Research Article

Multiple Positive Solutions of Nonhomogeneous Elliptic Equations in Unbounded Domains

Tsing-San Hsu

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We will show that under suitable conditions on f and h , there exists a positive number λ^* such that the nonhomogeneous elliptic equation $-\Delta u + u = \lambda(f(x, u) + h(x))$ in Ω , $u \in H_0^1(\Omega)$, $N \geq 2$, has at least two positive solutions if $\lambda \in (0, \lambda^*)$, a unique positive solution if $\lambda = \lambda^*$, and no positive solution if $\lambda > \lambda^*$, where Ω is the entire space or an exterior domain or an unbounded cylinder domain or the complement in a strip domain of a bounded domain. We also obtain some properties of the set of solutions.

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1. Introduction

Let $2^* = 2N/(N - 2)$ for $N \geq 3$, $2^* = \infty$ for $N = 2$. In this paper, we study the existence, nonexistence, and multiplicity of solutions of the equation

$$-\Delta u + u = \lambda(f(x, u) + h(x)) \text{ in } \Omega, \quad u \in H_0^1(\Omega), \quad u > 0 \text{ in } \Omega, \quad N \geq 2, \quad (1.1)_\lambda$$

where $\lambda > 0$, $N = m + n \geq 2$, $n \geq 1$, $0 \in \omega \subseteq \mathbb{R}^m$ is a smooth bounded domain, $\mathbb{S} = \omega \times \mathbb{R}^n$, D is a smooth bounded domain in $\mathbb{R}^N$ such that $D \subset \subset \mathbb{S}$, $\Omega = \mathbb{S} \setminus \overline{D}$ is the exterior of this domain in the strip.

Associated to $(1.1)_\lambda$, we consider the functional I , for $u \in H_0^1(\Omega)$,

$$I(u) = \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + u^2) dx - \lambda \int_{\Omega} F(x, u^+) dx - \lambda \int_{\Omega} h(x)u dx, \quad (1.1)$$

where $F(x, t) = \int_0^t f(x, s) ds$.

2 Abstract and Applied Analysis

It is assumed that $h(x) \in L^2(\Omega) \cap L^{q_0}(\Omega)$ for some $q_0 > N/2$ if $N \geq 4$, $q_0 = 2$ if $N = 2, 3$, $h(x) \geq 0$, $h(x) \not\equiv 0$, and $f(x, t)$ satisfies the following conditions:

(f1) $f(x, \cdot) \in C^1([0, +\infty), \mathbb{R}^+)$, $f(x, t) \equiv 0$ for $x \in \mathbb{S}$, $t \leq 0$, and $\lim_{t \rightarrow 0} (f(x, t)/t) = 0$ uniformly for $x \in \mathbb{S}$;

(f2) there exists a positive constant C such that for all $x \in \mathbb{S}$ and $t \in \mathbb{R}$,

$$0 < \frac{\partial}{\partial t} f(x, t) \leq C(1 + |t|^{p-2}), \quad (1.2)$$

where $2 < p < 2^*$;

(f3) there exists a number $\theta \in [1/p, 1)$ such that

$$\theta t \frac{\partial}{\partial t} f(x, t) \geq f(x, t) > 0 \quad \forall x \in \mathbb{S}, t > 0; \quad (1.3)$$

(f4) there exists $\bar{f} : \mathbb{R} \rightarrow \mathbb{R}$ such that $\lim_{|x| \rightarrow \infty} f(x, t) = \bar{f}(t)$ uniformly for bounded $t > 0$, $f(x, t) \geq \bar{f}(t)$, for all $x \in \mathbb{S}$, $t \geq 0$, and $\lim_{t \rightarrow \infty} (f(x, t)/t) = \infty$ uniformly for $x \in \mathbb{S}$;

(f5) $f(x, \cdot) \in C^2(0, +\infty)$ and $(\partial^2/\partial t^2)f(x, t) \geq 0$ for all $x \in \mathbb{S}$, $t \geq 0$.

Given $\varepsilon > 0$, by (f1) and (f2), there exists a $C_\varepsilon > 0$ such that

$$0 \leq f(x, u) \leq \varepsilon u + C_\varepsilon |u|^{p-1}, \quad (1.4)$$

$$0 \leq F(x, u) \leq \varepsilon u^2 + C_\varepsilon |u|^p. \quad (1.5)$$

If $\Omega = \mathbb{R}^N$ or $\Omega = \mathbb{R}^N \setminus \bar{D}$ ($m = 0$ in our case), then the homogeneous case of problem $(1.1)_\lambda$ (i.e., the case $h(x) \equiv 0$) has been studied by many authors; see Cao [1] and the references therein. For the nonhomogeneous case ($h(x) \not\equiv 0$), Zhu-Zhou [2] have studied the multiplicity of positive solutions of equations similar to $(1.1)_\lambda$. Recently, Chen [3] showed that there exists a $\lambda^* > 0$ such that $(1.1)_\lambda$ has exactly two positive solutions if $\lambda \in (0, \lambda^*)$, and $(1.1)_\lambda$ has no positive solution when $\lambda \in (\lambda^*, \infty)$. However, her method cannot determine whether λ^* is bounded or infinite (at least for general nonlinearity $f(x, u)$). In this paper, one of our results answers the question (see Theorem 1.1). Now, we state our main results.

THEOREM 1.1. *Let $\Omega = \mathbb{S} \setminus \bar{D}$ or $\Omega = \mathbb{R}^N \setminus \bar{D}$ or $\Omega = \mathbb{S}$ or $\Omega = \mathbb{R}^N$. Suppose $h(x) \geq 0$, $h(x) \not\equiv 0$, $h(x) \in L^2(\Omega) \cap L^{q_0}(\Omega)$ for some $q_0 > N/2$ if $N \geq 4$, $q_0 = 2$ if $N = 2, 3$, and $f(x, t)$ satisfies (f1)–(f5). Then there exists $\lambda^* > 0$, $0 < \lambda^* < \infty$, such that*

(i) *equation $(1.1)_\lambda$ has at least two positive solutions u_λ , U_λ and $u_\lambda < U_\lambda$ if $\lambda \in (0, \lambda^*)$;*

(ii) *equation $(1.1)_{\lambda^*}$ has a unique positive solution u_{λ^*} ;*

(iii) *equation $(1.1)_\lambda$ has no positive solutions if $\lambda > \lambda^*$,*

where u_λ is the minimal solution of $(1.1)_\lambda$ and U_λ is the second solution of $(1.1)_\lambda$ constructed in Section 4.

THEOREM 1.2. *Under the assumptions of Theorem 1.1, then*

(i) *u_λ is strictly increasing with respect to λ , u_λ is uniformly bounded in $L^\infty(\Omega) \cap H_0^1(\Omega)$ for all $\lambda \in (0, \lambda^*]$ and*

$$u_\lambda \longrightarrow 0 \quad \text{in } L^\infty(\Omega) \cap H_0^1(\Omega) \text{ as } \lambda \longrightarrow 0^+; \quad (1.6)$$

(ii) U_λ is unbounded in $L^\infty(\Omega) \cap H_0^1(\Omega)$ for $\lambda \in (0, \lambda^*)$, that is,

$$\lim_{\lambda \rightarrow 0^+} \|U_\lambda\| = \lim_{\lambda \rightarrow 0^+} \|U_\lambda\|_\infty = \infty, \quad (1.7)$$

where $\|U_\lambda\| = (\int_\Omega (|\nabla U|^2 + U^2) dx)^{1/2}$ and $\|U_\lambda\|_\infty = \sup_{x \in \Omega} |U(x)|$.

First of all, we list some properties of $f(x, t)$. The proof can be found in Zhu-Zhou [2, Lemma 2.1].

LEMMA 1.3. Assume (f1), (f3), and (f5) hold, then

- (i) $tf(x, t) \geq \nu F(x, t)$ for all $x \in \mathbb{S}$, $t > 0$ and $\nu = 1 + \theta^{-1} \in (2, p + 1]$;
- (ii) $t^{-1/\theta} f(x, t)$ is monotone nondecreasing and $t^{-1} f(x, t)$ is strictly monotone increasing for all $x \in \mathbb{S}$, $t > 0$;
- (iii) $f(x, t_1 + t_2) \geq f(x, t_1) + f(x, t_2)$ and $f(x, t_1 + t_2) \neq f(x, t_1) + f(x, t_2)$ for all $x \in \mathbb{S}$, $t_1, t_2 > 0$.

2. Asymptotic behavior of solutions

Throughout this paper, let $x = (y, z)$ be the generic point of $\mathbb{R}^N$ with $y \in \mathbb{R}^m$, $z \in \mathbb{R}^n$, $N = m + n \geq 2$, $n \geq 1$. We denote by C and C_i ($i = 1, 2, \dots$) universal constants, maybe the constants here should be allowed to depend on n and p , unless some statement is given, and denote $(\partial/\partial t)f(x, t)$ and $(\partial^2/\partial t^2)f(x, t)$ by $f'(x, t)$ and $f''(x, t)$, respectively, in what follows.

We define

$$\begin{aligned} \|u\| &= \left(\int_\Omega (|\nabla u|^2 + u^2) dx \right)^{1/2}, \\ \|u\|_p &= \left(\int_\Omega |u|^p dx \right)^{1/p}, \quad 2 \leq p < \infty, \\ \|u\|_\infty &= \sup_{x \in \Omega} |u(x)|. \end{aligned} \quad (2.1)$$

Now, we introduce the equation at infinity associated with (1.1) $_\lambda$ on an unbounded cylinder domain $\mathbb{S}$,

$$\begin{aligned} -\Delta u + u &= \lambda \bar{f}(u) \quad \text{in } \mathbb{S}, \\ u &\in H_0^1(\mathbb{S}), \quad N \geq 2. \end{aligned} \quad (2.1)_\lambda$$

P. L. Lions has studied the following minimization problem closely related to (2.1) $_\lambda$:

$$S^\infty = \inf \{ I^\infty(u) : u \in H_0^1(\mathbb{S}), u \neq 0, I^{\infty'}(u) = 0 \} > 0, \quad (2.2)$$

where $I^\infty(u) = (1/2) \int_\mathbb{S} (|\nabla u|^2 + u^2) dx - \lambda \int_\mathbb{S} \bar{F}(u^+) dx$, $\bar{F}(t) = \int_0^t \bar{f}(s) ds$. For this problem, also a minimum exists and is realized by a ground state solution $w > 0$ in $\mathbb{S}$ such that

$$S^\infty = I^\infty(w) = \sup_{t \geq 0} I^\infty(tw). \quad (2.3)$$

4 Abstract and Applied Analysis

In order to get the asymptotic behavior of solutions of $(1.1)_\lambda$ and $(2.1)_\lambda$, we need the following Lemmas 2.3 and 2.5. First, we quote two regularity lemmas (see Hsu [4] for the proof). Now, let $\mathbb{X}$ be a $C^{1,1}$ domain in $\mathbb{R}^N$ (typically the domains considered in the introduction).

LEMMA 2.1. *Let $f : \mathbb{X} \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function such that for almost every $x \in \mathbb{X}$, there holds*

$$|f(x, u)| \leq C(|u| + |u|^{p-1}) \quad \text{uniformly in } x \in \mathbb{X}, \quad (2.4)$$

where $2 < p < 2^*$. If $u \in H_0^1(\mathbb{X})$ is a weak solution of equation $-\Delta u = f(x, u) + h(x)$ in $\mathbb{X}$, where $h \in L^{N/2}(\mathbb{X}) \cap L^2(\mathbb{X})$, then $u \in L^q(\mathbb{X})$ for $q \in [2, \infty)$.

LEMMA 2.2. *Let $g \in L^2(\mathbb{X}) \cap L^q(\mathbb{X})$ for some $q \in [2, \infty)$ and let $u \in H_0^1(\mathbb{X})$ be a weak solution of the equation $-\Delta u + u = g$ in $\mathbb{X}$. Then $u \in W^{2,q}(\mathbb{X})$ satisfies*

$$\|u\|_{W^{2,q}(\mathbb{X})} \leq C(\|u\|_{L^q(\mathbb{X})} + \|g\|_{L^q(\mathbb{X})}), \quad (2.5)$$

where $C = C(N, q, \partial\mathbb{X})$.

By Lemmas 2.1 and 2.2, we obtain the first asymptotic behavior of solution of $(1.1)_\lambda$.

LEMMA 2.3 (asymptotic lemma 1). *Let (f1), (f2) hold and let u be a weak solution of $(1.1)_\lambda$, then $u(y, z) \rightarrow 0$ as $|z| \rightarrow \infty$ uniformly for $y \in \omega$. Moreover, there exist positive constants C_1 and C_2 such that*

$$\|u\|_\infty \leq C_1 \|u\|_{q_0} + \lambda C_2 \left(\|u\|_{(p-1)q_0}^{p-1} + \|h\|_{q_0} \right). \quad (2.6)$$

Proof. Suppose that u is a solution of $(1.1)_\lambda$, then $-\Delta u + u = \lambda(f(x, u) + h(x))$ in Ω . Since $h \in L^2(\Omega) \cap L^{q_0}(\Omega)$ for some $q_0 > N/2$ if $N \geq 4$, $q_0 = 2$ if $N = 2, 3$, this implies $h \in L^2(\Omega) \cap L^{N/2}(\Omega)$ for $N \geq 2$. By (1.4) and Lemma 2.1, we conclude that

$$u \in L^q(\Omega) \quad \text{for } q \in [2, \infty). \quad (2.7)$$

Hence $\lambda(f(x, u) + h(x)) \in L^2(\Omega) \cap L^{q_0}(\Omega)$ and by Lemma 2.2, we have

$$u \in W^{2,2}(\Omega) \cap W^{2,q_0}(\Omega), \quad q_0 > N/2 \quad \text{if } N \geq 4, \quad q_0 = 2 \quad \text{if } N = 2, 3. \quad (2.8)$$

Now, by the Sobolev embedding theorem, we obtain that $u \in C_b(\overline{\Omega})$. It is well known that the Sobolev embedding constants are independent of domains (see Adams [5]). Thus there exists a constant C such that for $R > 0$,

$$\|u\|_{L^\infty(\Omega \setminus B_R)} \leq C \|u\|_{W^{2,q_0}(\Omega \setminus B_R)} \quad \text{for } N \geq 2, \quad (2.9)$$

where $B_R = \{x = (y, z) \in \Omega \mid |z| \leq R\}$. From this, we conclude that $u(y, z) \rightarrow 0$ as $|z| \rightarrow \infty$ uniformly for $y \in \omega$. By Lemma 2.2 and (1.4), we also have that

$$\begin{aligned} \|u\|_\infty &\leq C \|u\|_{W^{2,q_0}(\Omega)} \\ &\leq C \left(\|u\|_{q_0} + \|\lambda f(x, u) + \lambda h(x)\|_{q_0} \right) \\ &\leq C_1 \|u\|_{q_0} + \lambda C_2 \left(\|u\|_{(p-1)q_0}^{p-1} + \|h\|_{q_0} \right), \end{aligned} \quad (2.10)$$

where C_1, C_2 are constants independent of λ . $\square$

Remark 2.4. Let w be a positive solution of $(2.1)_\lambda$. If $h(x) \equiv 0$ and $f(x, t) \equiv \bar{f}(t)$ for all $x \in \mathbb{S}$, $t \in \mathbb{R}$, by Lemma 2.3, then we have that $w(y, z) \rightarrow 0$ as $|z| \rightarrow \infty$ uniformly for $y \in \omega$.

We use Lemma 2.3, and modify the proof in Hsu [6], we obtain a precise asymptotic behavior of solutions of $(2.1)_\lambda$ at infinity and the second asymptotic behavior of solutions of $(1.1)_\lambda$.

LEMMA 2.5 (asymptotic lemma 2). *Let w be a positive solution of $(2.1)_\lambda$, let u be a positive solution of $(1.1)_\lambda$ and let φ be the first positive eigenfunction of the Dirichlet problem $-\Delta\varphi = \mu_1\varphi$ in ω , then for any $\varepsilon > 0$ with $0 < \varepsilon < 1 + \mu_1$, there exist constants $C, C_\varepsilon > 0$ such that*

$$\begin{aligned} w(y, z) &\leq C_\varepsilon \varphi(y) \exp\left(-\sqrt{1 + \mu_1 - \varepsilon}|z|\right), \\ w(y, z) &\geq C \varphi(y) \exp\left(-\sqrt{1 + \mu_1}|z|\right) |z|^{-(n-1)/2} \quad \text{as } |z| \rightarrow \infty, y \in \bar{\omega}, \\ u(y, z) &\geq C \varphi(y) \exp\left(-\sqrt{1 + \mu_1}|z|\right) |z|^{-(n-1)/2}. \end{aligned} \quad (2.11)$$

Proof. (i) First, we claim that for any $\varepsilon > 0$ with $0 < \varepsilon < 1 + \mu_1$, there exists $C_\varepsilon > 0$ such that

$$w(y, z) \leq C_\varepsilon \varphi(y) \exp\left(-\sqrt{1 + \mu_1 - \varepsilon}|z|\right) \quad \text{as } |z| \rightarrow \infty, y \in \bar{\omega}. \quad (2.12)$$

Without loss of generality, we may assume $\varepsilon < 1$. Now given $\varepsilon > 0$, by (f1), (f4), and Remark 2.4, we may choose R_0 large enough such that

$$\lambda \bar{f}(w(y, z)) \leq \lambda f(x, w(y, z)) \leq \varepsilon w(y, z) \quad \text{for } |z| \geq R_0. \quad (2.13)$$

Let $q = (q_y, q_z)$, $q_y \in \partial\omega$, $|q_z| = R_0$, and B a small ball in Ω such that $q \in \partial B$. Since $\varphi(y) > 0$ for $x = (y, z) \in B$, $\varphi(q_y) = 0$, $w(x) > 0$ for $x \in B$, $w(q) = 0$, by the strong maximum principle $(\partial\varphi/\partial y)(q_y) < 0$, $(\partial w/\partial x)(q) < 0$. Thus

$$\lim_{\substack{x \rightarrow q \\ |z| = R_0}} \frac{w(x)}{\varphi(y)} = \frac{(\partial w/\partial x)(q)}{(\partial\varphi/\partial y)(q_y)} > 0. \quad (2.14)$$

Note that $w(x)\varphi^{-1}(y) > 0$ for $x = (y, z)$, $y \in \omega$, $|z| = R_0$. Thus $w(x)\varphi^{-1}(y) > 0$ for $x = (y, z)$, $y \in \bar{\omega}$, $|z| = R_0$. Since $\varphi(y) \exp(-\sqrt{1 + \mu_1 - \varepsilon}|z|)$ and $w(x)$ belong to $C^1(\bar{\omega} \times \partial B_{R_0}(0))$, if set

$$C_\varepsilon = \sup_{y \in \bar{\omega}, |z| = R_0} \left(w(x)\varphi^{-1}(y) \exp\left(\sqrt{1 + \mu_1 - \varepsilon}R_0\right) \right), \quad (2.15)$$

then $C_\varepsilon > 0$ and

$$C_\varepsilon \varphi(y) \exp\left(-\sqrt{1+\mu_1-\varepsilon}R_0\right) \geq w(x) \quad \text{for } y \in \bar{\omega}, |z| = R_0. \quad (2.16)$$

Let $\Phi_1(x) = C_\varepsilon \varphi(y) \exp(-\sqrt{1+\mu_1-\varepsilon}|z|)$ for $x \in \bar{\Omega}$. Then for $|z| \geq R_0$, we have

$$\begin{aligned} \Delta(w - \Phi_1)(x) - (w - \Phi_1)(x) &= -\lambda \bar{f}(w(x)) + \left(\varepsilon + \frac{\sqrt{1+\mu_1-\varepsilon}(n-1)}{|z|}\right) \Phi_1(x) \\ &\geq -\varepsilon w(x) + \varepsilon \Phi_1(x) \\ &= \varepsilon(\Phi_1 - w)(x). \end{aligned} \quad (2.17)$$

Hence $\Delta(w - \Phi_1)(x) - (1 - \varepsilon)(w - \Phi_1)(x) \geq 0$ for $|z| \geq R_0$.

The strong maximum principle implies that $w(x) - \Phi_1(x) \leq 0$ for $x = (y, z)$, $y \in \bar{\omega}$, $|z| \geq R_0$, and therefore we get this claim.

(ii) Let

$$\Psi(y, z) = \left(1 + \frac{1}{\sqrt{|z|}}\right) \varphi(y) \exp\left(-\sqrt{1+\mu_1}|z|\right) |z|^{-(n-1)/2} \quad \text{for } (y, z) \in \Omega. \quad (2.18)$$

It is very easy to show that

$$-\Delta\Psi + \Psi \leq 0 \quad \text{for } y \in \bar{\omega}, |z| \text{ large}. \quad (2.19)$$

Therefore, by means of the maximum principle, there exists a constant $C > 0$ such that

$$\begin{aligned} w(y, z) &\geq C\varphi(y) \exp\left(-\sqrt{1+\mu_1}|z|\right) |z|^{-(n-1)/2} \\ u(y, z) &\geq C\varphi(y) \exp\left(-\sqrt{1+\mu_1}|z|\right) |z|^{-(n-1)/2} \quad \text{as } |z| \rightarrow \infty, y \in \bar{\omega}. \end{aligned} \quad (2.20)$$

This completes the proof of Lemma 2.5. $\square$

3. Existence of the minimal solution

We now prove the existence of minimal positive solutions of $(1.1)_\lambda$.

LEMMA 3.1. *If (f1) and (f2) hold, then for any given $\rho > 0$, there exists $\lambda_0 > 0$ such that for $\lambda \in (0, \lambda_0)$, one has $I(u) > 0$ for all $u \in S_\rho = \{u \in H_0^1(\Omega) \mid \|u\| = \rho\}$. Moreover, for any $\varepsilon \geq 0$, there exists $\delta > 0$ ($\delta \leq \rho$) such that $I(u) \geq -\varepsilon$ for all $u \in \{u \in H_0^1(\Omega) \mid \rho - \delta \leq \|u\| = \rho\}$.*

Proof. By (1.5), the Sobolev embedding theorem, and the Hölder inequality, we have that, for all $u \in S_\rho$,

$$\begin{aligned} I(u) &= \frac{1}{2} \|u\|^2 - \lambda \int_{\Omega} F(x, u^+) dx - \lambda \int_{\Omega} h u dx \\ &\geq \frac{1}{2} \|u\|^2 - \lambda \int_{\Omega} (\varepsilon |u|^2 + C_\varepsilon |u|^p) dx - \lambda \|h\|_2 \|u\| \\ &\geq \frac{1}{2} \|u\|^2 - \lambda C (\|u\|^2 + \|u\|^p) - \lambda \|h\|_2 \|u\| \\ &\geq \rho \left(\frac{1}{2} \rho - \lambda C (\rho + \rho^{p-1}) - \lambda \|h\|_2 \right), \end{aligned} \quad (3.1)$$

where $C > 0$ is a constant which is independent of λ, ρ . Hence by (3.1), there exists $\lambda_0 > 0$ such that for $\lambda \in (0, \lambda_0)$, we have $I(u) > 0$ for all $u \in S_\rho$.

Moreover, we can choose $\lambda_0 > 0$ small enough such that

$$\frac{\partial}{\partial \rho} \left(\frac{1}{2} \rho - \lambda C (\rho + \rho^{p-1}) \right) = \frac{1}{2} - \lambda (1 + (p-1) \rho^{p-2}) > 0 \quad \text{for } \lambda \in (0, \lambda_0). \quad (3.2)$$

Then for any $\varepsilon \geq 0$, there exists $\delta > 0$ ($\delta \leq \rho$) such that $I(u) \geq -\varepsilon$ for all $u \in \{u \in H_0^1(\Omega) \mid \rho - \delta \leq \|u\| \leq \rho\}$. $\square$

LEMMA 3.2. Assume (f1) and (f2) hold. If λ_0 is chosen as in Lemma 3.1 and $\lambda \in (0, \lambda_0)$, then there exists a $u_0 \in B_\rho$ such that u_0 is a positive solution of $(1.1)_\lambda$.

Proof. Since $h \not\equiv 0$ and $h \geq 0$, we can choose a function $\varphi \in H_0^1(\Omega)$ such that $\int_\Omega h \varphi > 0$. For $t \in (0, +\infty)$, then by (1.5),

$$\begin{aligned} I(t\varphi) &= \frac{t^2}{2} \int_\Omega (|\nabla \varphi|^2 + \varphi^2) - \lambda \int_{\mathbb{R}^N_+} F(x, t\varphi^+) - \lambda t \int_\Omega h \varphi \\ &\leq \frac{t^2}{2} \|\varphi\|^2 + \lambda C t^2 \int_\Omega (|\varphi|^2 + t^{p-2} |\varphi|^p) - \lambda t \int_\Omega h \varphi. \end{aligned} \quad (3.3)$$

Then for t small enough, $I(t\varphi) < 0$. So $\alpha = \inf\{I(u) \mid u \in \overline{B_\rho}\}$. Clearly $\alpha > -\infty$. By Lemma 3.1, there exists ρ' such that $0 < \rho' < \rho$ and $\alpha = \inf\{I(u) \mid u \in \overline{B_{\rho'}}\}$. By Ekeland's variational principle [7], there exists a $(PS)_\alpha$ -sequence $\{u_k\} \subset \overline{B_{\rho'}}$, that is, $I(u_k) = \alpha + o(1)$ and $I'(u_k) = o(1)$ strongly in $H^{-1}(\Omega)$ as $k \rightarrow \infty$. Then there exists a subsequence $\{u_k\}$ and $u_0 \in H_0^1(\Omega)$ such that $u_k \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$, $u_k \rightarrow u_0$ strongly in $L_{\text{loc}}^q(\Omega)$ for $2 \leq q < 2^*$ and $u_k \rightarrow u_0$ a.e. in Ω . Since $I'(u_k) = o(1)$ strongly in $H^{-1}(\Omega)$ as $k \rightarrow \infty$, and by (f1) and (f2), we have $I'(u_0) = 0$ in $H^{-1}(\Omega)$, that is, u_0 is a weak nonnegative solution of $(1.1)_\lambda$; and since $h \not\equiv 0$, by the maximum principle for weak solutions, we have $u_0 > 0$ in Ω . $\square$

By the standard barrier method, we prove the following lemma.

LEMMA 3.3. If (f1) and (f2) hold, then there exists $\lambda^* \in (0, +\infty]$ such that

- (i) for any $\lambda \in (0, \lambda^*)$, $(1.1)_\lambda$ has a minimal positive solution u_λ and u_λ is strictly increasing in λ ;
- (ii) if $\lambda > \lambda^*$, $(1.1)_\lambda$ has no positive solution.

Proof. Setting $Q_\lambda = \{0 < \lambda < +\infty \mid (1.1)_\lambda \text{ is solvable}\}$, by Lemma 3.2, we have Q_λ is non-empty. Denoting $\lambda^* = \sup Q_\lambda > 0$, we claim that $(1.1)_\lambda$ has at least one solution for all $\lambda \in (0, \lambda^*)$. In fact, for any $\lambda \in (0, \lambda^*)$, by the definition of λ^* , we know that there exists $\lambda' > 0$ and $0 < \lambda < \lambda' < \lambda^*$ such that $(1.1)_{\lambda'}$ has a solution $u_{\lambda'} > 0$, that is,

$$-\Delta u_{\lambda'} + u_{\lambda'} = \lambda' (f(x, u_{\lambda'}) + h(x)) \geq \lambda (f(x, u_{\lambda'}) + h(x)). \quad (3.4)$$

Then $u_{\lambda'}$ is a supersolution of $(1.1)_\lambda$. From $h(x) \geq 0$ and $h(x) \not\equiv 0$, it is easy to see that 0 is a subsolution of $(1.1)_\lambda$. By the standard barrier method, there exists a solution $u_\lambda > 0$ of $(1.1)_\lambda$ such that $0 \leq u_\lambda \leq u_{\lambda'}$. Since 0 is not a solution of $(1.1)_\lambda$ and $\lambda' > \lambda$, the maximum

principle implies that $0 < u_\lambda < u_{\lambda'}$. Again using a result of Amann [8, Theorem 9.4], we can choose a minimal positive solution u_λ of $(1.1)_\lambda$. $\square$

Let u_λ be the minimal positive solution of $(1.1)_\lambda$ for $\lambda \in (0, \lambda^*)$, we study the following eigenvalue problem

$$\begin{aligned} -\Delta v + v &= \sigma_\lambda f'(x, u_\lambda) v \quad \text{in } \Omega, \\ v &\in H_0^1(\Omega), \quad v > 0 \text{ in } \Omega, \end{aligned} \quad (3.5)$$

then we have the following.

LEMMA 3.4. Assume $(f1)$ – $(f5)$ hold, and let the first eigenvalue σ_λ of (3.5) be defined by

$$\sigma_\lambda = \inf \left\{ \int_\Omega (|\nabla v|^2 + v^2) dx \mid v \in H_0^1(\Omega), \int_\Omega f'(x, u_\lambda) v^2 dx = 1 \right\}. \quad (3.6)$$

Then

- (i) σ_λ is achieved;
- (ii) $\sigma_\lambda > \lambda$ and is strictly decreasing in λ , $\lambda \in (0, \lambda^*)$;
- (iii) $\lambda^* < +\infty$ and $(1.1)_{\lambda^*}$ has a minimal positive solution u_{λ^*} .

Proof. (i) Indeed, recall assumption $(f3)$, by the definition of σ_λ , we know that $0 < \sigma_\lambda < +\infty$. Let $\{v_k\} \subset H_0^1(\Omega)$ be a minimizing sequence of σ_λ , that is,

$$\int_\Omega f'(x, u_\lambda) v_k^2 dx = 1, \quad \int_\Omega (|\nabla v_k|^2 + v_k^2) dx \longrightarrow \sigma_\lambda \quad \text{as } k \longrightarrow \infty. \quad (3.7)$$

This implies that $\{v_k\}$ is bounded in $H_0^1(\Omega)$, then there exists a subsequence, still denoted by $\{v_k\}$ and some $v_0 \in H_0^1(\Omega)$ such that

$$\begin{aligned} v_k &\rightharpoonup v_0 \quad \text{weakly in } H_0^1(\Omega), \\ v_k &\longrightarrow v_0 \quad \text{almost everywhere in } \Omega, \\ v_k &\longrightarrow v_0 \quad \text{strongly in } L_{\text{loc}}^s(\Omega) \quad \text{for } 2 \leq s < 2^*. \end{aligned} \quad (3.8)$$

Thus

$$\int_\Omega (|\nabla v_0|^2 + v_0^2) dx \leq \liminf \int_\Omega (|\nabla v_k|^2 + v_k^2) dx = \sigma_\lambda. \quad (3.9)$$

By Lemma 2.3 and $(f1)$, we have $f'(x, u_\lambda) \rightarrow 0$ as $|x| \rightarrow \infty$, it is standard to show that v_0 achieves σ_λ . Clearly $|v_0|$ also achieves σ_λ . By (3.5) and the maximum principle, we may assume $v_0 > 0$ in Ω .

(ii) We now prove $\sigma_\lambda > \lambda$. Setting $\lambda' > \lambda > 0$ and $\lambda' \in (0, \lambda^*)$, by Lemma 3.3, $(1.1)_{\lambda'}$ has a positive solution $u_{\lambda'}$. Since u_λ is the minimal positive solution of $(1.1)_\lambda$, then $u_{\lambda'} > u_\lambda$ as $\lambda' > \lambda$. By virtue of $(1.1)_{\lambda'}$ and $(1.1)_\lambda$, we see that

$$-\Delta(u_{\lambda'} - u_\lambda) + (u_{\lambda'} - u_\lambda) = \lambda' f(x, u_{\lambda'}) - \lambda f(x, u_\lambda) + (\lambda' - \lambda)h. \quad (3.10)$$

Applying the Taylor expansion and noting that $\lambda' > \lambda$, $h(x) \geq 0$, and $f''(x, t) \geq 0$, $f(x, t) > 0$ for all $t > 0$, we get

$$\begin{aligned} -\Delta(u_{\lambda'} - u_\lambda) + (u_{\lambda'} - u_\lambda) &\geq (\lambda' - \lambda)f(x, u_\lambda) + \lambda' f'(x, u_\lambda)(u_{\lambda'} - u_\lambda) \\ &> \lambda f'(x, u_\lambda)(u_{\lambda'} - u_\lambda). \end{aligned} \quad (3.11)$$

Let $v_0 \in H_0^1(\Omega)$ and $v_0 > 0$ solves (3.5). Multiplying (3.11) by v_0 and noting (3.5), then we get

$$\sigma_\lambda \int_\Omega f'(x, u_\lambda)(u_{\lambda'} - u_\lambda)v_0 dx > \lambda \int_\Omega f'(x, u_\lambda)(u_{\lambda'} - u_\lambda)v_0 dx, \quad (3.12)$$

hence $\sigma_\lambda > \lambda$. Now, let v_λ be a minimizer of σ_λ , then

$$\int_\Omega f'(x, u_{\lambda'})v_\lambda^2 dx > \int_\Omega f'(x, u_\lambda)v_\lambda^2 dx = 1, \quad (3.13)$$

and there exists t , with $0 < t < 1$ such that

$$\int_\Omega f'(x, u_{\lambda'})(tv_\lambda)^2 dx = 1. \quad (3.14)$$

Therefore

$$\sigma_{\lambda'} \leq t^2 \|v_\lambda\|^2 < \|v_\lambda\|^2 = \sigma_\lambda \quad (3.15)$$

showing that σ_λ is strictly decreasing in λ for $\lambda \in (0, \lambda^*)$.

(iii) We show next that $\lambda^* < +\infty$. Let $\lambda_0 \in (0, \lambda^*)$ be fixed. For any $\lambda \geq \lambda_0$, we have $\sigma_\lambda > \lambda$ and by (3.15), then

$$\sigma_{\lambda_0} \geq \sigma_\lambda > \lambda \quad (3.16)$$

for all $\lambda \in [\lambda_0, \lambda^*)$. Thus $\lambda^* < +\infty$.

By (3.5) and $\sigma_\lambda > \lambda$, we have

$$\int_\Omega (|\nabla u_\lambda|^2 + |u_\lambda|^2) dx > \int_\Omega \lambda f'(x, u_\lambda)u_\lambda^2 dx, \quad (3.17)$$

and also we have

$$\int_\Omega (|\nabla u_\lambda|^2 + |u_\lambda|^2) dx - \int_\Omega \lambda f(x, u_\lambda)u_\lambda dx - \int_\Omega \lambda h(x)u_\lambda dx = 0. \quad (3.18)$$

By (f3) and (3.17), we have that

$$\begin{aligned} \int_\Omega (|\nabla u_\lambda|^2 + |u_\lambda|^2) dx &= \int_\Omega \lambda f(x, u_\lambda)u_\lambda dx + \int_\Omega \lambda h(x)u_\lambda dx \\ &\leq \theta \int_\Omega \lambda f'(x, u_\lambda)u_\lambda^2 dx + \lambda \|h\|_2 \|u_\lambda\| \\ &\leq \theta \|u_\lambda\|^2 + \lambda \|h\|_2 \|u_\lambda\|. \end{aligned} \quad (3.19)$$

This implies that

$$\|u_\lambda\| \leq \frac{\lambda}{1-\theta} \|h\|_2 \quad (3.20)$$

for all $\lambda \in (0, \lambda^*)$. By Lemma 3.3(i), the solution u_λ is strictly increasing with respect to λ ; we may suppose that

$$u_\lambda \rightharpoonup u_{\lambda^*} \quad \text{weakly in } H_0^1(\Omega) \text{ as } \lambda \rightarrow \lambda^*, \quad (3.21)$$

and by (1.4), we obtain that

$$\begin{aligned} \int_{\Omega} (\nabla u_\lambda \cdot \nabla \varphi + u_\lambda \varphi) dx &\rightarrow \int_{\Omega} (\nabla u_{\lambda^*} \cdot \nabla \varphi + u_{\lambda^*} \varphi) dx, \\ \lambda \int_{\Omega} (f(x, u_\lambda) + h) \varphi dx &\rightarrow \lambda^* \int_{\Omega} (f(x, u_{\lambda^*}) + h) \varphi dx \end{aligned} \quad \text{as } \lambda \rightarrow \lambda^* \quad (3.22)$$

for all $\varphi \in H_0^1(\Omega)$. Hence u_{λ^*} is a minimal positive solution of $(1.1)_{\lambda^*}$. This completes the proof of Lemma 3.4. $\square$

4. Existence of second solution

When $\lambda \in (0, \lambda^*)$, we know that $(1.1)_\lambda$ has a minimal positive solution u_λ by Lemma 3.3, then we need only to prove that $(1.1)_\lambda$ has another positive solution in the form of $U_\lambda = u_\lambda + \bar{v}$, where $\bar{v}$ is a solution of the following equation:

$$\begin{aligned} -\Delta v + v &= \lambda(f(x, u_\lambda + v) - f(x, u_\lambda)) \quad \text{in } \Omega, \\ v &> 0 \quad \text{in } \Omega, \quad v \in H_0^1(\Omega). \end{aligned} \quad (4.1)$$

We define the energy functional $J : H_0^1(\Omega) \rightarrow \mathbb{R}$ as follows:

$$J(v) = \frac{1}{2} \int_{\Omega} (|\nabla v|^2 + v^2) dx - \lambda \int_{\Omega} (F(x, u_\lambda + v^+) - F(x, u_\lambda) - f(x, u_\lambda) v^+) dx. \quad (4.2)$$

Using the monotonicity of f and the maximum principle, we know that the nontrivial critical points of energy functional J are the positive solutions of (4.1).

First, we give an inequality about concerning f and u_λ .

LEMMA 4.1. *If (f_1) and (f_2) hold, then for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$f(x, u_\lambda + s) - f(x, u_\lambda) - f'(x, u_\lambda)s \leq \varepsilon s + C_\varepsilon s^{p-1}, \quad s \geq 0, \text{ uniformly } \forall x \in \mathbb{S}, \quad (4.3)$$

where $1 < p < 2^* - 1$ and u_λ is the minimal solution of $(1.1)_\lambda$.

Proof. By (f1), (f2), (1.4), and Lemma 2.3, we obtain $u_\lambda \in L^\infty(\Omega)$ and

$$\begin{aligned} \lim_{s \rightarrow 0} \frac{f(x, u_\lambda + s) - f(x, u_\lambda) - f'(x, u_\lambda)s}{s} &= 0, \\ 0 \leq \limsup_{s \rightarrow \infty} \frac{f(x, u_\lambda + s) - f(x, u_\lambda) - f'(x, u_\lambda)s}{s^{p-1}} &\leq C_\varepsilon \end{aligned} \quad (4.4)$$

uniformly for all $x \in \mathbb{S}$. Thus, it is clear that Lemma 4.1 holds. $\square$

LEMMA 4.2. *If (f1)–(f5) hold, then there exist $\rho > 0$ and $\alpha > 0$ such that*

$$J(v) \geq \alpha > 0 \quad (4.5)$$

for all $v \in S_\rho = \{u \in H_0^1(\Omega) \mid \|u\| = \rho\}$.

Proof. By Lemma 3.4, it is easy to see that, for all $v \in H_0^1(\Omega)$,

$$\int_\Omega (|\nabla v|^2 + v^2) dx \geq \sigma_\lambda \int_\Omega f'(x, u_\lambda) v^2 dx. \quad (4.6)$$

Again by Lemma 4.1 and the Sobolev embedding theorem, we obtain that

$$\begin{aligned} J(v) &= \frac{1}{2} \int_\Omega (|\nabla v|^2 + v^2) dx - \lambda \int_\Omega (F(x, u_\lambda + v^+) - F(x, u_\lambda) - f(x, u_\lambda) v^+) dx \\ &= \frac{1}{2} \|v\|^2 - \frac{\lambda}{2} \int_\Omega f'(x, u_\lambda) |v^+|^2 dx \\ &\quad - \lambda \int_\Omega \int_0^{v^+} (f(x, u_\lambda + s) - f(x, u_\lambda) - f'(x, u_\lambda)s) ds dx \\ &\geq \frac{1}{2} \|v\|^2 - \frac{\lambda}{2} \int_\Omega f'(x, u_\lambda) |v^+|^2 dx - \frac{1}{2} \lambda \varepsilon \int_\Omega |v^+|^2 dx - \frac{1}{p} \lambda C_\varepsilon \int_\Omega |v^+|^p dx \\ &\geq \frac{1}{2} \|v\|^2 - \frac{\lambda}{2} \sigma^{-1} \|v\|^2 - \frac{1}{2} \lambda \varepsilon \|v\|^2 - \lambda C_\varepsilon \|v\|^p \\ &= \frac{1}{2} \sigma_\lambda^{-1} (\sigma_\lambda - \lambda - \lambda \sigma_\lambda \varepsilon) \|v\|^2 - \lambda C_\varepsilon \|v\|^p. \end{aligned} \quad (4.7)$$

Since $\sigma_\lambda > \lambda$, we may choose $\varepsilon > 0$ small enough such that $\sigma_\lambda - \lambda - \lambda \sigma_\lambda \varepsilon > 0$. If we take $\varepsilon = (\sigma_\lambda - \lambda)/2\lambda\sigma_\lambda$, then

$$J(v) \geq \frac{1}{4} \sigma_\lambda^{-1} (\sigma_\lambda - \lambda) \|v\|^2 - C \|v\|^p. \quad (4.8)$$

Hence, there exist $\rho > 0$ and $\alpha > 0$ such that $J(v) \geq \alpha > 0$ for all $v \in S_\rho = \{u \in H_0^1(\Omega) \mid \|u\| = \rho\}$. $\square$

PROPOSITION 4.3. *Assume (f1)–(f4) hold. Let $\{v_k\}$ be a $(PS)_c$ -sequence of J . Then there exists a subsequence (still denoted by $\{v_k\}$) for which the following holds: there exist an integer $l \geq 0$, sequences $\{x_k^i\} \subseteq \mathbb{R}^N$, $1 \leq i \leq l$, $k \in \mathbb{N}$, of the form $(0, z_k^i) \in \mathbb{S}$, a solution $\bar{v}$ of (4.1), and solutions u^i of (2.1) $_\lambda$, $1 \leq i \leq l$, such that, for some subsequence $\{v_k\}$, as $k \rightarrow \infty$,*

one has

$$\begin{aligned}
v_k &\rightharpoonup \bar{v} \quad \text{weakly in } H_0^1(\Omega), \\
J(v_k) &\longrightarrow J(\bar{v}) + \sum_{i=1}^l I^\infty(u^i), \\
v_k - \left(\bar{v} + \sum_{i=1}^l u^i(x - x_k^i) \right) &\longrightarrow 0 \quad \text{strong in } H_0^1(\mathbb{S}), \\
|x_k^i| &\longrightarrow \infty, \quad |x_k^i - x_k^j| \longrightarrow \infty, \quad 1 \leq i \neq j \leq l,
\end{aligned} \tag{4.9}$$

where one agrees that in the case $l = 0$ the above holds without u^i, x_k^i .

Proof. This result can be derived from the arguments in [9] (see also [10–12]). Here we omit it. $\square$

Now, let δ be small enough, D^δ a δ -tubular neighborhood of D such that $D^\delta \subset \subset \mathbb{S}$. Let $\eta(x) : \mathbb{S} \rightarrow [0, 1]$ be a C^∞ cutoff function such that $0 \leq \eta \leq 1$ and

$$\eta(x) = \begin{cases} 0, & \text{if } x \in D; \\ 1, & \text{if } x \in \mathbb{S} \setminus \bar{D}^\delta. \end{cases} \tag{4.10}$$

Let $e_N = (0, 0, \dots, 0, 1) \in \mathbb{R}^N$, denote

$$\begin{aligned}
\tau_0 &= 2 \sup_{x \in D^\delta} |x| + 1, \\
w_\tau(x) &= w(x - \tau e_N), \quad \tau \in [0, \infty),
\end{aligned} \tag{4.11}$$

where w is a ground state solution of (2.1) $_\lambda$.

LEMMA 4.4. *If (f1)–(f5) hold, then*

- (i) *there exists $t_0 > 0$ such that $J(t\eta w_\tau) < 0$ for $t \geq t_0, \tau \geq \tau_0$,*
- (ii) *there exists $\tau_* > 0$ such that the following inequality holds for $\tau \geq \tau_*$:*

$$0 < \sup_{t \geq 0} J(t\eta w_\tau) < I^\infty(w) = S^\infty. \tag{4.12}$$

Proof. (i) By the definition of η and Lemma 1.3(iii), we have

$$\begin{aligned}
J(t\eta w_\tau) &= \frac{1}{2} \int_{\Omega} (|\nabla(t\eta w_\tau)|^2 + (t\eta w_\tau)^2) dx - \lambda \int_{\Omega} \int_0^{t\eta w_\tau} (f(x, u_\lambda + s) - f(x, u_\lambda)) ds dx \\
&\leq \frac{t^2}{2} \int_{\Omega} (|\nabla(\eta w_\tau)|^2 + (\eta w_\tau)^2) dx - \lambda \int_{\mathbb{S} \setminus \bar{D}^\delta} F(x, t w_\tau) dx.
\end{aligned} \tag{4.13}$$

From Lemma 1.3(ii), we have that $F(x, u)/(\nu^{-1}u^\nu)$ is monotone nondecreasing for $u > 0$, where $\nu = 1 + \theta^{-1} > 2$. Thus for any given constant $C > 0$, there exists $u_0 \geq 0$ such that

$$F(x, u) \geq C u^\nu \quad \forall u \geq u_0. \tag{4.14}$$

Let r_0 be a positive constant such that $B^m(0; r_0) = \{y \mid |y| \leq r_0\} \subset\subset \omega$, $B^n(0; 1) = \{z \mid |z| \leq 1\}$, $\Omega_1 = B^m(0; r_0) \times B^n(0; 1)$, and $\Omega_{1\tau} = B^m(0; r_0) \times \{z + \tau e_N \mid |z| \leq 1\}$. By the definition of τ_0 , we have that $\Omega_{1\tau} \subset\subset \Omega \setminus \overline{D}^\delta$ for all $\tau \geq \tau_0$. This also implies that there exists $t_0 \geq 0$, as $t \geq t_0$, we have

$$F(x, tw_\tau) \geq Ct^\nu w^\nu \quad \forall \tau \geq \tau_0, \forall x \in \Omega_{1\tau}. \quad (4.15)$$

Therefore as $t > t_0$ and $\tau \geq \tau_0$,

$$\begin{aligned} J(t\eta w_\tau) &\leq \frac{t^2}{2} \int_{\Omega} (|\nabla(\eta w_\tau)|^2 + (\eta w_\tau)^2) dx - \lambda Ct^\nu \int_{\Omega_{1\tau}} w_\tau^\nu dx \\ &\leq \frac{t^2}{2} \|\eta w_\tau\|^2 - \lambda Ct^\nu \int_{\Omega_1} w^\nu dx. \end{aligned} \quad (4.16)$$

Since $\nu > 2$, we can choose $t_0 > 0$ large enough such that (i) holds.

(ii) By (i), J is continuous on $H_0^1(\Omega)$, $J(0) = 0$, and Lemma 4.2, we know that there exists t_1 with $0 < t_1 < t_0$ such that

$$\sup_{t \geq 0} J(t\eta w_\tau) = \sup_{t_1 \leq t \leq t_0} J(t\eta w_\tau) \quad \forall \tau \geq \tau_0. \quad (4.17)$$

Now, we define $\eta_\tau(x) = \eta(x + \tau e_N)$ for all $x \in \mathbb{S}$. For $\tau \geq \tau_0$, $t_1 \leq t \leq t_0$, by (f4), (1.4), (2.3), Lemmas 1.3 and 2.5, we have

$$\begin{aligned} J(t\eta w_\tau) &= \frac{t^2}{2} \int_{\Omega} (|\nabla(\eta w_\tau)|^2 + (\eta w_\tau)^2) dx - \lambda \int_{\Omega} F(x, t\eta w_\tau) dx \\ &\quad - \lambda \int_{\Omega} \int_0^{t\eta w_\tau} (f(x, u_\lambda + s) - f(x, u_\lambda) - f(x, s)) ds dx \\ &\leq \frac{t^2}{2} \int_{\mathbb{S}} (-\Delta w + w)(\eta_\tau^2 w) dx + \frac{t^2}{2} \int_{\mathbb{S}} |\nabla \eta_\tau|^2 |w|^2 dx - \lambda \int_{\mathbb{S}} F(x, tw_\tau) dx \\ &\quad + \lambda \int_{\mathbb{S}} \int_{t\eta w_\tau}^{tw_\tau} f(x, s) ds dx - \lambda \int_{\Omega} \int_0^{t\eta w_\tau} (f(x, u_\lambda + s) - f(x, u_\lambda) - f(x, s)) ds dx \\ &\leq \frac{t^2}{2} \int_{\mathbb{S}} (|\nabla w|^2 + w^2) dx - \lambda \int_{\mathbb{S}} \bar{F}(tw_\tau) dx + \frac{t_0^2}{2} \int_{D^\delta \setminus D} |\nabla \eta|^2 |w_\tau|^2 dx \\ &\quad + \lambda \int_{D^\delta} \int_0^{tw_\tau} f(x, s) ds dx - \lambda \int_{\Omega} \int_0^{t\eta w_\tau} (f(x, u_\lambda + s) - f(x, u_\lambda) - f(x, s)) ds dx \\ &\leq S^\infty + C_\varepsilon \exp\left(-2\sqrt{1 + \mu_1 - \varepsilon\tau}\right) + \lambda C \int_{D^\delta} \left[\frac{(tw_\tau)^2}{2} + \frac{(tw_\tau)^p}{p}\right] dx \\ &\quad - \lambda \int_{\Omega} \int_0^{t\eta w_\tau} (f(x, u_\lambda + s) - f(x, u_\lambda) - f(x, s)) ds dx \\ &\leq S^\infty + C_\varepsilon \exp\left(-2\sqrt{1 + \mu_1 - \varepsilon\tau}\right) \\ &\quad - \lambda \int_{\Omega} \int_0^{t\eta w_\tau} (f(x, u_\lambda + s) - f(x, u_\lambda) - f(x, s)) ds dx, \end{aligned} \quad (4.18)$$

where $0 < \varepsilon < 1 + \mu_1$ and C_ε is independent of τ .

It follows from Taylor's expansion that

$$f(x, u_\lambda + s) = f(x, s) + f'(x, s)u_\lambda + \frac{1}{2}f''(x, \xi)u_\lambda^2, \quad \xi \in (s, u_\lambda + s). \quad (4.19)$$

From (f5) and the above formula, for $t_1 \leq t \leq t_0$, we obtain that

$$\begin{aligned} & \int_0^{t\eta w_\tau} (f(x, u_\lambda + s) - f(x, u_\lambda) - f(x, s)) ds \\ & \geq \int_0^{t_1\eta w_\tau} (f'(x, s)u_\lambda - f(x, u_\lambda)) ds \\ & = [(t_1 w_\tau)^{-1} f(x, t_1 \eta w_\tau) - \eta u_\lambda^{-1} f(x, u_\lambda)] t_1 w_\tau u_\lambda. \end{aligned} \quad (4.20)$$

Since $w_\tau > 0$ in $\mathbb{S}$, there exists $\gamma_1 > 0$ such that

$$w_\tau \geq \gamma_1 \quad \text{in } \Omega_{1\tau}. \quad (4.21)$$

By the definition of w_τ and $u_\lambda(x) \rightarrow 0$ as $|x| \rightarrow \infty$, we see that for τ large enough,

$$t_1 w_\tau \geq u_\lambda \quad \text{in } \Omega_{1\tau}, \quad (4.22)$$

then Lemma 1.3(ii) implies that there exist $\gamma_2 > 0$ and $\tau_1 > 0$ such that, for $\tau \geq \tau_1$,

$$(t_1 w_\tau)^{-1} f(x, t_1 w_\tau) - u_\lambda^{-1} f(x, u_\lambda) > \gamma_2 \quad \text{in } \Omega_{1\tau}. \quad (4.23)$$

Now by Lemma 2.5, for $\tau \geq \max(\tau_0, \tau_1)$ and $t_1 \leq t \leq t_0$, we obtain that

$$\begin{aligned} & \int_{\Omega_{1\tau}} \int_0^{t\eta w_\tau} (f(x, u_\lambda + s) - f(x, u_\lambda) - f(x, s)) ds dx \\ & \geq \int_{\Omega_{1\tau}} [(t_1 w_\tau)^{-1} f(x, t_1 w_\tau) - u_\lambda^{-1} f(x, u_\lambda)] t_1 w_\tau u_\lambda dx \\ & \geq \gamma_1 \gamma_2 \int_{\Omega_{1\tau}} t_1 u_\lambda dx \\ & \geq C_2 \exp\left(-\sqrt{1 + \mu_1 \tau}\right), \end{aligned} \quad (4.24)$$

where C_2 is independent of τ .

Therefore we obtain that

$$J(t\eta w_\tau) \leq S^\infty + C_\varepsilon \exp\left(-2\sqrt{1 + \mu_1 - \varepsilon \tau}\right) - \lambda C_2 \exp\left(-\sqrt{1 + \mu_1 \tau}\right), \quad (4.25)$$

for $t \in [t_1, t_0]$ and $\tau \geq \max(\tau_0, \tau_1)$.

Now, let $\varepsilon = (1 + \mu_1)/2$, then we can find some τ_* large enough such that

$$C_\varepsilon \exp\left(-\sqrt{2(1 + \mu_1)\tau}\right) - \lambda C_2 \exp\left(-\sqrt{1 + \mu_1 \tau}\right) < 0, \quad (4.26)$$

for all $\tau \geq \tau_*$ and we complete the proof. $\square$

THEOREM 4.5. *If (f1)–(f5) hold, then (4.1) has a positive solution $\bar{v}$ if $\lambda \in (0, \lambda^*)$.*

Proof. Now, set

$$\begin{aligned} \Gamma &= \{p \in C([0, 1], H_0^1(\Omega)) \mid p(0) = 0, p(1) = t_0 \eta w_{\tau_*}\}, \\ c &= \inf_{p \in \Gamma} \max_{s \in [0, 1]} J(p(s)). \end{aligned} \quad (4.27)$$

By Lemmas 4.2 and 4.4, we have

$$0 < \alpha \leq c < S^\infty. \quad (4.28)$$

Applying the mountain pass theorem of Ambrosetti-Rabinowitz [13], there exists a $(PS)_c$ -sequence $\{v_k\}$, $k \in \mathbb{N}$, such that

$$\begin{aligned} J(v_k) &\longrightarrow c, \\ J'(v_k) &\longrightarrow 0 \quad \text{strong in } H^{-1}(\Omega). \end{aligned} \quad (4.29)$$

By Proposition 4.3, there exist a sequence (still denoted by $\{v_k\}$), an integer $l \geq 0$, sequence $\{x_k^i\}$ in $\mathbb{S}$, $1 \leq i \leq l$, a solution $\bar{v}$ of (4.1), and solutions u^i of $(2.1)_\lambda$ such that

$$c = J(\bar{v}) + \sum_{i=0}^l I^\infty(u^i). \quad (4.30)$$

By the strong maximum principle, to complete the proof, we only need to prove $\bar{v} \not\equiv 0$ in Ω . In fact, we have

$$c = J(\bar{v}) \geq \alpha > 0 \quad \text{if } l = 0, \quad S^\infty > c \geq J(\bar{v}) + S^\infty \quad \text{if } l \geq 1. \quad (4.31)$$

This implies $\bar{v} \not\equiv 0$ in Ω . □

5. Properties of solutions

Denote by $A = \{(\lambda, u) \mid u \text{ solves problem } (1.1)_\lambda\}$, the set of solutions of $(1.1)_\lambda$, $\lambda \in (0, \lambda^*]$. For each $(\lambda, u) \in A$, let $\sigma_\lambda(u)$ denote the number defined by

$$\sigma_\lambda(u) = \inf \left\{ \int_\Omega (|\nabla v|^2 + v^2) dx \mid v \in H_0^1(\Omega), \int_\Omega f'(x, u) v^2 dx = 1 \right\}, \quad (5.1)$$

which is the smallest eigenvalue of the following problem:

$$\begin{aligned} -\Delta v + v^2 &= \sigma_\lambda(u) f'(x, u) v \quad \text{in } \Omega, \\ v &> 0, \quad v \in H_0^1(\Omega). \end{aligned} \quad (5.2)$$

In this section, we always assume that (f1)–(f5) hold. By Lemma 2.3, we have $A \subset \mathbb{R} \times L^\infty(\mathbb{R}^N) \cap H_0^1(\Omega)$.

LEMMA 5.1. *Let u be a solution and u_λ be the minimal solution of $(1.1)_\lambda$ for $\lambda \in (0, \lambda^*)$. Then*

- (i) $\sigma_\lambda(u) > \lambda$ if and only if $u = u_\lambda$;
- (ii) $\sigma_\lambda(U_\lambda) < \lambda$, where U_λ is the second solution of $(1.1)_\lambda$ constructed in Section 4.

Proof. Now, let $\psi \geq 0$ and $\psi \in H_0^1(\Omega)$. Since u and u_λ solve $(1.1)_\lambda$, then

$$\begin{aligned} & \int_{\Omega} \nabla \psi \cdot \nabla (u_\lambda - u) dx + \int_{\Omega} \psi (u_\lambda - u) dx \\ &= \lambda \int_{\Omega} (f(x, u_\lambda) - f(x, u)) \psi dx = \lambda \int_{\Omega} \left(\int_u^{u_\lambda} f'(x, t) dt \right) \psi dx \\ &\geq \lambda \int_{\Omega} f'(x, u) (u_\lambda - u) \psi dx. \end{aligned} \quad (5.3)$$

Let $\psi = (u - u_\lambda)^+ \geq 0$ and $\psi \in H_0^1(\Omega)$. If $\psi \not\equiv 0$, then (5.3) implies

$$- \int_{\Omega} (|\nabla \psi|^2 + \psi^2) dx \geq -\lambda \int_{\Omega} f'(x, u) \psi^2 dx \quad (5.4)$$

and, therefore, the definition of $\sigma_\lambda(u)$ implies

$$\begin{aligned} \int_{\Omega} (|\nabla \psi|^2 + \psi^2) dx &\leq \lambda \int_{\Omega} f'(x, u) \psi^2 dx \\ &< \sigma_\lambda(u) \int_{\Omega} f'(x, u) \psi^2 dx \\ &\leq \int_{\Omega} (|\nabla \psi|^2 + \psi^2) dx, \end{aligned} \quad (5.5)$$

which is impossible. Hence $\psi \equiv 0$, and $u = u_\lambda$ in Ω . On the other hand, by Lemma 3.4, we also have that $\sigma_\lambda(u_\lambda) > \lambda$. This completes the proof of (i).

By (i), we get that $\sigma_\lambda(U_\lambda) \leq \lambda$ for $\lambda \in (0, \lambda^*)$. We claim that $\sigma_\lambda(U_\lambda) = \lambda$ cannot occur. We proceed by contradiction. Set $w = U_\lambda - u_\lambda$; we have

$$-\Delta w + w = \lambda [f(x, U_\lambda) - f(x, U_\lambda - w)], \quad w > 0 \text{ in } \Omega. \quad (5.6)$$

By $\sigma_\lambda(U_\lambda) = \lambda$, we have that the problem

$$-\Delta \phi + \phi = \lambda f'(x, U_\lambda) \phi, \quad \phi \in H_0^1(\Omega) \quad (5.7)$$

possesses a positive solution ϕ_1 .

Multiplying (5.6) by ϕ_1 and (5.7) by w , integrating and subtracting we deduce that

$$\begin{aligned} 0 &= \int_{\Omega} \lambda [f(x, U_\lambda) - f(x, U_\lambda - w) - f'(x, U_\lambda) w] \phi_1 dx \\ &= -\frac{1}{2} \int_{\Omega} \lambda f''(\xi_\lambda) w^2 \phi_1 dx, \end{aligned} \quad (5.8)$$

where $\xi_\lambda \in (u_\lambda, U_\lambda)$. Thus $w \equiv 0$, that is $U_\lambda = u_\lambda$ for $\lambda \in (0, \lambda^*)$. This is a contradiction. Hence we have that $\sigma_\lambda(U_\lambda) < \lambda$ for $\lambda \in (0, \lambda^*)$. $\square$

THEOREM 5.2. *Suppose u_{λ^*} is a solution of $(1.1)_{\lambda^*}$, then $\sigma_{\lambda^*}(u_{\lambda^*}) = \lambda^*$ and the solution u_{λ^*} is unique.*

Proof. Define $\mathcal{F} : \mathbb{R} \times H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ by

$$\mathcal{F}(\lambda, u) = \Delta u - u + \lambda(f(x, u) + h(x)). \quad (5.9)$$

Since $\sigma_\lambda(u_\lambda) > \lambda$ for $\lambda \in (0, \lambda^*)$, we have $\sigma_{\lambda^*}(u_{\lambda^*}) \geq \lambda^*$. If $\sigma_{\lambda^*}(u_{\lambda^*}) > \lambda^*$, the equation $\mathcal{F}_u(\lambda^*, u_{\lambda^*})\phi = 0$ has no nontrivial solution. By the standard argument, we can prove that $\mathcal{F}_u$ maps $\mathbb{R} \times H_0^1(\Omega)$ onto $H^{-1}(\Omega)$. Applying the implicit function theorem to $\mathcal{F}$, we can find a neighborhood $(\lambda^* - \delta, \lambda^* + \delta)$ of λ^* such that $(1.1)_\lambda$ possesses a solution u_λ if $\lambda \in (\lambda^* - \delta, \lambda^* + \delta)$. This is contradictory to the definition of λ^* . Hence we obtain that $\sigma_{\lambda^*}(u_{\lambda^*}) = \lambda^*$.

Next, we are going to prove that u_{λ^*} is unique. In fact, suppose $(1.1)_{\lambda^*}$ has another solution $U_{\lambda^*} \geq u_{\lambda^*}$. Set $w = U_{\lambda^*} - u_{\lambda^*}$; we have

$$-\Delta w + w = \lambda^*[f(w + u_{\lambda^*}) - f(x, u_{\lambda^*})], \quad w > 0 \text{ in } \Omega. \quad (5.10)$$

By $\sigma_{\lambda^*}(u_{\lambda^*}) = \lambda^*$, we have that the problem

$$-\Delta \phi + \phi = \lambda^* f'(x, u_{\lambda^*})\phi, \quad \phi \in H_0^1(\Omega) \quad (5.11)$$

possesses a positive solution ϕ_1 .

Multiplying (5.10) by ϕ_1 and (5.11) by w , integrating and subtracting we deduce that

$$\begin{aligned} 0 &= \int_{\Omega} \lambda^*[f(w + u_{\lambda^*}) - f(x, u_{\lambda^*}) - f'(x, u_{\lambda^*})w]\phi_1 dx \\ &= \frac{1}{2} \int_{\Omega} \lambda^* f''(\xi_{\lambda^*}) w^2 \phi_1 dx, \end{aligned} \quad (5.12)$$

where $\xi_{\lambda^*} \in (u_{\lambda^*}, u_{\lambda^*} + w)$. Thus $w \equiv 0$. □

PROPOSITION 5.3. *Let u_λ be the minimal solution of $(1.1)_\lambda$. Then u_λ is uniformly bounded in $L^\infty(\Omega) \cap H_0^1(\Omega)$ for all $\lambda \in (0, \lambda^*]$, and*

$$u_\lambda \longrightarrow 0 \quad \text{in } L^\infty(\Omega) \cap H_0^1(\Omega) \text{ as } \lambda \longrightarrow 0^+. \quad (5.13)$$

Proof. By (3.20), we have that

$$\|u_\lambda\| \leq \frac{\lambda}{1 - \theta} \|h\|_2 \quad (5.14)$$

for $\lambda \in (0, \lambda^*)$, and u_λ is strictly increasing with respect to λ , we can easily deduce that u_λ is uniformly bounded in $L^\infty(\Omega) \cap H_0^1(\Omega)$ for $\lambda \in (0, \lambda^*]$ and $u_\lambda \rightarrow 0$ in $H_0^1(\Omega)$ as $\lambda \rightarrow 0^+$.

By (2.6) and the fact that u_λ is uniformly bounded in $L^\infty(\Omega) \cap H_0^1(\Omega)$, we have that

$$\begin{aligned} \|u_\lambda\|_\infty &\leq C_1 \|u_\lambda\|_{q_0} + \lambda C_2 (\|u_\lambda\|_{(p-1)q_0}^{p-1} + \|h\|_{q_0}) \\ &\leq C_1 \|u_\lambda\|_\infty^{(q_0-2)/q_0} \|u_\lambda\|_2^{2/q_0} + C_3 \lambda \\ &\leq C(\lambda^{2/q_0} + \lambda), \end{aligned} \quad (5.15)$$

where C is independent of λ , and $\lambda \in (0, \lambda^*]$. Hence we obtain that $u_\lambda \rightarrow 0$ in $L^\infty(\Omega)$ as $\lambda \rightarrow 0^+$. $\square$

PROPOSITION 5.4. *If $\lambda \in (0, \lambda^*)$, then U_λ is unbounded in $L^\infty(\Omega) \cap H_0^1(\Omega)$, and*

$$\lim_{\lambda \rightarrow 0^+} \|U_\lambda\| = \lim_{\lambda \rightarrow 0^+} \|U_\lambda\|_\infty = \infty. \quad (5.16)$$

Proof. Let φ_λ be a minimizer of $\sigma_\lambda(U_\lambda)$ for $\lambda \in (0, \lambda^*)$, that is

$$\int_{\Omega} f'(x, U_\lambda) \varphi_\lambda^2 = 1, \quad \|\varphi_\lambda\|^2 = \sigma_\lambda(U_\lambda). \quad (5.17)$$

(i) First, we show that $\{U_\lambda : \lambda \in (0, \lambda_0)\}$ is unbounded in $L^\infty(\Omega)$ for any $\lambda_0 \in (0, \lambda^*)$. We proceed by contradiction. Assume to the contrary that there exists $C_0 > 0$ such that

$$\|U_\lambda\|_\infty \leq C_0 < \infty \quad \forall \lambda \in (0, \lambda_0). \quad (5.18)$$

By (f1) and (5.18), there exists a constant M independent of λ , such that $f'(x, U_\lambda(x)) \leq M$ for all $\lambda \in (0, \lambda_0)$ and $x \in \Omega$. Hence, by (5.17) and $\sigma_\lambda(U_\lambda) < \lambda$ for all $\lambda \in (0, \lambda_0)$, we obtain that

$$1 = \int_{\Omega} f'(x, U_\lambda) \varphi_\lambda^2 \leq M \|\varphi_\lambda\|^2 = M \sigma_\lambda(U_\lambda) < M \lambda. \quad (5.19)$$

This is a contradiction for all $\lambda < 1/M$. Hence, for any $\lambda_0 \in (0, \lambda^*)$, we have that $\{U_\lambda : \lambda \in (0, \lambda^*)\}$ is unbounded in $L^\infty(\Omega)$. From this result, it is easy to see that $\lim_{\lambda \rightarrow 0^+} \|U_\lambda\|_\infty = \infty$.

(ii) Now, we show that $\{U_\lambda : \lambda \in (0, \lambda_0)\}$ is unbounded in $H_0^1(\Omega)$ for any $\lambda_0 \in (0, \lambda^*)$. If not, then there exists a constant M independent of λ , such that

$$\|U_\lambda\| \leq M \quad \forall \lambda \in (0, \lambda_0). \quad (5.20)$$

By (5.17), (5.20), (f2), the Hölder inequality, the Sobolev embedding theorem, and $\sigma_\lambda(U_\lambda) < \lambda$ for all $\lambda \in (0, \lambda^*)$, we have that

$$\begin{aligned} 1 &= \int_{\Omega} f'(x, U_\lambda) \varphi_\lambda^2 \leq C_1 \int_{\Omega} (1 + U_\lambda^{p-2}) \varphi_\lambda^2 \leq C_1 \|\varphi_\lambda\|^2 + C_1 \|U_\lambda\|_p^{p-2} \|\varphi_\lambda\|_p^2 \\ &\leq C_1 \|\varphi_\lambda\|^2 + C_2 \|U_\lambda\|^{p-2} \|\varphi_\lambda\|^2 \leq C_3 \|\varphi_\lambda\|^2 = C_3 \sigma_\lambda(U_\lambda) < C_3 \lambda, \end{aligned} \quad (5.21)$$

where C_1 , C_2 , and C_3 are constants independent of λ . Now, let $\lambda \rightarrow 0^+$, then we obtain a contradiction. Hence $\{U_\lambda : \lambda \in (0, \lambda^*)\}$ is unbounded in $H_0^1(\Omega)$ and $\lim_{\lambda \rightarrow 0^+} \|U_\lambda\| = +\infty$. $\square$

Proof of Theorems 1.1 and 1.2. First, we consider the case $\Omega = \mathbb{S} \setminus \overline{D}$. Theorem 1.1 now follows from Lemmas 3.3, 3.4, and Theorems 4.5, 5.2. Theorem 1.2 follows immediately from Lemma 3.4, and Propositions 5.3, 5.4. $\square$

With the same argument, we also have that Theorems 1.1 and 1.2 hold for $\Omega = \mathbb{R}^N \setminus \overline{D}$ or $\Omega = \mathbb{S}$ or $\Omega = \mathbb{R}^N$.

References

- [1] D. M. Cao, "Eigenvalue problems and bifurcation of semilinear elliptic equation in $\mathbb{R}^N$," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 24, no. 4, pp. 529–554, 1995.
- [2] X. P. Zhu and H. S. Zhou, "Existence of multiple positive solutions of inhomogeneous semilinear elliptic problems in unbounded domains," *Proceedings of the Royal Society of Edinburgh. Section A*, vol. 115, no. 3-4, pp. 301–318, 1990.
- [3] K.-J. Chen, "Exactly two entire positive solutions for a class of nonhomogeneous elliptic equations," *Differential Integral Equations*, vol. 17, no. 1-2, pp. 1–16, 2004.
- [4] T.-S. Hsu, "Existence of multiple positive solutions of subcritical semilinear elliptic problems in exterior strip domains," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 64, no. 6, pp. 1203–1228, 2006.
- [5] R. A. Adams, *Sobolev Spaces*, vol. 65 of *Pure and Applied Mathematics*, Academic Press, New York, NY, USA, 1975.
- [6] T.-S. Hsu, "Multiple solutions for semilinear elliptic equations in unbounded cylinder domains," *Proceedings of the Royal Society of Edinburgh. Section A*, vol. 134, no. 4, pp. 719–731, 2004.
- [7] I. Ekeland, "Nonconvex minimization problems," *Bulletin of the American Mathematical Society. New Series*, vol. 1, no. 3, pp. 443–474, 1979.
- [8] H. Amann, "Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces," *SIAM Review*, vol. 18, no. 4, pp. 620–709, 1976.
- [9] A. Bahri and P.-L. Lions, "On the existence of a positive solution of semilinear elliptic equations in unbounded domains," *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire*, vol. 14, no. 3, pp. 365–413, 1997.
- [10] P.-L. Lions, "The concentration-compactness principle in the calculus of variations. The locally compact case. I," *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire*, vol. 1, no. 2, pp. 109–145, 1984.
- [11] P.-L. Lions, "The concentration-compactness principle in the calculus of variations. The locally compact case. II," *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire*, vol. 1, no. 4, pp. 223–283, 1984.
- [12] P.-L. Lions, "On positive solutions of semilinear elliptic equations in unbounded domains," in *Nonlinear Diffusion Equations and Their Equilibrium States, II (Berkeley, CA, 1986)*, W. M. Ni, L. A. Peletier, and J. Serrin, Eds., vol. 13 of *Math. Sci. Res. Inst. Publ.*, pp. 85–122, Springer, New York, NY, USA, 1988.
- [13] A. Ambrosetti and P. H. Rabinowitz, "Dual variational methods in critical point theory and applications," *Journal of Functional Analysis*, vol. 14, no. 4, pp. 349–381, 1973.

Tsing-San Hsu: Center for General Education, Chang Gung University, Kwei-Shan,
Tao-Yuan 333, Taiwan

Email address: tshsu@mail.cgu.edu.tw