Research Article

On a Cubic Equation and a Jensen-Quadratic Equation

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We obtain the general solutions of the cubic functional equation $3[g(x+y) + g(x-y) + 6g(x)] = 2g(2x+y) + 2g(2x-y) + g(-x-y) + g(-x+y) + 6g(-x)$ and the Jensen-quadratic functional equation $f((x+y)/2, z+w) + f((x+y)/2, z-w) = f(x, z) + f(x, w) + f(y, z) + f(y, w)$.

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1. Introduction

Throughout this paper, let X and Y be vector spaces.

A mapping $g : X \rightarrow Y$ is called a *Jensen* (resp., *quadratic*) mapping if g satisfies the functional equation $2g((x+y)/2) = g(x) + g(y)$ (resp., $g(x+y) + g(x-y) = 2g(x) + 2g(y)$).

Definition 1.1. A mapping $f : X \times X \rightarrow Y$ is called *Jensen-quadratic* if f satisfies the system of equations

$$\begin{aligned} 2f\left(\frac{x+y}{2}, z\right) &= f(x, z) + f(y, z), \\ f(x, y+z) + f(x, y-z) &= 2f(x, y) + 2f(x, z). \end{aligned} \tag{1.1}$$

When $X = Y = \mathbb{R}$, the function $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x, y) := axy^2 + by^2$ is a solution of (1.1). In particular, letting $x = y$, we get a cubic function $g : \mathbb{R} \rightarrow \mathbb{R}$ given by $g(x) := f(x, x) = ax^3 + bx^2$.

2 Abstract and Applied Analysis

For a mapping $g : X \rightarrow Y$, consider the cubic functional equation

$$\begin{aligned} & 3[g(x+y) + g(x-y) + 6g(x)] \\ & = 2g(2x+y) + 2g(2x-y) + g(-x-y) + g(-x+y) + 6g(-x). \end{aligned} \quad (1.2)$$

For a mapping $f : X \times X \rightarrow Y$, consider the functional equation

$$f\left(\frac{x+y}{2}, z+w\right) + f\left(\frac{x+y}{2}, z-w\right) = f(x, z) + f(x, w) + f(y, z) + f(y, w). \quad (1.3)$$

In this paper, we find out the general solutions of (1.2) and (1.3), and investigate the relation between them.

For more detailed definitions of the functional equation and the Hyers-Ulam stability, we refer the reader to [1–4].

2. Solution of the cubic functional equation (1.2)

LEMMA 2.1. *A mapping $g : X \rightarrow Y$ satisfies*

$$g(x+y) + g(x-y) = 2g(x) + 2g(y) \quad (2.1)$$

for all $x, y \in X$ if and only if

$$g(2x+y) + g(2x-y) = g(x+y) + g(x-y) + 6g(x) \quad (2.2)$$

for all $x, y \in X$.

Proof. If g satisfies (2.1), then $g(0) = 0$. Letting $y = x$ in (2.1), we get $g(2x) = 4g(x)$ for all $x \in X$. Thus one can easily see that

$$\begin{aligned} g(2x+y) + g(2x-y) &= 2g(2x) + 2g(y) \\ &= 8g(x) + 2g(y) \\ &= 2g(x) + 2g(y) + 6g(x) \\ &= g(x+y) + g(x-y) + 6g(x) \end{aligned} \quad (2.3)$$

for all $x, y \in X$.

Conversely, if g satisfies (2.2), then $g(0) = 0$. Setting $y = 0$ and $y = x$ in (2.2), respectively, we gain

$$g(2x) = 4g(x), \quad g(3x) = 9g(x) \quad (2.4)$$

for all $x \in X$. Letting $y = 2x$ in (2.2) and using (2.4), we know that g is even. Replacing y by $2y$ in (2.2), we get

$$g(2x+2y) + g(2x-2y) = g(x+2y) + g(x-2y) + 6g(x) \quad (2.5)$$

for all $x, y \in X$. By (2.2) and the above equality, we have

$$\begin{aligned}
 g(2x+2y) + g(2x-2y) &= g(2y+x) + g(2y-x) + 6g(x) \\
 &= g(y+x) + g(y-x) + 6g(y) + 6g(x) \\
 &= g(x+y) + g(x-y) + 6g(x) + 6g(y)
 \end{aligned} \tag{2.6}$$

for all $x, y \in X$. By (2.4), we obtain that g satisfies (2.1). $\square$

THEOREM 2.2. *A mapping $g : X \rightarrow Y$ satisfies (1.2) if and only if there exist a symmetric multiadditive mapping $M : X \times X \times X \rightarrow Y$ and a symmetric biadditive mapping $B : X \times X \rightarrow Y$ such that $g(x) = M(x, x, x) + B(x, x)$ for all $x \in X$.*

Proof. Assume that g satisfies (1.2). Let $g_e(x) := (g(x) + g(-x))/2$ and $g_o(x) := (g(x) - g(-x))/2$ for all $x \in X$. Then g_e and g_o also satisfy (1.2). Since g_e satisfies (1.2) and is even, g_e satisfies (2.2). By Lemma 2.1, g_e satisfies (2.1). By [5], there exists a symmetric biadditive mapping $B : X \times X \rightarrow Y$ such that $g_e(x) = B(x, x)$ for all $x \in X$, where the mapping B is given by

$$B(x, y) = \frac{1}{4} [g_e(x+y) - g_e(x-y)] \tag{2.7}$$

for all $x, y \in X$. Since g_o satisfies (1.2) and is odd,

$$g_o(2x+y) + g_o(2x-y) = 2g_o(x+y) + 2g_o(x-y) + 12g_o(x) \tag{2.8}$$

for all $x \in X$. By [6], there exists a symmetric multiadditive mapping $M : X \times X \times X \rightarrow Y$ such that $g_o(x) = M(x, x, x)$ for all $x \in X$, where the mapping B is given by

$$M(x, y, z) = \frac{1}{24} [g_o(x+y+z) - g_o(x+y-z) - g_o(x-y+z) + g_o(x-y-z)] \tag{2.9}$$

for all $x, y, z \in X$. Obviously, we obtain that

$$g(x) = g_o(x) + g_e(x) = M(x, x, x) + B(x, x) \tag{2.10}$$

for all $x \in X$.

Conversely, we may assume that there exist a symmetric multiadditive mapping $M : X \times X \times X \rightarrow Y$ and a symmetric biadditive mapping $B : X \times X \rightarrow Y$ such that $g(x) = M(x, x, x) + B(x, x)$ for all $x \in X$.

4 Abstract and Applied Analysis

Then we obtain that

$$\begin{aligned}
 & 3[g(x+y) + g(x-y) + 6g(x)] \\
 &= 3[M(x+y, x+y, x+y) + B(x+y, x+y) \\
 &\quad + M(x-y, x-y, x-y) + B(x-y, x-y) + 6M(x, x, x) + 6B(x, x)] \\
 &= 24M(x, x, x) + 18M(x, y, y) + 24B(x, x) + 6B(y, y) \\
 &= 2M(2x+y, 2x+y, 2x+y) + 2B(2x+y, 2x+y) \\
 &\quad + 2M(2x-y, 2x-y, 2x-y) + 2B(2x-y, 2x-y) \\
 &\quad + M(-x-y, -x-y, -x-y) + B(-x-y, -x-y) \\
 &\quad + M(-x+y, -x+y, -x+y) + B(-x+y, -x+y) \\
 &\quad + 6M(-x, -x, -x) + 6B(-x, -x) \\
 &= 2g(2x+y) + 2g(2x-y) + g(-x-y) + g(-x+y) + 6g(-x)
 \end{aligned} \tag{2.11}$$

for all $x, y \in X$. □

3. Solution of the Jensen-quadratic functional equation (1.3)

LEMMA 3.1. *If a mapping $g : X \rightarrow Y$ satisfies (2.1) for all $x, y \in X$, then it also satisfies*

$$g(x+y+z) - g(x+y-z) = g(x+z) - g(x-z) + g(y+z) - g(y-z) \tag{3.1}$$

for all $x, y \in X$.

Proof. Replacing x and y by $x+y$ and z in (2.1), respectively, we gain

$$g(x+y+z) + g(x+y-z) = 2g(x+y) + 2g(z) \tag{3.2}$$

for all $x, y, z \in X$. Replacing y by $y-z$ in (2.1), we get

$$g(x+y-z) + g(x-y+z) = 2g(x) + 2g(y-z) \tag{3.3}$$

for all $x, y, z \in X$. Subtracting the latter from the former, we have

$$g(x+y+z) - g(x-y+z) = 2g(x+y) + 2g(z) - 2g(x) - 2g(y-z) \tag{3.4}$$

for all $x, y, z \in X$. Exchanging y and z in the above equation and using the fact that g is even, we obtain

$$\begin{aligned}
 & g(x+y+z) - g(x+y-z) \\
 &= 2g(x+z) + 2g(y) - 2g(x) - 2g(y-z) \\
 &= g(x+z) - g(x-z) + g(y+z) - g(y-z) \\
 &\quad + g(x+z) + g(x-z) + 2g(y) - g(y+z) - g(y-z) - 2g(x) \\
 &= g(x+z) - g(x-z) + g(y+z) - g(y-z) \\
 &\quad + 2g(x) + 2g(z) + 2g(y) - 2g(y) - 2g(z) - 2g(x) \\
 &= g(x+z) - g(x-z) + g(y+z) - g(y-z)
 \end{aligned} \tag{3.5}$$

for all $x, y, z \in X$. □

THEOREM 3.2. *A mapping $f : X \times X \rightarrow Y$ satisfies (1.1) if and only if there exist a multiadditive mapping $M : X \times X \times X \rightarrow Y$ and a symmetric biadditive mapping $B : X \times X \rightarrow Y$ such that $f(x, y) = M(x, y, y) + B(y, y)$ and $M(x, y, z) = M(x, z, y)$ for all $x, y, z \in X$.*

Proof. We first assume that f is a solution of (1.1). Let $x \in X$ be arbitrarily fixed. Define $g_x : X \rightarrow Y$ by $g_x(y) = f(x, y)$ for all $y \in X$. Then g_x is a quadratic mapping. By [5], there exists a symmetric biadditive mapping $B_x : X \times X \rightarrow Y$ such that $g_x(y) = B_x(y, y)$, where the mapping B_x is given by

$$B_x(y, z) = \frac{1}{4} [g_x(y+z) - g_x(y-z)] \quad (3.6)$$

for all $y, z \in X$.

Define

$$M(x, y, z) := B_x(y, z) - B_0(y, z) \quad (3.7)$$

for all $x, y, z \in X$. Replacing x by $2x$ and letting $y = 0$ in the first equation in (1.1), we get

$$f(2x, z) = 2f(x, z) - f(0, z) \quad (3.8)$$

for all $x, z \in X$. By the first equation in (1.1) and the above equation, we know that

$$\begin{aligned} & M(x+y, z, w) \\ &= B_{x+y}(z, w) - B_0(z, w) \\ &= \frac{1}{4} [g_{x+y}(z+w) - g_{x+y}(z-w)] - \frac{1}{4} [g_0(z+w) - g_0(z-w)] \\ &= \frac{1}{4} [f(x+y, z+w) - f(x+y, z-w)] - \frac{1}{4} [f(0, z+w) - f(0, z-w)] \\ &= \frac{1}{8} [f(2x, z+w) + f(2y, z+w) - f(2x, z-w) - f(2y, z-w)] \\ &\quad - \frac{1}{4} [f(0, z+w) - f(0, z-w)] \\ &= \frac{1}{4} [f(x, z+w) - f(x, z-w)] + \frac{1}{4} [f(y, z+w) - f(y, z-w)] \\ &\quad - \frac{1}{2} [f(0, z+w) - f(0, z-w)] \\ &= \frac{1}{4} [g_x(z+w) - g_x(z-w)] + \frac{1}{4} [g_y(z+w) - g_y(z-w)] \\ &\quad - \frac{1}{2} [g_0(z+w) - g_0(z-w)] \\ &= B_x(z, w) + B_y(z, w) - 2B_0(z, w) \\ &= M(x, z, w) + M(y, z, w) \end{aligned} \quad (3.9)$$

for all $x, y, z, w \in X$. By the second equation in (1.1) and Lemma 3.1, we see that

$$\begin{aligned}
 M(x, y+z, w) &= B_x(y+z, w) - B_0(y+z, w) \\
 &= \frac{1}{4} [g_x(y+z+w) - g_x(y+z-w)] - \frac{1}{4} [g_0(y+z+w) - g_0(y+z-w)] \\
 &= \frac{1}{4} [f(x, y+z+w) - f(x, y+z-w)] - \frac{1}{4} [f(0, y+z+w) - f(0, y+z-w)] \\
 &= \frac{1}{4} [f(x, y+w) - f(x, y-w) + f(x, z+w) - f(x, z-w)] \\
 &\quad - \frac{1}{4} [f(0, y+w) - f(0, y-w) + f(0, z+w) - f(0, z-w)] \\
 &= \frac{1}{4} [g_x(y+w) - g_x(y-w)] + \frac{1}{4} [g_x(z+w) - g_x(z-w)] \\
 &\quad - \frac{1}{4} [g_0(y+w) - g_0(y-w)] - \frac{1}{4} [g_0(z+w) - g_0(z-w)] \\
 &= B_x(y, w) + B_x(z, w) - B_0(y, w) - B_0(z, w) \\
 &= M(x, y, w) + M(x, z, w)
 \end{aligned} \tag{3.10}$$

for all $x, y, z, w \in X$. Since g_x is quadratic for all $x \in X$, one can easily obtain that $M(x, y, z) = M(x, z, y)$ for all $x, y, z \in X$. Thus M is multiadditive.

Conversely, we assume that there exist a multiadditive mapping $M : X \times X \times X \rightarrow Y$ and a symmetric biadditive mapping $B : X \times X \rightarrow Y$ such that $f(x, y) = M(x, y, y) + B(y, y)$ and $M(x, y, z) = M(x, z, y)$ for all $x, y, z \in X$. Since M is additive in the first variable,

$$\begin{aligned}
 2f\left(\frac{x+y}{2}, z\right) &= 2M\left(\frac{x+y}{2}, z, z\right) + 2B(z, z) \\
 &= M(x, z, z) + M(y, z, z) + 2B(z, z) \\
 &= f(x, z) + f(y, z)
 \end{aligned} \tag{3.11}$$

for all $x, y, z \in X$. Since M is multiadditive and odd in each variable,

$$\begin{aligned}
 f(x, y+z) + f(x, y-z) &= M(x, y+z, y+z) + M(x, y-z, y-z) \\
 &\quad + B(y+z, y+z) + B(y-z, y-z) \\
 &= 2M(x, y, y) + 2M(x, z, z) + 2B(y, y) + 2B(z, z) \\
 &= 2f(x, y) + 2f(x, z)
 \end{aligned} \tag{3.12}$$

for all $x, y, z \in X$. □

THEOREM 3.3. *A mapping $f : X \times X \rightarrow Y$ satisfies (1.1) if and only if it satisfies (1.3).*

Proof. If f satisfies (1.1), then

$$\begin{aligned}
 & f\left(\frac{x+y}{2}, z+w\right) + f\left(\frac{x+y}{2}, z-w\right) \\
 &= \frac{1}{2}[f(x, z+w) + f(y, z+w)] + \frac{1}{2}[f(x, z-w) + f(y, z-w)] \\
 &= \frac{1}{2}[f(x, z+w) + f(x, z-w)] + \frac{1}{2}[f(y, z+w) + f(y, z-w)] \\
 &= f(x, z) + f(x, w) + f(y, z) + f(y, w)
 \end{aligned} \tag{3.13}$$

for all $x, y, z, w \in X$.

Conversely, assume that f satisfies (1.3). Choosing $x = y = z = w = 0$ in (1.3), we have $f(0, 0) = 0$. Letting $y = x$ and $z = w = 0$ in (1.3), we get $f(x, 0) = 0$ for all $x \in X$. Putting $w = 0$ in (1.3), we obtain

$$2f\left(\frac{x+y}{2}, z\right) = f(x, z) + f(y, z) \tag{3.14}$$

for all $x, y, z \in X$. Taking $y = x$ in (1.3) and replacing z by y and w by z , we have

$$f(x, y+z) + f(x, y-z) = 2f(x, y) + 2f(x, z) \tag{3.15}$$

for all $x, y, z \in X$. □

4. The relation between (1.2) and (1.3)

THEOREM 4.1. *Let $g : X \rightarrow Y$ be a mapping satisfying (1.2) and let $f : X \times X \rightarrow Y$ be the mapping given by*

$$f(x, y) := \frac{1}{6}[g(x+y) + g(x-y) - 2g(x) + 2g(y) + 2g(-y)] \tag{4.1}$$

for all $x, y \in X$. Then f satisfies (1.3) and

$$g(x) = f(x, x) \tag{4.2}$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (1.2), $g(0) = 0$. Putting $y = 0$ in (1.2), we have

$$g(2x) = 6g(x) - 2g(-x) \tag{4.3}$$

for all $x \in X$. Setting $y = x$ in (4.1), (4.2) holds by (4.3). By Theorem 2.2, there exist a symmetric multiadditive mapping $M : X \times X \times X \rightarrow Y$ and a symmetric biadditive mapping $B : X \times X \rightarrow Y$ such that

$$g(x) = M(x, x, x) + B(x, x) \tag{4.4}$$

for all $x \in X$. By (4.4), we obtain that

$$\begin{aligned}
& g\left(\frac{x+y}{2} + z + w\right) + g\left(\frac{x+y}{2} - z - w\right) - 2g\left(\frac{x+y}{2}\right) + 2g(z+w) + 2g(-z-w) \\
& \quad + g\left(\frac{x+y}{2} + z - w\right) + g\left(\frac{x+y}{2} - z + w\right) - 2g\left(\frac{x+y}{2}\right) + 2g(z-w) + 2g(-z+w) \\
& = M\left(\frac{x+y}{2} + z + w, \frac{x+y}{2} + z + w, \frac{x+y}{2} + z + w\right) + B\left(\frac{x+y}{2} + z + w, \frac{x+y}{2} + z + w\right) \\
& \quad + M\left(\frac{x+y}{2} - z - w, \frac{x+y}{2} - z - w, \frac{x+y}{2} - z - w\right) + B\left(\frac{x+y}{2} - z - w, \frac{x+y}{2} - z - w\right) \\
& \quad - 2M\left(\frac{x+y}{2}, \frac{x+y}{2}, \frac{x+y}{2}\right) - 2B\left(\frac{x+y}{2}, \frac{x+y}{2}\right) \\
& \quad + 2M(z+w, z+w, z+w) + 2B(z+w, z+w) \\
& \quad + 2M(-z-w, -z-w, -z-w) + 2B(-z-w, -z-w) \\
& \quad + M\left(\frac{x+y}{2} + z - w, \frac{x+y}{2} + z - w, \frac{x+y}{2} + z - w\right) + B\left(\frac{x+y}{2} + z - w, \frac{x+y}{2} + z - w\right) \\
& \quad + M\left(\frac{x+y}{2} - z + w, \frac{x+y}{2} - z + w, \frac{x+y}{2} - z + w\right) + B\left(\frac{x+y}{2} - z + w, \frac{x+y}{2} - z + w\right) \\
& \quad - 2M\left(\frac{x+y}{2}, \frac{x+y}{2}, \frac{x+y}{2}\right) - 2B\left(\frac{x+y}{2}, \frac{x+y}{2}\right) \\
& \quad + 2M(z-w, z-w, z-w) + 2B(z-w, z-w) \\
& \quad + 2M(-z+w, -z+w, -z+w) + 2B(-z+w, -z+w) \\
& = 6[M(x, z, z) + M(x, w, w) + M(y, z, z) + M(y, w, w) + 2B(z, z) + 2B(w, w)] \\
& = M(x+z, x+z, x+z) + B(x+z, x+z) + M(x-z, x-z, x-z) + B(x-z, x-z) \\
& \quad - 2M(x, x, x) - 2B(x, x) + 2M(z, z, z) + 2B(z, z) + 2M(-z, -z, -z) + 2B(-z, -z) \\
& \quad + M(x+w, x+w, x+w) + B(x+w, x+w) + M(x-w, x-w, x-w) + B(x-w, x-w) \\
& \quad - 2M(x, x, x) - 2B(x, x) + 2M(w, w, w) + 2B(w, w) + 2M(-w, -w, -w) + 2B(-w, -w) \\
& \quad + M(y+z, y+z, y+z) + B(y+z, y+z) + M(y-z, y-z, y-z) + B(y-z, y-z) \\
& \quad - 2M(y, y, y) - 2B(y, y) + 2M(z, z, z) + 2B(z, z) + 2M(-z, -z, -z) + 2B(-z, -z) \\
& \quad + M(y+w, y+w, y+w) + B(y+w, y+w) + M(y-w, y-w, y-w) + B(y-w, y-w) \\
& \quad - 2M(y, y, y) - 2B(y, y) + 2M(w, w, w) + 2B(w, w) + 2M(-w, -w, -w) + 2B(-w, -w)
\end{aligned}$$

$$\begin{aligned}
&= g(x+z) + g(x-z) - 2g(x) + 2g(z) + 2g(-z) \\
&\quad + g(x+w) + g(x-w) - 2g(x) + 2g(w) + 2g(-w) \\
&\quad + g(y+z) + g(y-z) - 2g(y) + 2g(z) + 2g(-z) \\
&\quad + g(y+w) + g(y-w) - 2g(y) + 2g(w) + 2g(-w)
\end{aligned} \tag{4.5}$$

for all $x, y, z, w \in X$.

By (4.1) and the above equality, f satisfies (1.3). $\square$

THEOREM 4.2. *Let $f : X \times X \rightarrow Y$ be a mapping satisfying (1.3) and let $g : X \rightarrow Y$ be the mapping given by (4.2) for all $x, y \in X$. If f and g satisfy (4.1) for all $x, y \in X$, then g satisfies (1.2).*

Proof. Letting $x = y = z = w = 0$ in (1.3) and then using (4.2), we have $g(0) = 0$. Putting $y = x$ in (4.1) and then using (4.2), we obtain that g satisfies (4.3). By (1.3) and (4.1), we know that

$$\begin{aligned}
&g\left(\frac{x+y}{2} + z + w\right) + g\left(\frac{x+y}{2} - z - w\right) - 2g\left(\frac{x+y}{2}\right) + 2g(z+w) + 2g(-z-w) \\
&+ g\left(\frac{x+y}{2} + z - w\right) + g\left(\frac{x+y}{2} - z + w\right) - 2g\left(\frac{x+y}{2}\right) + 2g(z-w) + 2g(-z+w) \\
&= g(x+z) + g(x-z) - 2g(x) + 2g(z) + 2g(-z) \\
&\quad + g(x+w) + g(x-w) - 2g(x) + 2g(w) + 2g(-w) \\
&\quad + g(y+z) + g(y-z) - 2g(y) + 2g(z) + 2g(-z) \\
&\quad + g(y+w) + g(y-w) - 2g(y) + 2g(w) + 2g(-w)
\end{aligned} \tag{4.6}$$

for all $x, y, z, w \in X$. Replacing x, y, z , and w by $0, 0, x$, and y , respectively, in (4.6), we obtain that

$$g(x+y) + g(x-y) + g(-x+y) + g(-x-y) = 2[g(x) + g(-x) + g(y) + g(-y)] \tag{4.7}$$

for all $x, y \in X$. Replacing x, y, z , and w by $2x, 0, y$, and 0 , respectively, in (4.6), we see that, by (4.3),

$$2[g(x+y) + g(x-y) + 4g(x)] = g(2x+y) + g(2x-y) + g(y) + g(-y) + 4g(-x) \tag{4.8}$$

for all $x, y \in X$. From the above two equalities, we conclude that

$$3[g(x+y) + g(x-y) + 6g(x)] = 2g(2x+y) + 2g(2x-y) + g(-x-y) + g(-x+y) + 6g(-x) \tag{4.9}$$

for all $x, y \in X$. $\square$

References

- [1] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, USA, 2002.
- [2] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, vol. 34 of *Progress in Nonlinear Differential Equations and Their Applications*, Birkhäuser, Boston, Mass, USA, 1998.
- [3] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, Fla, USA, 2001.
- [4] Th. M. Rassias, *Functional Equations and Inequalities*, vol. 518 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.
- [5] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, vol. 31 of *Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, Cambridge, UK, 1989.
- [6] K.-W. Jun and H.-M. Kim, "The generalized Hyers-Ulam-Rassias stability of a cubic functional equation," *Journal of Mathematical Analysis and Applications*, vol. 274, no. 2, pp. 867–878, 2002.

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