2)} \left[b_{\sup} S^p(\Omega)\right]^{1/(2-p)}, \tag{1.19}$$

then (1.1) in Ω has at least two positive solutions.

This paper is organized as follows. In Section 2, we describe various preliminaries. In Section 3, we use the shape of the domain Ω to prove that (1.5) in Ω has a ground-state solution. In Section 4, we modify the proof of Adachi and Tanaka [21], Tarantello [23], Cao and Zhu [18], and Zhu [17] to prove that (1.1) in Ω has at least two positive solutions.

2. Preliminary

We define the Palais-Smale (PS) sequences, (PS) values, and (PS) conditions in $H_0^1(\Omega)$ for J_h^b as follows.

Definition 2.1. (i) For $\beta \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J_h^b if $J_h^b(u_n) = \beta + o(1)$ and $(J_h^b)'(u_n) = o(1)$ strongly in $H^{-1}(\Omega)$ as $n \to \infty$;

(ii) $\beta \in \mathbb{R}$ is a (PS) value in $H_0^1(\Omega)$ for J_h^b if there is a $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J_h^b;

(iii) J_h^b satisfies the $(\mathrm{PS})_\beta$-condition in $H_0^1(\Omega)$ if every $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J_h^b contains a convergent subsequence;

(iv) J_h^b satisfies the (PS) condition in $H_0^1(\Omega)$ if for every $\beta \in \mathbb{R}$, J_h^b satisfies the $(\mathrm{PS})_\beta$-condition in $H_0^1(\Omega)$.

We need the following lemmas.

Lemma 2.2. *Let $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$. Then there exists a subsequence $\{u_n\}$ such that*

(i) *$\{u_n\}$ is bounded in $H_0^1(\Omega)$ and $\|u\|_{H^1} \leq \liminf_{n\to\infty} \|u_n\|_{H^1}$;*

(ii) *$u_n \rightharpoonup u$, $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(\Omega)$, and $u_n \to u$ a.e. in Ω;*

(iii) $\|u_n - u\|_{H^1}^2 = \|u_n\|_{H^1}^2 - \|u\|_{H^1}^2 + o(1)$.

The proof is clear by the routine arguments, and hence is omitted here.

Lemma 2.3 (Brézis-Lieb lemma). *Suppose that $u_n \to u$ a.e. in Ω and there exists $c > 0$ such that $\|u_n\|_{L^p} \leq c$ for $n = 1, 2, \ldots$. Then*

(i) $\|u_n - u\|_{L^p}^p = \|u_n\|_{L^p}^p - \|u\|_{L^p}^p + o(1)$;

(ii) $|u_n - u|^{p-2}(u_n - u) - |u_n|^{p-2}u_n + |u|^{p-2}u = o(1)$ *in* $L^{p/(p-1)}(\Omega)$.

For the proof, see Brézis and Lieb [24].

Lemma 2.4. *Let $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and*

$$(J_h^b)'(u_n) = -\Delta u_n + u_n - b(x)\,|u_n|^{p-2}u_n + h(x) = o(1) \quad \text{in } H^{-1}(\Omega). \tag{2.1}$$

Then

(i) $|u_n - u|^{p-2}(u_n - u) - |u_n|^{p-2}u_n + |u|^{p-2}u = o(1)$ *in* $H^{-1}(\Omega)$;

(ii) $(J_0^\infty)'(w_n) = -\Delta w_n + w_n - b^\infty |w_n|^{p-2}w_n = o(1)$ *in* $H^{-1}(\Omega)$, *where* $w_n = u_n - u$;

(iii) *if* $\{u_n\}$ *is a* $(PS)_\beta$*-sequence in* $H_0^1(\Omega)$ *for* J_h^b *then* $\{w_n\}$ *is a* $(PS)_{(\beta - J_h^b(u))}$*-sequence in* $H_0^1(\Omega)$ *for* J_0^∞.

Proof. For (i), (ii), see Bahri and Lions [11]. (iii) Since $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $\{u_n\}$ is a $(PS)_\beta$-sequence for J_h^b in $H_0^1(\Omega)$, by Lemmas 2.2, 2.3, and the Sobolev embedding theorem, there exists a subsequence $\{u_n\}$ such that $w_n \rightharpoonup 0$ in $H_0^1(\Omega)$,

$$\begin{aligned} \|w_n\|_{H^1}^2 &= \|u_n\|_{H^1}^2 - \|u\|_{H^1}^2 + o(1), \\ \|w_n\|_{L^p}^p &= \|u_n\|_{L^p}^p - \|u\|_{L^p}^p + o(1). \end{aligned} \tag{2.2}$$

Thus,

$$J_0^\infty(w_n) = J_h^b(w_n) + o(1) = J_h^b(u_n) - J_h^b(u) + o(1) = \beta - J_h^b(u) + o(1). \tag{2.3}$$

Therefore, by part (ii), $\{p_n\}$ is a $(PS)_{(\beta - J_h^b(u))}$-sequence in $H_0^1(\Omega)$ for J_0^∞. □

We need the following useful results.

Lemma 2.5. *Let* $\{u_n\}$ *be a sequence in* $H_0^1(\Omega)$. *Then* $\{u_n\}$ *is a* $(PS)_{\alpha_0^b(\Omega)}$*-sequence for* J_0^b *if and only if* $J_0^b(u_n) = \alpha_0^b(\Omega) + o(1)$ *and* $\int_\Omega |\nabla u_n|^2 + u_n^2 = \int_\Omega b(x)|u_n|^p + o(1)$. *In particular, every minimizing sequence* $\{u_n\}$ *in* $\mathbf{M}_0^b(\Omega)$ *of* $\alpha_0^b(\Omega)$ *is a* $(PS)_{\alpha_0^b(\Omega)}$*-sequence in* $H_0^1(\Omega)$ *for* J_0^b.

The proof is almost the same as that by Wang and Wu in [4, Lemma 7], and is omitted here.

We introduce the Nehari minimization problem for (1.1) as

$$\alpha_h^b(\Omega) = \inf_{u \in \mathbf{M}_h^b(\Omega)} J_h^b(u), \tag{2.4}$$

where $\mathbf{M}_h^b(\Omega) = \{u \in H_0^1(\Omega)\backslash\{0\} \mid \langle (J_h^b)'(u), u\rangle = 0\}$. Define

$$\psi(u) = \langle (J_h^b)'(u), u\rangle = \|u\|_{H^1}^2 - \int_\Omega b(x)|u|^p - \int_\Omega h(x)u. \tag{2.5}$$

Then we have the following result.

Lemma 2.6. *If* $\|h\|_{H^{-1}} < (p-2)(1/(p-1))^{(p-1)/(p-2)}[b_{\sup}S^p(\Omega)]^{1/(2-p)}$, *then for each* $u \in \mathbf{M}_h^b(\Omega)$,

$$\langle \psi'(u), u\rangle = \|u\|_{H^1}^2 - (p-1)\int_\Omega b(x)|u|^p \neq 0. \tag{2.6}$$

Proof. For $u \in \mathbf{M}_h^b(\Omega)$, we have

$$\|u\|_{H^1}^2 - \int_\Omega b(x)|u|^p - \int_\Omega h(x)u = 0. \tag{2.7}$$

Then

$$\begin{aligned}\langle \psi'(u), u\rangle &= 2\|u\|_{H^1}^2 - p\int_\Omega b(x)|u|^p - \int_\Omega h(x)u \\ &= \|u\|_{H^1}^2 - (p-1)\int_\Omega b(x)|u|^p.\end{aligned} \tag{2.8}$$

We claim that if $\|h\|_{H^{-1}} < (p-2)(1/(p-1))^{(p-1)/(p-2)}[b_{\sup}S^p(\Omega)]^{1/(2-p)}$, then $\langle \psi'(u), u\rangle \neq 0$ for all $u \in \mathbf{M}_h^b(\Omega)$. Let $I : \mathbf{M}_h^b(\Omega) \to \mathbb{R}$ be given by

$$I(u) = K(p)\left(\frac{\|u\|_{H^1}^{2p-2}}{\int_\Omega b(x)|u|^p}\right)^{1/(p-2)} - \int_\Omega h(x)u, \tag{2.9}$$

where $K(p) = (p-2)(1/(p-1))^{(p-1)/(p-2)}$. Then we have for $u \in \mathbf{M}_h^b(\Omega)$,

$$\begin{aligned} I(u) &= K(p)\left(\frac{\|u\|_{H^1}^{2p-2}}{\int_\Omega b(x)|u|^p}\right)^{1/(p-2)} - \int_\Omega h(x)u \\ &\geq K(p)\left(\frac{\|u\|_{H^1}^{2p-2}}{\int_\Omega b(x)|u|^p}\right)^{1/(p-2)} - \|h\|_{H^{-1}}\|u\|_{H^1} \\ &= \|u\|_{H^1}\left(K(p)\left(\frac{\|u\|_{H^1}^{p}}{\int_\Omega b(x)|u|^p}\right)^{1/(p-2)} - \|h\|_{H^{-1}}\right)\end{aligned} \tag{2.10}$$

since

$$\left(\frac{\|u\|_{H^1}^{p}}{\int_\Omega b(x)|u|^p}\right)^{1/(p-2)} \geq [b_{\sup}S^p(\Omega)]^{1/(2-p)} \quad \forall u \in H_0^1(\Omega)\backslash\{0\}. \tag{2.11}$$

Thus, for $\|h\|_{H^{-1}} < K(p)[b_{\sup}S^p(\Omega)]^{1/(2-p)}$, we have

$$I(u) > 0 \quad \forall u \in \mathbf{M}_h^b(\Omega). \tag{2.12}$$

Assume that there is a $w \in \mathbf{M}_h^b(\Omega)$ such that $\langle \psi'(w), w\rangle = 0$, then we have

$$\begin{aligned} \|w\|_{H^1}^2 &= (p-1)\int_\Omega b(x)|w|^p, \\ \int_\Omega h(x)w &= \|w\|_{H^1}^2 - \int_\Omega b(x)|w|^p = (p-2)\int_\Omega b(x)|w|^p.\end{aligned} \tag{2.13}$$

From (2.12) and (2.13),

$$\begin{aligned} 0 < I(w) &= K(p)\left(\frac{\|w\|_{H^1}^{2p-2}}{\int_\Omega b(x)|w|^p}\right)^{1/(p-2)} - \int_\Omega h(x)w \\ &= \left(\frac{1}{p-1}\right)^{(p-1)/(p-2)}(p-2)\left(\frac{(p-1)^{p-1}\left[\int_\Omega b(x)|w|^p\right]^{p-1}}{\int_\Omega b(x)|w|^p}\right)^{1/(p-2)} - (p-2)\int_\Omega h(x)w = 0,\end{aligned} \tag{2.14}$$

which is a contradiction. Thus, we can conclude that for

$$\|h\|_{H^{-1}} < (p-2)\left(\frac{1}{p-1}\right)^{(p-1)/(p-2)}\left[b_{\sup}S^p(\Omega)\right]^{1/(2-p)}, \tag{2.15}$$

we have $\langle \psi'(u), u\rangle \neq 0$ for all $u \in \mathbf{M}_h^b(\Omega)$. □

By Lemma 2.6, we write $\mathbf{M}_h^b(\Omega) = \mathbf{M}_h^{b+}(\Omega) \cup \mathbf{M}_h^{b-}(\Omega)$, where

$$\begin{aligned} \mathbf{M}_h^{b+}(\Omega) &= \left\{u \in \mathbf{M}_h^b(\Omega) \mid \|u\|_{H^1}^2 - (p-1)\int_\Omega b(x)|u|^p > 0\right\}, \\ \mathbf{M}_h^{b-}(\Omega) &= \left\{u \in \mathbf{M}_h^b(\Omega) \mid \|u\|_{H^1}^2 - (p-1)\int_\Omega b(x)|u|^p < 0\right\}, \end{aligned} \tag{2.16}$$

and define

$$\alpha_h^{b+}(\Omega) = \inf_{u\in\mathbf{M}_h^{b+}(\Omega)} J_h^b(u), \qquad \alpha_h^{b-}(\Omega) = \inf_{u\in\mathbf{M}_h^{b-}(\Omega)} J_h^b(u). \tag{2.17}$$

For each $u \in H_0^1(\Omega)\backslash\{0\}$, we write

$$t_{\max} = \left(\frac{\|u\|_{H^1}^2}{(p-1)\int_\Omega b(x)|u|^p}\right)^{1/(p-2)} > 0. \tag{2.18}$$

Similar as the proof of some results by Tarantello in [23], we have the following two lemmas.

Lemma 2.7. *For each $u \in H_0^1(\Omega)\backslash\{0\}$,*

(i) *there is a unique $t^- = t^-(u) > t_{\max} > 0$ such that $t^-u \in \mathbf{M}_h^{b-}(\Omega)$ and $J_h^b(t^-u) = \max_{t\geq t_{\max}} J_h^b(tu)$;*

(ii) *$t^-(u)$ is a continuous function for nonzero u;*

(iii) *$\mathbf{M}_h^{b-}(\Omega) = \{u \in H_0^1(\Omega)\backslash\{0\} \mid (1/\|u\|_{H^1})t^-(u/\|u\|_{H^1}) = 1\}$;*

(iv) *if $\int_\Omega hu > 0$, then there is a unique $0 < t^+ = t^+(u) < t_{\max}$ such that $t^+u \in \mathbf{M}_h^{b+}(\Omega)$ and $J_h^b(t^+u) = \min_{0\leq t\leq t^-} J_h^b(tu)$.*

Lemma 2.8. (i) *For each $u \in \mathbf{M}_h^{b+}(\Omega)$, $\int_\Omega h(x)u > 0$ and $J_h^b(u) < 0$. In particular, $\alpha_h(\Omega) \leq \alpha_h^+(\Omega) < 0$;*

(ii) *J_h^b is coercive and bounded below on $\mathbf{M}_h^b(\Omega)$.*

Proof. (i) For each $u \in \mathbf{M}_h^{b+}(\Omega)$, $\|u\|_{H^1}^2 - (p-1)\int_\Omega b(x)|u|^p > 0$ and

$$\|u\|_{H^1}^2 = \int_\Omega b(x)|u|^p + \int_\Omega h(x)u. \tag{2.19}$$

Thus,

$$\int_\Omega h(x)u = \|u\|_{H^1}^2 - \int_\Omega b(x)|u|^p > (p-2)\int_\Omega b(x)|u|^p > 0, \tag{2.20}$$

and hence

$$\begin{aligned} J_h^b(u) &= \left(\frac{1}{2}-\frac{1}{p}\right)\int_\Omega b(x)|u|^p - \frac{1}{2}\int_\Omega h(x)u \\ &< \frac{p-2}{2p}\int_\Omega b(x)|u|^p - \frac{p-2}{2}\int_\Omega b(x)|u|^p \\ &= -\frac{(p-1)(p-2)}{2p}\int_\Omega b(x)|u|^p < 0. \end{aligned} \tag{2.21}$$

(ii) Is similar to the proof of Theorem 1 by Tarantello in [23]. □

3. Homogeneous problems

First, we present several (PS) conditions in $H_0^1(\Omega)$ for J_0^b which are used to prove our main results. As a consequence of Lemma 2.8(ii), for each $(\mathrm{PS})_\beta$-sequence $\{u_n\}$ in $H_0^1(\Omega)$ for J_0^b, there exist a subsequence $\{u_n\}$ and u_0 in $H_0^1(\Omega)$ such that $u_n \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$. Then u_0 is a solution of (1.5) in Ω. Moreover, we have the following lemma.

Let Ω be any unbounded domain and $\xi \in C^\infty([0,\infty))$ such that $0 \le \xi \le 1$ and

$$\xi(t) = \begin{cases} 0 & \text{for } t \in [0,1] \\ 1 & \text{for } t \in [2,\infty). \end{cases} \tag{3.1}$$

Let

$$\xi_n(z) = \xi\left(\frac{2|z|}{n}\right). \tag{3.2}$$

Then we have the following result.

Lemma 3.1. *Let $\{u_n\}$ be a $(PS)_\beta$-sequence in $H_0^1(\Omega)$ for J_0^b satisfying $u_n \rightharpoonup 0$ weakly in $H_0^1(\Omega)$ and let $v_n = \xi_n u_n$. Then there exists a subsequence $\{u_n\}$ such that*

(i) $\|u_n - v_n\|_{H^1} = o(1)$ *as* $n \to \infty$;

(ii) $\int_\Omega b(x)|u_n|^p = \int_\Omega b(x)|v_n|^p + o(1) = \int_\Omega b^\infty |v_n|^p + o(1)$;

(iii) $\int_\Omega |\nabla v_n|^2 + v_n^2 = \int_\Omega b^\infty |v_n|^p + o(1)$;

(iv) $\{v_n\}$ *is a $(PS)_\beta$-sequence in $H_0^1(\Omega)$ for J_0^∞.*

Proof. By the fact that

$$||u_n - v_n||_{H^1}^2 = ||u_n||_{H^1}^2 + ||v_n||_{H^1}^2 - 2\langle u_n, v_n\rangle_{H^1}, \tag{3.3}$$

thus it suffices to show that $\langle u_n, v_n\rangle_{H^1} = \|u_n\|_{H^1}^2 + o(1) = \|v_n\|_{H^1}^2 + o(1)$. Since

$$\langle u_n, v_n\rangle_{H^1} = \int_\Omega \nabla u_n \nabla v_n + u_n v_n = \int_\Omega \xi_n[\,|\nabla u_n|^2 + u_n^2] + \int_\Omega u_n \nabla u_n \nabla \xi_n, \tag{3.4}$$

$|\nabla \xi_n| \le c/n$ and $\{u_n\}$ is a $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J_0^b, it follows that

$$\int_\Omega \xi_n^q u_n \nabla u_n \nabla \xi_n = o(1) \quad \text{for } q > 0. \tag{3.5}$$

Hence,

$$\langle u_n, v_n \rangle_{H^1} = \int_\Omega \xi_n [\,|\nabla u_n|^2 + u_n^2] + o(1). \tag{3.6}$$

Similarly, we have

$$\|v_n\|_{H^1}^2 = \int_\Omega \xi_n^2 [\,|\nabla u_n|^2 + u_n^2] + o(1). \tag{3.7}$$

Given $r \geq 1$, since $\{\xi_n^r u_n\}$ is bounded in $H_0^1(\Omega)$, we have

$$\begin{aligned} o(1) &= \langle (J_0^b)'(u_n), \xi_n^r u_n \rangle \\ &= \int_\Omega (\xi_n^r |\nabla u_n|^2 + r\xi_n^{r-1} u_n \nabla \xi_n \nabla u_n + \xi_n^r u_n^2) - \int_\Omega b(x) \xi_n^r |u_n|^p. \end{aligned} \tag{3.8}$$

From (3.5), we can conclude that

$$\int_\Omega \xi_n^r (|\nabla u_n|^2 + u_n^2) = \int_\Omega b(x) \xi_n^r |u_n|^p + o(1). \tag{3.9}$$

Since $u_n \rightharpoonup 0$ weakly in $H_0^1(\Omega)$ and $b(x) \to b^\infty$ as $|x| \to \infty$, there exists a subsequence $\{u_n\}$ such that $u_n \to 0$ strongly in $L^p_{\mathrm{loc}}(\Omega)$, or there exists a subsequence $\{u_n\}$ such that

$$\int_{Q(n)} b(x) |u_n|^p = o(1), \tag{3.10}$$

where $Q(n) = \Omega \cap B^N(0;n)$. Clearly,

$$\int_\Omega b(x) |u_n|^p = \int_\Omega b(x) \xi_n^r |u_n|^p + o(1) = \int_\Omega b^\infty \xi_n^r |u_n|^p + o(1). \tag{3.11}$$

By (3.6), (3.7), (3.9), and (3.11),

$$\begin{aligned} \langle u_n, v_n \rangle_{H^1} &= \|u_n\|_{H^1}^2 + o(1) = \|v_n\|_{H^1}^2 + o(1), \\ \int_\Omega b(x) |u_n|^p &= \int_\Omega b(x) |v_n|^p + o(1) = \int_\Omega b^\infty |v_n|^p + o(1). \end{aligned} \tag{3.12}$$

Therefore, $\|u_n - v_n\|_{H^1} = o(1)$ as $n \to \infty$. The results of (iii) and (iv), from (i), (ii) and Lemmas 2.4, 2.5. □

We need the following compactness results.

Proposition 3.2. *Suppose that the domain Ω satisfies the conditions $(\Omega 1')$-$(\Omega 2')$. If $\{u_n\}$ is a $(PS)_\beta$-sequence in $H_0^1(\Omega)$ for J_0^b with*

$$\alpha_0^b(\Omega) \leq \beta < \min\{\alpha_0^\infty(\Omega) + \alpha_0^b(\Omega), \alpha_0^\infty(\Omega_1), \alpha_0^\infty(\Omega_2), \ldots, \alpha_0^\infty(\Omega_m)\}, \tag{3.13}$$

then there exist a subsequence $\{u_n\}$ and $u_0 \neq 0$ such that $u_n \to u_0$ strongly in $H_0^1(\Omega)$ and $J_0^b(u_0) = \beta$.

Proof. Let $\{u_n\}$ be a $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J_0^b with

$$\alpha_0^b(\Omega) \le \beta < \min\{\alpha_0^\infty(\Omega) + \alpha_0^b(\Omega), \alpha_0^\infty(\Omega_1), \alpha_0^\infty(\Omega_2), \dots, \alpha_0^\infty(\Omega_m)\}. \tag{3.14}$$

Since $\{u_n\}$ is bounded, there exist a subsequence $\{u_n\}$ and u_0 in $H_0^1(\Omega)$ such that $u_n \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$ and $u_n \to u_0$ a.e in Ω. Moreover, u_0 is a solution of (1.5) in Ω. If $u_0 \equiv 0$, by Lemma 3.1 there exists a subsequence $\{u_n\}$ such that $\{\xi_n u_n\}$ is a $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J_0^∞, where ξ_n is as in (3.2). Let $v_n = \xi_n u_n$, and we obtain

$$J_0^\infty(v_n) = \beta + o(1), \quad (J_0^\infty)'(v_n) = o(1) \quad \text{in } H^{-1}(\Omega). \tag{3.15}$$

Since $\Omega_i \cap \Omega_j$ is bounded for $i \ne j$ and Ω_l is also bounded for $m+1 \le l \le k$, there exists $n_0 \in \mathbb{N}$ such that $v_n = 0$ in $\overline{\Omega(n_0)}$ for $n > 2n_0$ and $\Omega_l \subset \Omega(n_0)$ for all $l \in \{m+1, m+2, \dots, k\}$, where $\Omega(n) = \Omega \cap B^N(0;n)$. Moreover, $v_n = v_n^1 + v_n^2 + \cdots + v_n^m$ and for $i = 1, 2, \dots, m$,

$$v_n^i(z) = \begin{cases} v_n(z) & \text{for } z \in \Omega_i, \\ 0, & \text{for } z \notin \Omega_i. \end{cases} \tag{3.16}$$

Then $v_n^i \in H_0^1(\Omega_i)$ and

$$\int_{\Omega_i} \left(|\nabla v_n^i|^2 + (v_n^i)^2 \right) = \int_{\Omega_i} b^\infty |v_n^i|^p + o(1). \tag{3.17}$$

By (3.15), we obtain

$$\begin{gathered} (J_0^\infty)'(v_n^i) = o(1) \quad \text{strongly in } H^{-1}(\Omega_i) \text{ for } i = 1, 2, \dots, m, \\ \beta = J_0^\infty(v_n) + o(1) = \sum_{i=1}^m J_0^\infty(v_n^i) + o(1). \end{gathered} \tag{3.18}$$

Assume that

$$J_0^\infty(v_n^i) = c_i + o(1) \quad \text{for } i = 1, 2, \dots, m, \tag{3.19}$$

then $c_1 + c_2 + \cdots + c_m = \beta$, since all of c_i are (PS)-values in $H_0^1(\Omega_i)$ for J_0^∞ and nonnegative. Thus, there exists $i_0 \in \{1, 2, \dots, m\}$ such that c_{i_0} are positive (PS)-values in $H_0^1(\Omega_i)$ for J_0^∞ and

$$\alpha_0^\infty(\Omega_{i_0}) \le c_{i_0} \le \beta, \tag{3.20}$$

which contradicts (3.14). Consequently, $u_0 \not\equiv 0$ and $\beta \ge J_0^b(u_0) \ge \alpha_0^b(\Omega)$. Let $p_n = u_n - u_0$. By Lemma 2.4, $\{p_n\}$ is a $(\mathrm{PS})_{(\beta - J_0^b(u_0))}$-sequence in $H_0^1(\Omega)$ for J_0^∞. Since $\beta < \alpha_0^\infty(\Omega) + \alpha_0^b(\Omega)$, $J_0^b(u_0) \ge \alpha_0^b(\Omega)$ and $\alpha_0^b(\Omega)$ is a smallest positive (PS)-value in $H_0^1(\Omega)$ for J_0^b. Thus, $\beta - J_0^b(u_0) = 0$. This implies that $u_n \to u_0$ strongly in $H_0^1(\Omega)$ and $J_0^b(u_0) = \beta$. □

Proposition 3.3. *Suppose that the unbounded domain Ω satisfies the condition $(\Omega 3')$. If $\{u_n\}$ is a (PS)$_\beta$-sequence in $H_0^1(\Omega)$ for J_0^b with*

$$\alpha_0^b(\Omega) \le \beta < \min\left\{\alpha_0^\infty(\Omega) + \alpha_0^b(\Omega),\ \lim_{r\to\infty} \alpha_0^\infty(\Omega^c(r))\right\}, \tag{3.21}$$

then there exist a subsequence $\{u_n\}$ and $u_0 \neq 0$ such that $u_n \to u_0$ strongly in $H_0^1(\Omega)$ and $J_0^b(u_0) = \beta$.

Proof. Let $\{u_n\}$ be a (PS)$_\beta$-sequence in $H_0^1(\Omega)$ for J_0^b with

$$\alpha_0^b(\Omega) \le \beta < \min\left\{\alpha_0^\infty(\Omega) + \alpha_0^b(\Omega),\ \lim_{r\to\infty} \alpha_0^\infty(\Omega^c(r))\right\}. \tag{3.22}$$

Since $\{u_n\}$ is bounded, there exist a subsequence $\{u_n\}$ and u_0 in $H_0^1(\Omega)$ such that $u_n \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$ and $u_n \to u_0$ a.e in Ω. Moreover, u_0 is a solution of (1.5) in Ω. If $u_0 \equiv 0$, by Lemma 3.1 there exists a subsequence $\{u_n\}$ such that $\{\xi_n u_n\}$ is a (PS)$_\beta$-sequence in $H_0^1(\Omega)$ for J_0^∞, where ξ_n is as in (3.2). Let $v_n = \xi_n u_n$, we obtain $v_n \in H_0^1(\Omega^c(n))$ for each n,

$$J_0^\infty(v_n) = \beta + o(1), \quad \left(J_0^\infty\right)'(v_n) = o(1) \quad \text{in } H^{-1}(\Omega). \tag{3.23}$$

Moreover, there is an $s_n > 0$ such that $s_n v_n \in \mathbf{M}^\infty(\Omega^c(n))$ and $s_n = 1 + o(1)$. Then

$$J_0^\infty(s_n v_n) \ge \alpha_0^\infty(\Omega^c(n)). \tag{3.24}$$

By (3.23), (3.24), we obtain

$$\beta \ge \lim_{n\to\infty} \alpha_0^\infty(\Omega^c(n)), \tag{3.25}$$

which contradicts (3.22). Consequently, $u_0 \not\equiv 0$ and $\beta \ge J_0^b(u_0) \ge \alpha_0^b(\Omega)$. Let $p_n = u_n - u_0$. By Lemma 2.4, $\{p_n\}$ is a (PS)$_{(\beta - J_0^b(u_0))}$-sequence in $H_0^1(\Omega)$ for J_0^∞. Since $\beta < \alpha_0^\infty(\Omega) + \alpha_0^b(\Omega)$, $J_0^b(u_0) \ge \alpha_0^b(\Omega)$ and $\alpha_0^b(\Omega)$ is smallest positive (PS)-value in $H_0^1(\Omega)$ for J_0^b. Thus, $\beta - J_0^b(u_0) = 0$. This implies that $u_n \to u_0$ strongly in $H_0^1(\Omega)$ and $J_0^b(u_0) = \beta$. □

Now, we begin to show the proof of Theorem 1.1: since the domain Ω satisfies the conditions $(\Omega 1')$-$(\Omega 2')$, we have (1.11), and there exists a ground-state solution u_0 such that $J_0^\infty(u_0) = \alpha_0^\infty(\Omega)$. Let $s_0 > 0$ with $s_0 u_0 \in \mathbf{M}_0^b(\Omega)$. Then

$$s_0^2 \int_\Omega \left(|\nabla u_0|^2 + u_0^2\right) = s_0^p \int_\Omega b(x)|u_0|^p. \tag{3.26}$$

Since $b(x) \ge b^\infty(\alpha_0^\infty(\Omega)/\min\{\alpha_0^\infty(\Omega_1), \alpha_0^\infty(\Omega_2), \ldots, \alpha_0^\infty(\Omega_m)\})^{(p-2)/2}$ and $b(x) \to b^\infty$ as $|x| \to \infty$, we apply (3.26) to obtain

$$s_0 < \left(\frac{\min\{\alpha_0^\infty(\Omega_1), \alpha_0^\infty(\Omega_2), \ldots, \alpha_0^\infty(\Omega_m)\}}{\alpha_0^\infty(\Omega)}\right)^{1/2}. \tag{3.27}$$

Thus,

$$\begin{aligned}\alpha_0^b(\Omega) \le J_0^b(s_0u_0) &= \left(\frac{1}{2}-\frac{1}{p}\right)s_0^2\int_\Omega\left(|\nabla u_0|^2+u_0^2\right)\\ &< \frac{\min\{\alpha_0^\infty(\Omega_1),\alpha_0^\infty(\Omega_2),\ldots,\alpha_0^\infty(\Omega_m)\}}{\alpha_0^\infty(\Omega)}\left(\frac{1}{2}-\frac{1}{p}\right)\int_\Omega\left(|\nabla u_0|^2+u_0^2\right)\\ &= \min\{\alpha_0^\infty(\Omega_1),\alpha_0^\infty(\Omega_2),\ldots,\alpha_0^\infty(\Omega_m)\}.\end{aligned} \tag{3.28}$$

By Proposition 3.2, (1.5) has a ground-state solution.

Now, we begin to show the proof of Theorem 1.3: since the domain Ω satisfies the condition $(\Omega 3')$, we have (1.11) in Ω, and there exists a ground-state solution u_0 such that $J_0^\infty(u_0)=\alpha_0^\infty(\Omega)$. Let $s_0>0$ with $s_0u_0\in\mathbf{M}_0^b(\Omega)$. Then

$$s_0^2\int_\Omega\left(|\nabla u_0|^2+u_0^2\right)=s_0^p\int_\Omega b(x)|u_0|^p. \tag{3.29}$$

Since $b(x)\ge b^\infty(\alpha_0^\infty(\Omega)/\lim_{r\to\infty}\alpha_0^\infty(\Omega^c(r)))^{(p-2)/2}$ and $b(x)\to b^\infty$ as $|x|\to\infty$, we apply (3.29) to obtain

$$s_0<\left(\frac{\lim_{r\to\infty}\alpha_0^\infty(\Omega^c(r))}{\alpha_0^\infty(\Omega)}\right)^{1/2}. \tag{3.30}$$

Thus,

$$\begin{aligned}\alpha_0^b(\Omega) \le J_0^b(s_0u_0) &= \left(\frac{1}{2}-\frac{1}{p}\right)s_0^2\int_\Omega\left(|\nabla u_0|^2+u_0^2\right)\\ &< \frac{\lim_{r\to\infty}\alpha_0^\infty(\Omega^c(r))}{\alpha_0^\infty(\Omega)}\left(\frac{1}{2}-\frac{1}{p}\right)\int_\Omega\left(|\nabla u_0|^2+u_0^2\right)\\ &= \lim_{r\to\infty}\alpha_0^\infty(\Omega^c(r)).\end{aligned} \tag{3.31}$$

By Proposition 3.3, (1.5) has a ground-state solution.

4. Nonhomogeneous problems

4.1. Existence of a local minimum. First, we establish the existence of a local minimum. Similar as the proof of Lemma 1.4 by Adachi and Tanaka in [21], we have the following lemma.

Lemma 4.1. *If* $\|h\|_{H^{-1}} < (p-2)(1/(p-1))^{(p-1)/(p-2)}[b_{\sup}S^p(\Omega)]^{1/(2-p)}$, *then*

(i) $\mathbf{M}_h^{b+}(\Omega) \subset B(0;r_0)$;

(ii) $J_h^b(u)$ *is strictly convex in* $B(0;r_0)$,

where $B(0;r_0) = \{u \in H^1(\Omega) \mid \|u\|_{H^1} < r_0\}$ *and* $r_0 = [(p-1)b_{\sup}S^p(\Omega)]^{1/(2-p)}$.

Furthermore, we have the following theorem.

Theorem 4.2. *If* r_0 *is as in Lemma 4.1, then the functional* J_h^b *has a unique critical point* $u_{\min}$ *in* $B(0;r_0)$ *and it satisfies*

(i) $u_{\min} \in \mathbf{M}_h^{b+}(\Omega)$ *and* $J_h^b(u_{\min}) = \alpha_h^{b+}(\Omega) = \alpha_h^b(\Omega)$;

(ii) $u_{\min}$ *is a positive solution of (1.1).*

Proof. Similar as the proof of Theorem 2.1 by Cao and Zhu in [18], there is a $u_{\min} \in \mathbf{M}_h^{b+}(\Omega)$ which is a critical point for J_h^b such that $J_h^b(u_{\min}) = \alpha_h^{b+} = \alpha_h^b$, since $\mathbf{M}_h^{b+}(\Omega) \subset B(0;r_0)$ and $J_h^b(u)$ is strictly convex in $B(0;r_0)$, so that $u_{\min}$ is a unique critical point of J_h^b in $B(0;r_0)$. Since $u_{\min}$ is a unique critical point of J_h^b in $B(0;r_0)$, we have that $u_{\min}$ is a nonnegative solution of (1.1). By the maximum principle, $u_{\min}$ is positive. □

4.2. Multiple positive solutions. Throughout this section, we let $u_{\min}$ be the local minimum for J_h^b in $H_0^1(\Omega)$ in Theorem 4.2 and

$$\|h\|_{H^{-1}} < (p-2)\left(\frac{1}{p-1}\right)^{(p-1)/(p-2)}[b_{\sup}S^p(\Omega)]^{1/(2-p)}. \tag{4.1}$$

Then we have the following restricted (PS) conditions.

Proposition 4.3. *Suppose that the domain* Ω *satisfies the conditions* $(\Omega 1')$-$(\Omega 2')$. *If* $\{u_n\}$ *is a* $(PS)_\beta$*-sequence in* $H_0^1(\Omega)$ *for* J_h^b *with*

$$\beta < \alpha_h^b(\Omega) + \min\{\alpha_0^\infty(\Omega_1), \alpha_0^\infty(\Omega_2), \ldots, \alpha_0^\infty(\Omega_m)\}, \tag{4.2}$$

then there exist a subsequence $\{u_n\}$ *and* u *in* $H_0^1(\Omega)$ *such that* $u_n \to u$ *strongly in* $H_0^1(\Omega)$ *and* $J_h^b(u) = \beta$.

Proof. Let $\{u_n\}$ be a $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J_h^b. By Lemma 2.8(ii), $\{u_n\}$ is bounded. Then there exist a subsequence $\{u_n\}$ and a nonzero solution u of (1.1) such that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$. Suppose that $u_n \not\to u$ strongly in $H_0^1(\Omega)$. Let $w_n = u_n - u$ for $n = 1,2,\ldots$. Then, by Lemma 2.4, $\{w_n\}$ is a $(\mathrm{PS})_{\beta - J_h^b(u)}$-sequence in $H_0^1(\Omega)$ for J_0^∞, since $w_n \rightharpoonup 0$ and $w_n \not\to 0$ strongly in $H_0^1(\Omega)$. Similar as the proof of Proposition 3.2,

$$\beta - J_h^b(u) \geq \min\{\alpha_0^\infty(\Omega_1), \alpha_0^\infty(\Omega_2), \ldots, \alpha_0^\infty(\Omega_m)\}, \tag{4.3}$$

which is a contradiction. Thus $u_n \to u$ strongly in $H_0^1(\Omega)$. □

Proposition 4.4. *Suppose that the domain* Ω *satisfies the condition* $(\Omega 3')$. *If* $\{u_n\}$ *is a* $(PS)_\beta$*-sequence in* $H_0^1(\Omega)$ *for* J_h^b *with*

$$\beta < \alpha_h(\Omega) + \lim_{r\to\infty} \alpha_0^\infty(\Omega^c(r)), \tag{4.4}$$

then there exist a subsequence $\{u_n\}$ *and* u *in* $H_0^1(\Omega)$ *such that* $u_n \to u$ *strongly in* $H_0^1(\Omega)$ *and* $J_h^b(u) = \beta$.

The proof is similar to the proof of Proposition 4.3.

Lemma 4.5. *Suppose that the domain* Ω *satisfies the conditions* $(\Omega 1')$-$(\Omega 2')$ *and the coefficient* $b(x)$ *satisfies the condition* $(b1)$. *Let* $\overline{u}$ *be a positive solution of (1.11) in* Ω *such that* $J_0^\infty(\overline{u}) = \alpha_0^\infty(\Omega)$ *and let* $u_{\min}$ *be a local minimum in Theorem 4.2. Then*

$$\sup_{t\geq 0} J_h^b(u_{\min} + t\overline{u}) < J_h^b(u_{\min}) + \min\{\alpha_0^\infty(\Omega_1), \alpha_0^\infty(\Omega_2), \ldots, \alpha_0^\infty(\Omega_m)\}. \tag{4.5}$$

Proof. Since $u_{\min}$ is a positive solution of (1.1). Let $f(s) = s^{p-1}$ for $s \geq 0$ and $F_b(u) = \int_\Omega b(x)\int_0^u f(s)ds\,dx = (1/p)\int_\Omega b(x)u^p$, then

$$\begin{aligned} J_h^b(u_{\min} + t\overline{u}) &= J_h^b(u_{\min}) + J_0^b(t\overline{u}) + t\left(\int_\Omega b(x)u_0^{p-1}\overline{u} + h(x)\overline{u}\right) - \int_\Omega h(x)t\overline{u} \\ &\quad + \frac{1}{p}\left[\int_\Omega b(x)u_0^p + \int_\Omega b(x)|t\overline{u}|^p - \int_\Omega b(x)\,|u_0 + t\overline{u}|^p\right] \\ &= J_h^b(u_{\min}) + J_0^b(t\overline{u}) - \int_\Omega b(x)\left\{\int_0^{t\overline{u}} [f(u_0+s) - f(s) - f(u_0)]ds\right\}. \end{aligned} \tag{4.6}$$

For $v > 0$ and $w > 0$, we have

$$\begin{aligned} f(v+w) &= (v+w)^{p-1} \\ &= (v+w)^{p-2}v + (v+w)^{p-2}w \\ &> v^{p-1} + w^{p-1} = f(v) + f(w). \end{aligned} \tag{4.7}$$

Thus, $J_h^b(u_{\min} + t\overline{u}) \leq J_h^b(u_{\min}) + J_0^b(t\overline{u})$. Since $J_0^b(t\overline{u}) \to -\infty$ as $t \to \infty$, there is a $t_0 > 0$ such that $J_h^b(u_{\min} + t\overline{u}) < J_h^b(u_0)$ for $t \geq t_0$. Hence,

$$\sup_{t\geq 0} J_h^b(u_{\min} + t\overline{u}) = \sup_{0\leq t\leq t_0} J_h^b(u_{\min} + t\overline{u}). \tag{4.8}$$

Let $g_1(t) = J_h^b(u_{\min} + t\overline{u})$ for $t \geq 0$. By the continuity of $g_1(t)$, given

$$\varepsilon = \frac{1}{2}\min\{\alpha_0^\infty(\Omega_1), \alpha_0^\infty(\Omega_2), \ldots, \alpha_0^\infty(\Omega_m)\} > 0, \tag{4.9}$$

there exists $t_1 \in (0, t_0)$ such that

$$g_1(t) < g_1(0) + \frac{1}{2}\min\{\alpha_0^\infty(\Omega_1), \alpha_0^\infty(\Omega_2), \ldots, \alpha_0^\infty(\Omega_m)\} \quad \text{for } t \in [0, t_1). \tag{4.10}$$

Then

$$\begin{aligned} \sup_{0\leq t\leq t_1} J_h^b(u_{\min} + t\overline{u}) &\leq J_h^b(u_{\min}) + \frac{1}{2}\min\{\alpha_0^\infty(\Omega_1), \alpha_0^\infty(\Omega_2), \ldots, \alpha_0^\infty(\Omega_m)\} \\ &< J_h^b(u_{\min}) + \min\{\alpha_0^\infty(\Omega_1), \alpha_0^\infty(\Omega_2), \ldots, \alpha_0^\infty(\Omega_m)\}. \end{aligned} \tag{4.11}$$

Now, we only need to show that

$$\sup_{t_1\le t\le t_0} J_h^b(u_{\min}+t\overline{u}) < J_h^b(u_{\min}) + \min\{\alpha_0^\infty(\Omega_1),\alpha_0^\infty(\Omega_2),\dots,\alpha_0^\infty(\Omega_m)\}. \tag{4.12}$$

Let $g_2(t) = J_0^b(t\overline{u})$ for $t \ge 0$. Then

$$\begin{aligned} g_2'(t) &= t\int_\Omega (|\nabla\overline{u}|^2+\overline{u}^2) - t^{p-1}\int_\Omega b(x)\overline{u}^p, \\ g_2''(t) &= \int_\Omega (|\nabla\overline{u}|^2+\overline{u}^2) - (p-1)t^{p-2}\int_\Omega b(x)\overline{u}^p. \end{aligned} \tag{4.13}$$

There is a unique $\overline{t} = [\int_\Omega(|\nabla\overline{u}|^2+\overline{u}^2)/\int_\Omega b(x)\overline{u}^p]^{1/(p-2)}$ such that $g_2'(\overline{t}) = 0$ and $g_2''(\overline{t}) < 0$. Thus, $g_2(t)$ has an absolutely maximum at $\overline{t}$. Since

$$b(x) \ge b^\infty\left(\frac{\alpha_0^\infty(\Omega)}{\min\{\alpha_0^\infty(\Omega_1),\alpha_0^\infty(\Omega_2),\dots,\alpha_0^\infty(\Omega_m)\}}\right)^{(p-2)/2}, \tag{4.14}$$

we have

$$\overline{t} \le \left(\frac{\min\{\alpha_0^\infty(\Omega_1),\alpha_0^\infty(\Omega_2),\dots,\alpha_0^\infty(\Omega_m)\}}{\alpha_0^\infty(\Omega)}\right)^{1/2}. \tag{4.15}$$

Therefore,

$$\begin{aligned} \sup_{t\ge0} J_0^b(t\overline{u}) = J_0^b(\overline{t}\overline{u}) &= \left(\frac{1}{2}-\frac{1}{p}\right)\overline{t}^2\int_\Omega(|\nabla\overline{u}|^2+\overline{u}^2) \\ &\le \min\{\alpha_0^\infty(\Omega_1),\alpha_0^\infty(\Omega_2),\dots,\alpha_0^\infty(\Omega_m)\}. \end{aligned} \tag{4.16}$$

By (4.6), (4.7), and (4.16), we obtain

$$\begin{aligned} &\sup_{t_1\le t\le t_0} J_h^b(u_{\min}+t\overline{u}) \\ &\quad\le J_h^b(u_{\min}) + \min\{\alpha_0^\infty(\Omega_1),\alpha_0^\infty(\Omega_2),\dots,\alpha_0^\infty(\Omega_m)\} \\ &\qquad - \inf_{t_1\le t\le t_0}\int_\Omega b(x)\left\{\int_0^{t\overline{u}}[f(u_{\min}+s)-f(s)-f(u_{\min})]ds\right\} \\ &\quad< J_h^b(u_{\min}) + \min\{\alpha_0^\infty(\Omega_1),\alpha_0^\infty(\Omega_2),\dots,\alpha_0^\infty(\Omega_m)\}. \end{aligned} \tag{4.17}$$

Thus, $\sup_{t\ge0} J_h^b(u_{\min}+t\overline{u}) < J_h^b(u_{\min}) + \min\{\alpha_0^\infty(\Omega_1),\alpha_0^\infty(\Omega_2),\dots,\alpha_0^\infty(\Omega_m)\}$. □

Lemma 4.6. *Suppose that the domain Ω satisfies the condition $(\Omega 3')$ and the coefficient $b(x)$ satisfies the condition $(b2)$. Let $\overline{u}$ be a positive solution of (1.11) in Ω such that $J_0^\infty(\overline{u}) = \alpha_0^\infty(\Omega)$ and let $u_{\min}$ be the local minimum in Theorem 4.2. Then*

$$\sup_{t\ge0} J_h^b(u_{\min}+t\overline{u}) < J_h^b(u_{\min}) + \lim_{r\to\infty}\alpha_0^\infty(\Omega^c(r)). \tag{4.18}$$

The proof is similar to the proof of Lemma 4.5.

Now, we begin to show the proof of Theorem 1.6: for $u \in H_0^1(\Omega)$ with $\|u\|_{H^1} = 1$, by Lemma 2.7 there is a unique $t^-(u) > 0$ such that $t^-(u)$, $u \in \mathbf{M}_h^{b-}(\Omega)$ and

$$J_h^b(t^-(u)u) = \max_{t \geq t_{\max}} J_h^b(tu). \tag{4.19}$$

By Lemma 2.7(ii) and (iii), we have that $t^-(u)$ is a continuous function for nonzero u and

$$\mathbf{M}_h^{b-}(\Omega) = \left\{ u \in H_0^1(\Omega) \backslash \{0\} \mid \frac{1}{\|u\|_{H^1}} t^- \left(\frac{u}{\|u\|_{H^1}} \right) = 1 \right\}. \tag{4.20}$$

Let

$$\begin{aligned} A_1 &= \left\{ u \in H_0^1(\Omega) \backslash \{0\} \mid \frac{1}{\|u\|_{H^1}} t^- \left(\frac{u}{\|u\|_{H^1}} \right) > 1 \right\} \cup \{0\}, \\ A_2 &= \left\{ u \in H_0^1(\Omega) \backslash \{0\} \mid \frac{1}{\|u\|_{H^1}} t^- \left(\frac{u}{\|u\|_{H^1}} \right) < 1 \right\}. \end{aligned} \tag{4.21}$$

Then $\mathbf{M}_h^{b-}(\Omega)$ disconnects $H_0^1(\Omega)$ in two connected components A_1 and A_2 and $H_0^1(\Omega) \backslash \mathbf{M}_h^{b-}(\Omega) = A_1 \cup A_2$. For each $u \in \mathbf{M}_h^{b+}(\Omega)$, we have

$$1 < t_{\max}(u) < t^-(u). \tag{4.22}$$

Since $t^-(u) = (1/\|u\|_{H^1}) t^-(u/\|u\|_{H^1})$, then $\mathbf{M}_h^{b+}(\Omega) \subset A_1$. In particular, $u_{\min} \in A_1$. We claim that there exists $t_0 > 0$ such that $u_{\min} + t_0 \overline{u} \in A_2$. First, we find a constant $c > 0$ such that $0 < t^-((u_{\min} + t\overline{u})/\|u_{\min} + t\overline{u}\|_{H^1}) < c$ for each $t \geq 0$. Otherwise, there exists a sequence $\{t_n\}$ such that $t_n \to \infty$ and $t^-((u_{\min} + t_n\overline{u})/\|u_{\min} + t_n\overline{u}\|_{H^1}) \to \infty$ as $n \to \infty$. Let $v_n = (u_{\min} + t_n\overline{u})/\|u_{\min} + t_n\overline{u}\|_{H^1}$. Since $t^-(v_n)$, $v_n \in \mathbf{M}_h^{b-}(\Omega) \subset \mathbf{M}_h^b(\Omega)$, and by the Lebesgue dominated convergence theorem,

$$\begin{aligned} \int_\Omega b(x) v_n^p &= \frac{1}{\|u_{\min} + t_n\overline{u}\|_{H^1}^p} \int_\Omega b(x) (u_{\min} + t_n\overline{u})^p \\ &= \frac{1}{\|u_{\min}/t_n + \overline{u}\|_{H^1}^p} \int_\Omega b(x) \left(\frac{u_{\min}}{t_n} + \overline{u} \right)^p \longrightarrow \frac{\int_\Omega b(x)\overline{u}^p}{\|\overline{u}\|_{H^1}^p} \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{4.23}$$

We have

$$\begin{aligned} J_h^b(t^-(v_n)v_n) &= \frac{1}{2}[t^-(v_n)]^2 - \frac{1}{p}[t^-(v_n)]^p \int_\Omega b(x) v_n^p \\ &\quad - t^-(v_n) \int_\Omega h v_n \longrightarrow -\infty \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{4.24}$$

But J_h^b is bounded below on $\mathbf{M}_h^b(\Omega)$, a contradiction. Let

$$t_0 = \frac{\left| c^2 - \|u_{\min}\|_{H^1}^2 \right|^{1/2}}{\|\overline{u}\|_{H^1}} + 1. \tag{4.25}$$

Then

$$\begin{aligned}\|u_{\min}+t_0\overline{u}\|_{H^1}^2 &= \|u_{\min}\|_{H^1}^2+t_0^2\|\overline{u}\|_{H^1}^2+2t_0\langle u_{\min},\overline{u}\rangle\\ &> \|u_{\min}\|_{H^1}^2+\left|c^2-\|u_{\min}\|_{H^1}^2\right|+2\int_\Omega b^\infty\overline{u}^{p-1}u_{\min}\\ &> c^2>\left[t^-\left(\frac{u_{\min}+t_0\overline{u}}{\|u_{\min}+t_0\overline{u}\|_{H^1}}\right)\right]^2,\end{aligned}\tag{4.26}$$

that is, $u_{\min}+t_0\overline{u}\in A_2$. Define a path $\gamma(s)=u_{\min}+st_0\overline{u}$ for $s\in[0,1]$, then

$$\gamma(0)=u_{\min}\in A_1,\qquad \gamma(1)=u_{\min}+t_0\overline{u}\in A_2,\tag{4.27}$$

and there exists $s_0\in(0,1)$ such that $u_{\min}+s_0t_0\overline{u}\in\mathbf{M}_h^{b-}(\Omega)$. Thus, by Lemma 4.5,

$$\begin{aligned}\alpha_h^-(\Omega)\le J_h^b(u_{\min}+s_0t_0\overline{u})&\le\max_{s\in[0,1]}J_h^b(\gamma(s))\\ &<J_h^b(u_{\min})+\min\{\alpha_0^\infty(\Omega_1),\alpha_0^\infty(\Omega_2),\dots,\alpha_0^\infty(\Omega_m)\}.\end{aligned}\tag{4.28}$$

By the Ekeland variational principle [25], there exists a sequence $\{u_n\}$ in $\mathbf{M}_h^{b-}(\Omega)$ such that

$$\begin{aligned}J_h^b(u_n)&=\alpha_h^{b-}(\Omega)+o(1),\\ (J_h^b)'(u_n)&=o(1)\quad\text{strongly in } H^{-1}(\Omega).\end{aligned}\tag{4.29}$$

Then by Proposition 4.3, there exist a subsequence $\{u_n\}$ and $u^0\in\mathbf{M}_h^b(\Omega)$ such that $u_n\to u^0$ strongly in $H_0^1(\Omega)$, u^0 is a solution of (1.1), and $J_h^b(u^0)=\alpha_h^{b-}(\Omega)$. By the Sobolev imbedding theorem, we have $u_n\to u^0$ strongly in $L^p(\Omega)$. Thus,

$$\|u^0\|_{H^1}^2-(p-1)\int_\Omega b(x)\,|u^0|^p\le 0.\tag{4.30}$$

Then $u^0\in\mathbf{M}_h^{b-}(\Omega)$ and

$$J_h^b(u^0)=\alpha_h^{b-}(\Omega).\tag{4.31}$$

This implies that $u_{\min}$ and u^0 are distinct. Finally, since $h\ge 0$, by Lemma 2.7 there exists $t^-(|u^0|)>0$ such that

$$\begin{aligned}&t^-(|u^0|)\,|u^0|\in\mathbf{M}_h^{b-}(\Omega),\qquad t^-(|u^0|)>t_{\max}(|u^0|)=t_{\max}(u^0),\\ &\alpha_h^{b-}(\Omega)\le J_h^b(t^-(|u^0|)\,|u^0|)\le J_h^b(t^-(|u^0|)u^0)\\ &\qquad\le\max_{t\ge t_{\max}(u^0)}J_h^b(tu^0)=J_h^b(u^0)=\alpha_h^{b-}(\Omega).\end{aligned}\tag{4.32}$$

Thus,

$$J_h^b(t^-(|u^0|)\,|u^0|)=J_h^b(t^-(|u^0|)u^0)=\alpha_h^{b-}(\Omega).\tag{4.33}$$

We concluded that $\int_\Omega hu^0 = \int_\Omega h|u^0|$. Let

$$u_+^0 = \max\{u^0, 0\}, \qquad u_-^0 = \max\{-u^0, 0\}, \tag{4.34}$$

then $\int_\Omega hu_-^0 = 0$. Since $h \geq 0$ and $u_-^0 \geq 0$, we have $u_-^0 = 0$. Hence, u^0 is nonnegative. By the maximum principle, u^0 is positive. We complete the proof of Theorem 1.6.

Remark 4.7. The proof of Theorem 1.7 similar to Theorem 1.6.

Acknowledgment

This work was partially supported by the National Science Council of Taiwan.

References

[1] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, vol. 65 of *CBMS Regional Conference Series in Mathematics*, American Mathematical Society, Providence, RI, USA, 1986.

[2] A. Ambrosetti and P. H. Rabinowitz, "Dual variational methods in critical point theory and applications," *Journal of Functional Analysis*, vol. 14, no. 4, pp. 349–381, 1973.

[3] M. Willem, *Minimax Theorems*, vol. 24 of *Progress in Nonlinear Differential Equations and Their Applications*, Birkhäuser, Boston, Mass, USA, 1996.

[4] H.-C. Wang and T.-F. Wu, "Symmetry breaking in a bounded symmetry domain," *NoDEA. Nonlinear Differential Equations and Applications*, vol. 11, no. 3, pp. 361–377, 2004.

[5] H. Berestycki and P.-L. Lions, "Nonlinear scalar field equations. I. Existence of a ground state," *Archive for Rational Mechanics and Analysis*, vol. 82, no. 4, pp. 313–345, 1983.

[6] W. C. Lien, S. Y. Tzeng, and H.-C. Wang, "Existence of solutions of semilinear elliptic problems on unbounded domains," *Differential and Integral Equations*, vol. 6, no. 6, pp. 1281–1298, 1993.

[7] K.-J. Chen and H.-C. Wang, "A necessary and sufficient condition for Palais-Smale conditions," *SIAM Journal on Mathematical Analysis*, vol. 31, no. 1, pp. 154–165, 1999.

[8] M. A. Del Pino and P. L. Felmer, "Local mountain passes for semilinear elliptic problems in unbounded domains," *Calculus of Variations and Partial Differential Equations*, vol. 4, no. 2, pp. 121–137, 1996.

[9] M. A. Del Pino and P. L. Felmer, "Least energy solutions for elliptic equations in unbounded domains," *Proceedings of the Royal Society of Edinburgh. Section A*, vol. 126, no. 1, pp. 195–208, 1996.

[10] M. K. Kwong, "Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^N$," *Archive for Rational Mechanics and Analysis*, vol. 105, no. 3, pp. 243–266, 1989.

[11] A. Bahri and P.-L. Lions, "On the existence of a positive solution of semilinear elliptic equations in unbounded domains," *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire*, vol. 14, no. 3, pp. 365–413, 1997.

[12] V. Benci and G. Cerami, "Positive solutions of some nonlinear elliptic problems in exterior domains," *Archive for Rational Mechanics and Analysis*, vol. 99, no. 4, pp. 283–300, 1987.

[13] M. J. Esteban and P.-L. Lions, "Existence and nonexistence results for semilinear elliptic problems in unbounded domains," *Proceedings of the Royal Society of Edinburgh. Section A*, vol. 93, no. 1-2, pp. 1–14, 1982-1983.

[14] D.-M. Cao, "Positive solution and bifurcation from the essential spectrum of a semilinear elliptic equation on $\mathbb{R}^N$," *Nonlinear Analysis*, vol. 15, no. 11, pp. 1045–1052, 1990.

[15] A. Bahri and Y. Y. Li, "On a min-max procedure for the existence of a positive solution for certain scalar field equations in $\mathbb{R}^N$," *Revista Matemática Iberoamericana*, vol. 6, no. 1-2, pp. 1–15, 1990.

[16] T.-F. Wu, "Multiplicity of single-bump solutions for semilinear elliptic equations in multi-bump domains," *Nonlinear Analysis*, vol. 59, no. 6, pp. 973–992, 2004.

[17] X. P. Zhu, "A perturbation result on positive entire solutions of a semilinear elliptic equation," *Journal of Differential Equations*, vol. 92, no. 2, pp. 163–178, 1991.

[18] D.-M. Cao and H.-S. Zhou, "Multiple positive solutions of nonhomogeneous semilinear elliptic equations in $\mathbb{R}^N$," *Proceedings of the Royal Society of Edinburgh. Section A*, vol. 126, no. 2, pp. 443–463, 1996.

[19] L. Jeanjean, "Two positive solutions for a class of nonhomogeneous elliptic equations," *Differential and Integral Equations*, vol. 10, no. 4, pp. 609–624, 1997.

[20] S. Adachi and K. Tanaka, "Multiple positive solutions for nonhomogeneous elliptic equations," *Nonlinear Analysis*, vol. 47, no. 6, pp. 3783–3793, 2001.

[21] S. Adachi and K. Tanaka, "Four positive solutions for the semilinear elliptic equation: $-\Delta u + u = a(x)u^p + f(x)$ in $\mathbb{R}^N$," *Calculus of Variations and Partial Differential Equations*, vol. 11, no. 1, pp. 63–95, 2000.

[22] H.-L. Lin, H.-C. Wang, and T.-F. Wu, "A Palais-Smale approach to Sobolev subcritical operators," *Topological Methods in Nonlinear Analysis*, vol. 20, no. 2, pp. 393–407, 2002.

[23] G. Tarantello, "On nonhomogeneous elliptic equations involving critical Sobolev exponent," *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire*, vol. 9, no. 3, pp. 281–304, 1992.

[24] H. Brézis and E. Lieb, "A relation between pointwise convergence of functions and convergence of functionals," *Proceedings of the American Mathematical Society*, vol. 88, no. 3, pp. 486–490, 1983.

[25] I. Ekeland, "On the variational principle," *Journal of Mathematical Analysis and Applications*, vol. 47, no. 2, pp. 324–353, 1974.

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