Research Article

On Bloch-Type Functions with Hadamard Gaps

Stevo Stević

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We give some sufficient and necessary conditions for an analytic function \( f \) on the unit ball \( B \) with Hadamard gaps, that is, for \( f(z) = \sum_{k=1}^{\infty} P_k(z) \) (the homogeneous polynomial expansion of \( f \)) satisfying \( n_{k+1}/n_k \geq \lambda > 1 \) for all \( k \in \mathbb{N} \), to belong to the space \( \mathcal{B}_p(B) = \{ f \mid \sup_{0<r<1} (1-r^2)^a \| \mathcal{R} f_r \|_p < \infty, f \in H(B) \} \), \( p = 1, 2, \infty \) as well as to the corresponding little space. A remark on analytic functions with Hadamard gaps on mixed norm space on the unit disk is also given.

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1. Introduction

Let \( B = \{ z \in \mathbb{C}^n : |z| < 1 \} \) be the open unit ball of \( \mathbb{C}^n \), \( \partial B = \{ z \in \mathbb{C}^n : |z| = 1 \} \) its boundary, \( \mathbb{D} \) the unit disk in \( \mathbb{C} \), \( dv \) the normalized Lebesgue measure of \( B \) (i.e., \( v(B) = 1 \)), and \( d\sigma \) the normalized rotation invariant Lebesgue measure of \( S \) satisfying \( \sigma(\partial B) = 1 \). We denote the class of all holomorphic functions on the unit ball by \( H(B) \).

For \( f \in H(B) \) with the Taylor expansion \( f(z) = \sum_{|\beta|=0}^{\infty} a_\beta z^\beta \), let \( \mathcal{R} f(z) = \sum_{|\beta|=0}^{\infty} |\beta| a_\beta z^\beta \) be the radial derivative of \( f \), where \( \beta = (\beta_1, \beta_2, \ldots, \beta_n) \) is a multi-index and \( z^\beta = z_1^{\beta_1} \cdots z_n^{\beta_n} \).

It is well known that \( \mathcal{R} f(z) = \sum_{j=1}^{n} z_j (\partial f/\partial z_j)(z) = \sum_{k=0}^{\infty} kP_k(z) \), if \( f(z) = \sum_{k=0}^{\infty} P_k(z) \).

As usual, we write

\[
\| f_r \|_p = \left( \int_0^1 |f(r\zeta)|^p d\sigma(\zeta) \right)^{1/p}
\]

(1.1)

if \( p \in (0, \infty) \), and where \( f_r(\zeta) = f(r\zeta) \). If \( p = \infty \), then \( \| f \|_\infty = \sup_{z \in B} |f(z)| \).
2 Abstract and Applied Analysis

Let $\alpha > 0$. The $\alpha$-Bloch space $B^\alpha = B^\alpha(B)$ is the space of all holomorphic functions $f$ on $B$ such that

$$b_\alpha(f) = \sup_{z \in B} (1 - |z|^2)^\alpha |f(z)| < \infty.$$  \hspace{1cm} (1.2)

It is clear that $B^\alpha$ is a normed space under the norm $\|f\|_{\alpha} = |f(0)| + b_\alpha(f)$, and $B^\alpha_0 \subset B^\alpha$ for $\alpha_1 < \alpha_2$. Let $B^\alpha_0$ denote the subspace of $B^\alpha$ consisting of those $f \in B^\alpha$ for which $(1 - |z|^2)^\alpha |f(z)| \to 0$ as $|z| \to 1$. This space is called the little $\alpha$-Bloch space. For $\alpha = 1$, the $\alpha$-Bloch space and the little $\alpha$-Bloch space become Bloch space $B$ and the little Bloch space $B_0$. Some characterizations of these spaces can be found, for example, in the following papers [1–6].

We say that an analytic function $f$ on the unit disk $\mathbb{D}$ has Hadamard gaps if $f(z) = \sum_{k=1}^\infty a_k z^k n_k$ where $n_k + 1/n_k \geq \lambda > 1$, for all $k \in \mathbb{N}$.

In [7], Yamashita proved the following result.

**Theorem 1.1.** Assume that $f$ is an analytic function on $\mathbb{D}$ with Hadamard gaps. Then for $\alpha > 0$, the following two propositions hold:

(a) $f \in B^\alpha(\mathbb{D})$ if and only if $\limsup_{k \to \infty} |a_k n_k^{1-\alpha} < \infty$;

(b) $f \in B^\alpha_0(\mathbb{D})$ if and only if $\lim_{k \to \infty} |a_k n_k^{1-\alpha} = 0$.

An analytic function on $B$ with the homogeneous expansion $f(z) = \sum_{k=1}^\infty P_n(z)$ (here, $P_n$ is a homogeneous polynomial of degree $n_k$) is said to have Hadamard gaps if $n_{k+1}/n_k \geq \lambda > 1$, for all $k \in \mathbb{N}$. In [8], among others, Choa generalizes the main result in [9], proving the following result.

**Theorem 1.2.** Assume that $p \in (0, \infty)$ and $f(z) = \sum_{k=1}^\infty P_n(z)$ is an analytic function on $B$ with Hadamard gaps. Then the following statements are equivalent:

(a) $\|f\|_{X_p} = (\int_B |f(z)|^p (1 - |z|^2)^{p-1} dv(z))^{1/p} < \infty$;

(b) $\sum_{k=1}^\infty \|P_n\|_p < \infty$.

This result motivates us to find some characterizations for certain function spaces of analytic functions on the unit ball, in terms of the sequence $\left(\|P_n\|_p\right)_{k \in \mathbb{N}}$.

Now note that the quantity $b_\alpha$ in the definition of the $\alpha$-Bloch spaces can be written in the following form:

$$b_\alpha(f) = \sup_{0<r<1} (1 - r^2)^\alpha \sup_{\zeta \in S} |f(\zeta)| = \sup_{0<r<1} (1 - r^2)^\alpha M_\infty(\mathcal{R} f, r).$$ \hspace{1cm} (1.3)

On the other hand, the quantity $b_\alpha$ can be considered as the limit case of the following quantities:

$$\|f\|_{\mathcal{R}^\alpha_p} = \sup_{0<r<1} (1 - r^2)^\alpha \|\mathcal{R} f_r\|_p,$$ \hspace{1cm} (1.4)

as $p \to \infty$. Note that for every $f \in H(B)$ and $p \in (0, \infty)$,

$$\sup_{0<r<1} (1 - r^2)^\alpha \|\mathcal{R} f_r\|_p \leq \sup_{0<r<1} (1 - r^2)^\alpha \|\mathcal{R} f_r\|_\infty.$$ \hspace{1cm} (1.5)
Hence, in this paper we also consider analytic functions with Hadamard gaps on the following spaces:

\[
\mathcal{B}_p^a = \left\{ f \mid \sup_{0 < r < 1} (1 - r^2)^a \| Rf \|_p < \infty, f \in H(B) \right\},
\]

\[
\mathcal{B}_{p,0}^a = \left\{ f \mid \lim_{r \to 1} (1 - r^2)^a \| Rf \|_p = 0, f \in H(B) \right\}.
\] (1.6)

Motivated by Theorem 1.1 in this paper, we study analytic functions with Hadamard gaps, which belong to \( \mathcal{B}_p^a \) or \( \mathcal{B}_{p,0}^a \) space when \( p = 1, 2, \infty \). Some characterizations for these classes of functions on the unit ball are given in terms of the sequence \( (\| P_{n_k} \|_p)_{k \in \mathbb{N}} \).

The following are the main results.

**Theorem 1.3.** Assume that \( \alpha > 0, p = 1, 2, \infty, \) and \( f(z) = \sum_{k=1}^{\infty} P_{n_k}(z) \) is an analytic function on \( B \) with Hadamard gaps. Then the following statements are equivalent:

(a) \( f \in \mathcal{B}_p^a \);

(b) \( \limsup_{k \to \infty} \| P_{n_k} \|_p n_k^{1-a} < \infty. \)

**Theorem 1.4.** Assume that \( \alpha > 0, p = 1, 2, \infty, \) and \( f(z) = \sum_{k=1}^{\infty} P_{n_k}(z) \) is an analytic function on \( B \) with Hadamard gaps. Then the following statements are equivalent:

(a) \( f \in \mathcal{B}_{p,0}^a \);

(b) \( \lim_{k \to \infty} \| P_{n_k} \|_p n_k^{1-a} = 0. \)

Throughout this paper, constants are denoted by \( C \), they are positive and may differ from one occurrence to the other. The notation \( A \asymp B \) means that there is a positive constant \( C \) such that \( B/C \leq A \leq CB \).

### 2. Proof of main results

Before proving the main results of this paper we quote two auxiliary results which are incorporated in the lemmas which follow (see [9, 10]).

**Lemma 2.1.** Assume that \( p \in (0, \infty) \). If \( (n_k) \) is an increasing sequence of positive integers satisfying \( n_{k+1}/n_k \geq \lambda > 1 \), for all \( k \), then there is a positive constant \( A \) depending only on \( p \) and \( \lambda \) such that

\[
\frac{1}{A} \left( \sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2} \leq \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=1}^{\infty} a_k e^{in\theta} \right|^p d\theta \right)^{1/p} \leq A \left( \sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2} \] (2.1)

for any number \( a_k, k \in \mathbb{N} \).

**Lemma 2.2.** Assume that \( \alpha > 0, p > 0, n \in \mathbb{N}_0, (a_n)_{n \in \mathbb{N}_0} \) is the sequence of nonnegative numbers, \( I_n = \{ k \mid 2^n \leq k < 2^{n+1}, k \in \mathbb{N} \}, t_n = \sum_{k \in I_n} a_k, \) and \( g(x) = \sum_{n=1}^{\infty} a_n x^n \). Then there is a positive constant \( K \) depending only on \( p \) and \( \alpha \) such that

\[
\frac{1}{K} \sum_{n=0}^{\infty} \frac{t_n^p}{2^{na}} \leq \int_0^1 (1 - x)^{\alpha - 1} g^p(x) dx \leq K \sum_{n=0}^{\infty} \frac{t_n^p}{2^{na}}. \] (2.2)
Proof of Theorem 1.3. (a)⇒(b) (Case p = 1). Let \( f \in \mathbb{H}_1^+ \). Let \( f(\zeta w) = f(w) \), \( \zeta \in \mathbb{S} \), where \( \zeta \) is fixed and \( w \in \mathbb{D} \), be a slice function. By some calculation we see that

\[
f'_{\zeta}(w) = \zeta \frac{\partial f}{\partial z_1}(w\zeta) + \cdots + \zeta_n \frac{\partial f}{\partial z_n}(w\zeta) = \frac{1}{w} \mathcal{R} f(w\zeta).
\] (2.3)

From (2.3) and since \( f'_{\zeta}(w) = \sum_{k=1}^{\infty} n_k P_{n_k}(\zeta) w^{n_k-1} \), we have that

\[
\int_{\mathbb{S}} n_k |P_{n_k}(\zeta)| d\sigma(\zeta) = \int_{\mathbb{S}} \left| \frac{1}{2\pi i} \int_{\partial D} \frac{\eta f'_{\zeta}(\eta)}{\eta^{n_k+1}} d\eta \right| d\sigma(\zeta)
\]

\[
\leq \frac{1}{2\pi} \int_{\partial D} \int_{\mathbb{S}} \left| \frac{\mathcal{R} f(\zeta \eta)}{\eta^{n_k+1}} \right| d\sigma(\zeta) |d\eta|
\]

\[
\leq \frac{\|f\|_{\mathbb{A}^+_1}}{(1-r)^{\alpha r^{n_k}}},
\]

which implies that

\[
n_k r^{n_k} \|P_{n_k}\|_1 \leq \frac{\|f\|_{\mathbb{A}^+_1}}{(1-r)^{\alpha}},
\] (2.5)

for every \( k \in \mathbb{N} \) and \( r \in (0,1) \). Choosing \( r = 1 - (1/n_k) \), we obtain \( n_k^{1-\alpha} \|P_{n_k}\|_1 \leq C \), as desired.

(b)⇒(a) (Case p = 1). Assume \( \limsup_{k \to \infty} \|P_{n_k}\|_1 n_k^{1-\alpha} < \infty \). We have that

\[
\|f\|_{\mathbb{A}^+_1} = \sup_{0 < r < 1} (1-r^2)^{\alpha} \int_{\mathbb{S}} |\mathcal{R} f(r\zeta)| d\sigma(\zeta)
\]

\[
= \sup_{0 < r < 1} (1-r^2)^{\alpha} \int_{\mathbb{S}} \left| \sum_{k=1}^{\infty} n_k P_{n_k}(\zeta) r^{n_k} \right| d\sigma(\zeta)
\]

\[
\leq \sup_{0 < r < 1} (1-r^2)^{\alpha} \sum_{k=1}^{\infty} n_k \|P_{n_k}\|_1 r^{n_k}
\]

\[
\leq \sup_{0 < r < 1} (1-r^2)^{\alpha+1} \sum_{n=1}^{\infty} \left( \sum_{n_k \leq n} n_k \right) r^n
\]

\[
\leq C \sup_{0 < r < 1} (1-r^2)^{\alpha+1} \sum_{n=1}^{\infty} n^n r^n \leq C,
\]

(2.6)

where we have used the fact that there is a positive constant \( C \) independent of \( n \) such that \( \sum_{n_k \leq n} n_k^{\alpha} \leq C n^{\alpha} \) (here is used the assumption that \( n_{k+1}/n_k \geq \lambda > 1 \)) and the following well-known estimate:

\[
\sum_{n=1}^{\infty} n^n r^n \leq C(1-r)^{-(\alpha+1)},
\] (2.7)

\( \alpha > 0, r \in [0,1) \); see, for example, [11].
Case \( p = 2 \). Since
\[
\| f \|_{B^2_p} = \sup_{0 < r < 1} (1 - r^2)^2 \left( \sum_{k=1}^{\infty} n_k^2 ||P_{n_k}||_2^2 r^{2n_k} \right)^{1/2}
\] (2.8)
we have that
\[
\sup_{0 < r < 1} (1 - r^2)^2 n_k ||P_{n_k}||_2 r^{n_k} \leq \| f \|_{B^2_p} \leq \sup_{0 < r < 1} (1 - r^2)^2 \sum_{k=1}^{\infty} n_k ||P_{n_k}||_2 r^{n_k},
\] (2.9)
from which the result follows similar to the case \( p = 1 \).

Now we show that \((a) \Leftrightarrow (b)\) for case \( p = \infty \). As above, the function \( f_\zeta(w) = \sum_{k=1}^{\infty} P_{n_k}(\zeta) w^{n_k} \), where \( w = r e^{i\theta} \), is a lacunary series in \( D \) and
\[
(1 - r^2)^a R f(r\zeta) = r e^{i\theta} (1 - r^2)^a f_\zeta e^{-i\theta}(r e^{i\theta}),
\] (2.10)
from which by Theorem 1.1 the equivalence follows. \( \square \)

Proof of Theorem 1.4. \((a) \Rightarrow (b)\) (Case \( p = 1 \)). Let \( f \in \mathcal{B}_1^a \), then for every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that
\[
(1 - r^2)^a \left( \int S |R f(r\zeta)| d\sigma(\zeta) < \varepsilon, \right.
\] (2.11)
whenever \( \delta < r < 1 \). From (2.4), (2.11), and rotational invariance of \( d\sigma \), we have that
\[
\int_S n_k |P_{n_k}(\zeta)| d\sigma(\zeta) \leq \frac{1}{2\pi} \int_{\partial D} \int_S \left| \frac{R f(\zeta\eta)}{|\eta^{n_{k+1}}|} \right| d\sigma(\zeta) |d\eta|
\leq \frac{1}{2\pi} \int_{\partial D} \int_S \frac{(1 - r^2)^a |R f(\zeta\eta)|}{(1 - r^2)^a r^{n_{k+1}}} d\sigma(\zeta) |d\eta|
\leq \frac{\varepsilon}{(1 - r)^a r^{n_k}},
\] (2.12)
which implies that
\[
n_k r^{n_k} ||P_{n_k}||_1 \leq \frac{\varepsilon}{(1 - r)^a}
\] (2.13)
for every \( k \in \mathbb{N} \) and \( r \in (\delta, 1) \). Choosing \( r = 1 - (1/n_k) \), we obtain
\[
n_k ||P_{n_k}||_1 \leq C\varepsilon n_k^a,
\] (2.14)
from which \((b) \Rightarrow (a)\) follows in this case.

\((b) \Rightarrow (a)\) (Case \( p = 1 \)). Assume that \( \lim_{k \to \infty} ||P_{n_k}||_1 n_k^{1-a} = 0 \), then for every \( \varepsilon > 0 \) there is a \( k_0 \in \mathbb{N} \) such that
\[
||P_{n_k}||_1 \leq \varepsilon n_k^{a-1}, \quad \text{for } k \geq k_0.
\] (2.15)
We may assume that $k_0 = 1$. From this and by the proof of Theorem 1.3, $(b) \Rightarrow (a)$ (Case $p = 1$), we have that

\[
(1 - r^2)^\alpha \| \mathcal{R} f_r \|_1 \leq \sup_{0 < r < 1} (1 - r^2)^\alpha + 1 \sum_{n=1}^{\infty} \left( \sum_{k \in \mathbb{N}} n_k \| P_{nk} \|_1 \right) r^n
\]

\[
\leq C \varepsilon \sup_{0 < r < 1} (1 - r^2)^\alpha + 1 \sum_{n=1}^{\infty} n_k^2 r^n \tag{2.16}
\]

\[
\leq C \varepsilon \sup_{0 < r < 1} (1 - r^2)^\alpha + 1 \sum_{n=1}^{\infty} n^\alpha r^n \leq C \varepsilon,
\]

from which the implication follows.

**Case $p = 2$.** By using (2.9) the result follows similar to the Case $p = 1$. The proof is omitted.

Finally, in view of (2.10) and employing Theorem 1.1 (b) it is easy to see that $(a) \Leftrightarrow (b)$ for case $p = \infty$. □

### 3. The case of mixed norm space

In this section, we give a note concerning analytic functions with Hadamard gaps on the mixed norm space. The mixed norm space $H_{p,q,\alpha}(B)$, $p, q > 0$, and $\alpha \in (-1, \infty)$, consists of all $f \in H(B)$ such that

\[
\| f \|_{p,q,\alpha} = \left( \int_0^{1/2} \left| f(r\zeta) \right|^q (1 - r)^{\alpha} dr \right)^{1/q} < \infty. \tag{3.1}
\]

From [12, Theorem 4] the following result holds.

**Theorem 3.1.** Assume that $p \in (0, \infty)$, $\alpha > -1$ and $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ is an analytic function on $\mathbb{D}$ with Hadamard gaps. Then $f^{(m)} \in H_{p,q,\alpha}(\mathbb{D})$ if and only if $\sum_{k=0}^{\infty} n_k^{q m - \alpha - 1} |a_k|^q < \infty$.

**Proof.** First we consider the case $m = 0$. Similar to the proof of [12, Theorem 4] and by Lemmas 2.1 and 2.2, we have that

\[
\| f \|^q_{H_{p,q,\alpha}} = \int_0^1 \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=1}^{\infty} a_k r^{n_k} e^{in_k \theta} \right|^p d\theta \right)^{q/p} (1 - r)^{\alpha} dr
\]

\[
\approx \int_0^1 \left( \sum_{k=1}^{\infty} |a_k|^2 r^{2n_k} \right)^{q/2} (1 - r)^{\alpha} dr
\]

\[
\approx \int_0^1 \left( \sum_{k=1}^{\infty} |a_k|^2 \rho^{n_k} \right)^{q/2} (1 - \rho)^{\alpha} d\rho
\]

\[
\approx \sum_{k=0}^{\infty} \frac{1}{2^{(\alpha+1)k}} \left( \sum_{m \in I_k} |a_m|^2 \right)^{q/2} \sum_{k=0}^{\infty} \frac{|a_k|^q}{n_k^{\alpha+1}},
\]

from which the result follows in this case.
Since $f$ has Hadamard gaps and $f^{(m)}(z) = \sum_{k=1}^{\infty} a_k(n_k - 1) \cdots (n_k - m + 1)z^{n_k-m}$, it follows that $f^{(m)}$ has Hadamard gaps too. Applying the just proved result to the function $f^{(m)}$, we obtain that $f^{(m)} \in H_{p,q,\alpha}(\mathbb{D})$ if and only if
\[
\sum_{k=0}^{\infty} \frac{|n_k(n_k - 1) \cdots (n_k - m + 1) a_k|^q}{n_k^{\alpha+1-mq}} < \infty,
\] finishing the proof. \qed

Remark 3.2. Motivated by [12, Theorems 3 and 4], we can conjecture that if $p \in (0, \infty)$, $\alpha > -1$, and $f(z) = \sum_{k=1}^{\infty} P_{n_k}(z)$ is an analytic function on $B$ with Hadamard gaps, then $\mathcal{R}^{(m)} f \in H_{p,q,\alpha}(B)$ if and only if $\sum_{k=0}^{\infty} n_k^{q-\alpha-1} \|P_{n_k}\|^q < \infty$. Note that the result is true for the case of the weighted Bergman space, that is, when $p = q$, see [12, Corollary 1]. It is also expected that Theorems 1.3 and 1.4 hold for every $p \in [1; \infty]$ (for the case $n = 1$, see [13]).

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References

Abstract and Applied Analysis


Stevo Stević: Mathematical Institute of the Serbian Academy of Science, Knez Mihailova 35/I, 11000 Beograd, Serbia

Email addresses: sstevic@ptt.yu; sstevo@matf.bg.ac.yu
Submit your manuscripts at http://www.hindawi.com