Research Article

A Tauberian Theorem with a Generalized One-Sided Condition

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We prove a Tauberian theorem to recover moderate oscillation of a real sequence $u = (u_n)$ out of Abel limitability of the sequence $(V_n^{(1)}(\Delta u))$ and some additional condition on the general control modulo of oscillatory behavior of integer order of $u = (u_n)$.

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1. Introduction

Let $u = (u_n)$ be a sequence of real numbers. Throughout this paper the symbols $u_n = o(1)$ and $u_n = O(1)$ mean, respectively, that $u_n \rightarrow 0$ as $n \rightarrow \infty$ and that (u_n) is bounded for large enough n . Denote by $\omega_n^{(0)}(u) = n\Delta u_n$ the classical control modulo of the oscillatory behavior of (u_n) . For each integer $m \geq 1$ and for all nonnegative integer n , define by

$$\omega_n^{(m)}(u) = \omega_n^{(m-1)}(u) - \sigma_n(\omega^{(m-1)}(u)) \quad (1.1)$$

the general control modulo of the oscillatory behavior of order m . For a sequence $u = (u_n)$,

$$u_n - \sigma_n(u) = V_n^{(0)}(\Delta u), \quad n = 0, 1, 2, \dots, \quad (1.2)$$

where $\sigma_n(u) = (1/(n+1)) \sum_{k=0}^n u_k$, $V_n^{(0)}(\Delta u) = (1/(n+1)) \sum_{k=0}^n k \Delta u_k$, and

$$\Delta u_n = \begin{cases} u_n - u_{n-1}, & n \geq 1, \\ u_0, & n = 0. \end{cases} \quad (1.3)$$

For each integer $m \geq 1$ and for all nonnegative integer n , define

$$V_n^{(m)}(\Delta u) = \sigma_n(V^{(m-1)}(\Delta u)). \quad (1.4)$$

A sequence $u = (u_n)$ is said to be left one-sidedly bounded if $u_n \geq -C$ for all nonnegative integers n and for some $C \geq 0$. A sequence $u = (u_n)$ is said to be left one-sidedly bounded with respect to sequence (C_n) if $u_n \geq -C_n$ for all nonnegative integers n . A sequence (u_n) is said to be Abel limitable if the limit

$$\lim_{x \rightarrow 1^-} (1-x) \sum_{n=0}^{\infty} u_n x^n = A(u) \quad (1.5)$$

exists and is finite. A classical Tauberian theorem of Hardy and Littlewood [1] says that if $(\omega_n^{(0)}(u))$ is left one-sidedly bounded and (1.5) exists, then $\lim_n u_n = A(u)$. Dik [2] improved Hardy and Littlewood's theorem [1] by proving that if $(\omega_n^{(1)}(u))$ is left one-sidedly bounded and (1.5) exists, then $\lim_n u_n = A(u)$.

Č. V. Stanojević and V. B. Stanojević [3] proved the following theorem.

THEOREM 1.1. *For the real sequence $u = (u_n)$, let there exist a nonnegative sequence $M = (M_n)$ such that*

$$\left(\sum_{k=1}^n \frac{M_k}{k} \right) \text{ is slowly oscillating} \quad (1.6)$$

and $(\omega_n^{(2)}(u))$ is left one-sidedly bounded with respect to the sequence (M_n) . If

$$\lim_{x \rightarrow 1^-} (1-x) \sum_{n=0}^{\infty} V_n^{(1)}(\Delta u) x^n = A(V^{(1)}(\Delta u)) \quad (1.7)$$

exists, then $u = (u_n)$ is slowly oscillating.

We remind the reader that a sequence (u_n) is slowly oscillating [4] if

$$\lim_{\lambda \rightarrow 1^+} \limsup_n \max_{n+1 \leq k \leq [\lambda n]} \left| \sum_{j=n+1}^k \Delta u_j \right| = 0, \quad (1.8)$$

and more generally, it is moderately oscillating [4] if, for $\lambda > 1$,

$$\limsup_n \max_{n+1 \leq k \leq [\lambda n]} \left| \sum_{j=n+1}^k \Delta u_j \right| < \infty, \quad (1.9)$$

where $[\lambda n]$ denotes the integer part of λn .

An equivalent definition of slowly oscillating sequence (u_n) is given by Dik [2] in terms of $(V_n^{(0)}(\Delta u))$. A sequence $u = (u_n)$ is slowly oscillating if and only if $(V_n^{(0)}(\Delta u))$ is slowly oscillating and bounded. Clearly, (1.9) implies that $V_n^{(0)}(\Delta u) = O(1)$.

2. The main theorem

The main goal of this paper is to generalize Č. V. Stanojević and V. B. Stanojević's [3] result for the general control modulo of the oscillatory behavior of the order m , where m is any integer greater than or equal to 1.

THEOREM 2.1. *For the real sequence $u = (u_n)$, let there exist a nonnegative sequence $M = (M_n)$ such that*

$$\left(\sum_{k=1}^n \frac{M_k}{k} \right) \text{ is moderately oscillating} \quad (2.1)$$

and for some integer $m \geq 1$,

$$(\omega_n^{(m)}(u)) \text{ is left one-sidedly bounded with respect to the sequence } (M_n). \quad (2.2)$$

If (1.7) exists, then $u = (u_n)$ is moderately oscillating.

There would be some cases that for some integer $m \geq 1$, $(\omega_n^{(m)}(u))$ is not left one-sidedly bounded with respect to any nonnegative sequence (M_n) with the property (1.6). In this case, we cannot get any information related to the asymptotic behavior of the sequence (u_n) out of (2.2) and (1.5). But for an integer k greater than m , $(\sigma_n(\omega^{(k)}(u)))$ could be left one-sidedly bounded with respect to some nonnegative sequence (M_n) with the property (1.6) as provided in the following example.

Example 2.2. For the sequence (u_n) defined by

$$u_n = \begin{cases} 1, & n = 2^j, j = 1, 2, 3, \dots, \\ 0, & \text{for other values of } n, \end{cases} \quad (2.3)$$

we have

$$n\Delta u_n = \begin{cases} j, & n = 2^j, j = 1, 2, 3, \dots, \\ -j, & n = 2^j + 1, j = 1, 2, 3, \dots, \\ 0, & \text{for other values of } n. \end{cases} \quad (2.4)$$

Since the sequence (u_n) has two subsequences $((u_{2^n})$ and $(u_{2^n+1}))$ converging to different values (1 and 0, resp.), (u_n) does not converge. Consider the series $\sum_{n=1}^{\infty} \Delta u_n x^n$. We may rewrite this series as $f(\Delta u, x) = \sum_{n=1}^{\infty} (x^{2^n} - x^{2^n+1})$. Notice that if $0 \leq x < 1$, then $f(\Delta u, x) \geq 0$. Hence, it follows that

$$\liminf_{x \rightarrow 1^-} f(\Delta u, x) \geq 0. \quad (2.5)$$

Also, observe that from the rewritten form of $f(\Delta u, x)$, we have

$$f(\Delta u, x) = (1-x) \sum_{n=1}^{\infty} x^{2^n} \leq (1-x) \left(x^2 + x^4 + x^8 + C \left(\sqrt{\ln \left(\frac{1}{x} \right)} \right)^{-1} \right). \quad (2.6)$$

Since $\ln(1/x) \sim 1 - x$ as $x \rightarrow 1^-$, we have

$$\limsup_{x \rightarrow 1^-} f(\Delta u, x) \leq 0. \quad (2.7)$$

From (2.5) and (2.7), it follows that (u_n) is Abel limitable to zero.

It is clear that $(\omega_n^{(0)}(u))$ is not left one-sidedly bounded with respect to any nonnegative sequence (M_n) with the property (1.6). Indeed, there were such a nonnegative sequence (M_n) with the property (1.6), we would have $-1 = \liminf_n \Delta u_n \geq -\lim_n (M_n/n) = 0$. We also note that for any integer $m \geq 1$, $(\omega_n^{(m)}(u))$ is not left one-sidedly bounded with respect to any nonnegative sequence (M_n) with the property (1.6). If $(\omega_n^{(m)}(u))$ is not left one-sidedly bounded with respect to any sequence (M_n) with the property (1.6) and $A(u)$ exists, then $(\omega_n^{(m+1)}(u))$ is not left one-sidedly bounded with respect to the nonnegative sequence (M_n) with the property (1.6). Suppose that $(\omega_n^{(m+1)}(u))$ is left one-sidedly bounded with respect to any nonnegative sequence (M_n) with the property (1.6) and $A(u)$ exists. Then by Corollary 2.9, the sequence (u_n) converges and this implies that $(\omega_n^{(m)}(u))$ is left one-sidedly bounded with respect to some nonnegative sequence (M_n) with the property (1.6), which is contrary to the fact that $(\omega_n^{(m)}(u))$ is not left one-sidedly bounded with respect to any nonnegative sequence (M_n) with the property (1.6).

Since $(V_n^{(0)}(\Delta u))$ is bounded, then $V_n^{(0)}(\Delta u) \geq -C$ and

$$\omega_n^{(1)}(\sigma(u)) = n\Delta V_n^{(0)}(\Delta \sigma(u)) = n\Delta V_n^{(1)}(\Delta u) \geq -C \quad (2.8)$$

for some $C \geq 0$. Since $(\sigma_n(u))$ is Abel limitable, by Corollary 2.9, we obtain that $(\sigma_n(u))$ converges.

Remark 2.3. The condition $\omega_n^{(0)}(u) \geq -M_n$ with the properties (1.6) and (2.2) is a Tauberian condition for Abel limitable method, but $\sigma_n(\omega^{(0)}(u)) \geq -M_n$ is not. However,

$$\omega_n^{(1)}(u) = \omega_n^{(0)}(u) - \sigma_n(\omega^{(0)}(u)) \geq -M_n \quad (2.9)$$

is a Tauberian condition for Abel limitable method as proved in Theorem 2.1.

If (u_n) is slowly oscillating or moderately oscillating in the sense of Stanojević [4], then $(V_n^{(0)}(\Delta u))$ is bounded. Hence, for any integer $m \geq 1$, $(\sigma_n(\omega^{(m)}(u)))$ is left one-sidedly bounded with respect to the constant sequence $(M_n) = (C)$. Also, from the definition of slow oscillation, one obtains that the arithmetic means of $(\omega_n^{(m)}(u))$ is slowly oscillating. But, that the converse is not true is provided by example.

We need the following identities and observations for the proof of Theorem 2.1.

LEMMA 2.4 [2, 4]. (i) For $\lambda > 1$,

$$u_n - \sigma_n(u) = \frac{[\lambda n] + 1}{[\lambda n] - n} (\sigma_{[\lambda n]}(u) - \sigma_n(u)) - \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} \sum_{j=n+1}^k \Delta u_j, \quad (2.10)$$

where $[\lambda n]$ denotes the integer part of λn .

(i) For $1 < \lambda < 2$,

$$\begin{aligned} u_n - \sigma_{n-[(\lambda-1)n]-1}(u) &= \frac{n+1}{[(\lambda-1)n]+1} (\sigma_{n-[(\lambda-1)n]-1}(u) - \sigma_n(u)) \\ &+ \frac{1}{[(\lambda-1)n]+1} \sum_{k=n-[(\lambda-1)n]}^n \sum_{j=k+1}^n \Delta u_j, \end{aligned} \quad (2.11)$$

where $[\lambda n]$ denotes the integer part of λn .

Proof. (i) For $\lambda > 1$, define

$$\tau_{n, [\lambda n]}(u) = \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} u_k. \quad (2.12)$$

The difference $\tau_{n, [\lambda n]}(u) - \sigma_n(u)$ can be written as

$$\begin{aligned} \tau_{n, [\lambda n]}(u) - \sigma_n(u) &= \frac{([\lambda n] + 1)\sigma_{[\lambda n]}(u) - (n+1)\sigma_n(u)}{[\lambda n] - n} - \sigma_n(u) \\ &= \frac{[\lambda n] + 1}{[\lambda n] - n} (\sigma_{[\lambda n]}(u) - \sigma_n(u)). \end{aligned} \quad (2.13)$$

This completes the proof. □

(ii) Proof of Lemma 2.4(ii) is similar to that of Lemma 2.4(i).

For a sequence $u = (u_n)$, we define

$$(n\Delta)_m u_n = (n\Delta)_{m-1}((n\Delta)u_n) = n\Delta((n\Delta)_{m-1}u_n), \quad m = 1, 2, \dots, \quad (2.14)$$

where $(n\Delta)_0 u_n = u_n$ and $(n\Delta)_1 u_n = n\Delta u_n$.

LEMMA 2.5. For each integer $m \geq 1$,

$$\omega_n^{(m)}(u) = (n\Delta)_m V_n^{(m-1)}(\Delta u). \quad (2.15)$$

Proof. We do the proof by induction. By definition, for $m = 1$, we have

$$\omega_n^{(1)}(u) = \omega_n^{(0)}(u) - \sigma_n(\omega_n^{(0)}(u)) = n\Delta u_n - V_n^{(0)}(\Delta u) = n\Delta V_n^{(0)}(\Delta u). \quad (2.16)$$

Assume the observation is true for $m = k$. That is, assume that

$$\omega_n^{(k)}(u) = (n\Delta)_k V_n^{(k-1)}(\Delta u). \quad (2.17)$$

We must show that the observation is true for $m = k + 1$. That is, we must show that

$$\omega_n^{(k+1)}(u) = (n\Delta)_{k+1} V_n^{(k)}(\Delta u). \quad (2.18)$$

Again by definition,

$$\omega_n^{(k+1)}(u) = \omega_n^{(k)}(u) - \sigma_n(\omega_n^{(k)}(u)). \quad (2.19)$$

By (2.17),

$$\begin{aligned}
 \omega_n^{(k+1)}(u) &= (n\Delta)_k V_n^{(k-1)}(\Delta u) - (n\Delta)_k V_n^{(k)}(\Delta u) \\
 &= (n\Delta)_k (V_n^{(k-1)}(\Delta u) - V_n^{(k)}(\Delta u)) \\
 &= (n\Delta)_k ((n\Delta) V_n^{(k)}(\Delta u)) \\
 &= (n\Delta)_{k+1} V_n^{(k)}(\Delta u).
 \end{aligned} \tag{2.20}$$

Thus, we conclude that Lemma 2.5 is true for every positive integer m . $\square$

LEMMA 2.6. For each integer $m \geq 1$,

$$\omega_n^{(m)}(u) = \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} n\Delta V_n^{(j)}(\Delta u), \tag{2.21}$$

where $\binom{m-1}{j} = (m-1)(m-2) \cdots (m-j+1)/j!$.

Proof. We do the proof by induction. For $m = 1$, we have

$$\begin{aligned}
 \omega_n^{(1)}(u) &= n\Delta u_n - V_n^{(0)}(\Delta u) \\
 &= n\Delta V_n^{(0)}(\Delta u) \\
 &= \sum_{j=0}^0 (-1)^j \binom{0}{j} n\Delta V_n^{(j)}(\Delta u).
 \end{aligned} \tag{2.22}$$

Assume the observation is true for $m = k$. That is, assume that

$$\omega_n^{(k)}(u) = \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} n\Delta V_n^{(j)}(\Delta u). \tag{2.23}$$

We must show that the observation is true for $m = k + 1$. That is, we must show that

$$\omega_n^{(k+1)}(u) = \sum_{j=0}^k (-1)^j \binom{k}{j} n\Delta V_n^{(j)}(\Delta u). \tag{2.24}$$

By definition,

$$\omega_n^{(k+1)}(u) = \omega_n^{(k)}(u) - \sigma_n(\omega^{(k)}(u)). \tag{2.25}$$

By (2.23),

$$\begin{aligned}
 \omega_n^{(k+1)}(u) &= \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} n\Delta V_n^{(j)}(\Delta u) \\
 &\quad - \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} n\Delta V_n^{(j+1)}(\Delta u).
 \end{aligned} \tag{2.26}$$

Let $j + 1 = i$ in the second sum. Using this substitution,

$$\begin{aligned}\omega_n^{(k+1)}(u) &= \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} n\Delta V_n^{(j)}(\Delta u) \\ &\quad + \sum_{i=1}^k (-1)^i \binom{k-1}{i-1} n\Delta V_n^{(i)}(\Delta u).\end{aligned}\tag{2.27}$$

In the second sum of (2.27), we rename the index of summation j , split the first term off in the first sum and the last term in the second sum of (2.27), we have

$$\begin{aligned}\omega_n^{(k+1)}(u) &= (-1)^0 \binom{k-1}{0} n\Delta V_n^{(0)}(\Delta u) + \sum_{j=1}^{k-1} (-1)^j \binom{k-1}{j} n\Delta V_n^{(j)}(\Delta u) \\ &\quad + \sum_{j=1}^{k-1} (-1)^j \binom{k-1}{j-1} n\Delta V_n^{(j)}(\Delta u) + (-1)^k \binom{k-1}{k-1} n\Delta V_n^{(k)}(\Delta u).\end{aligned}\tag{2.28}$$

Rewritten (2.28), we have

$$\begin{aligned}\omega_n^{(k+1)}(u) &= (-1)^0 \binom{k-1}{0} n\Delta V_n^{(0)}(\Delta u) \\ &\quad + \sum_{j=1}^{k-1} (-1)^j \left[\binom{k-1}{j} + \binom{k-1}{j-1} \right] n\Delta V_n^{(j)}(\Delta u) \\ &\quad + (-1)^k \binom{k-1}{k-1} n\Delta V_n^{(k)}(\Delta u).\end{aligned}\tag{2.29}$$

Since $\binom{k-1}{j} + \binom{k-1}{j-1} = \binom{k}{j}$, the last identity can be written

$$\begin{aligned}\omega_n^{(k+1)}(u) &= (-1)^0 \binom{k-1}{0} n\Delta V_n^{(0)}(\Delta u) \\ &\quad + \sum_{j=1}^{k-1} (-1)^j \binom{k}{j} n\Delta V_n^{(j)}(\Delta u) \\ &\quad + (-1)^k \binom{k-1}{k-1} n\Delta V_n^{(k)}(\Delta u) \\ &= \sum_{j=1}^k (-1)^j \binom{k}{j} n\Delta V_n^{(j)}(\Delta u).\end{aligned}\tag{2.30}$$

Thus, we conclude that Lemma 2.6 is true for every positive integer m . $\square$

Corollary 2.7 is an improved version of the main theorem in [3, Theorem 3.1]. Corollaries 2.8 and 2.9 are analogous to classical Tauberian theorems.

COROLLARY 2.7. *For the real sequence $u = (u_n)$, let there exist a nonnegative sequence $M = (M_n)$ such that $(\sum_{k=1}^n (M_k/k))$ is slowly oscillating and condition (2.2) is satisfied. If (1.7) exists, then $u = (u_n)$ is slowly oscillating.*

Proof. Since slow oscillation of $(\sum_{k=1}^n (M_k/k))$ implies that $\sigma_n(M) = O(1)$, we have

$$\begin{aligned} \lim_n V_n^{(1)}(\Delta u) &= A(V^{(1)}(\Delta u)), \\ n\Delta V_n^{(0)}(\Delta u) &\geq -(M_n + C) \end{aligned} \quad (2.31)$$

for some constant C as in Theorem 2.1. Applying Lemma 2.4(i) to $(V_n^{(0)}(\Delta u))$ and noticing that

$$n\Delta V_n^{(0)}(\Delta u) \geq -(M_n + C) \quad (2.32)$$

for some constant C , we have

$$\begin{aligned} V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u) &\leq \frac{[\lambda n] + 1}{[\lambda n] - n} (V_{[\lambda n]}^{(1)}(\Delta u) - V_n^{(1)}(\Delta u)) \\ &\quad + \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} \sum_{j=n+1}^k \frac{M_j + C}{j} \\ &\leq \frac{[\lambda n] + 1}{[\lambda n] - n} (V_{[\lambda n]}^{(1)}(\Delta u) - V_n^{(1)}(\Delta u)) \\ &\quad + \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} \sum_{j=n+1}^k \frac{M_j}{j} + C_1 \log\left(\frac{[\lambda n]}{n}\right) \end{aligned} \quad (2.33)$$

for some constant C_1 . From the last inequality, we have

$$\begin{aligned} V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u) &\leq \frac{[\lambda n] + 1}{[\lambda n] - n} (V_{[\lambda n]}^{(1)}(\Delta u) - V_n^{(1)}(\Delta u)) \\ &\quad + \max_{n+1 \leq k \leq [\lambda n]} \sum_{j=n+1}^k \frac{M_j}{j} + C_1 \log\left(\frac{[\lambda n]}{n}\right). \end{aligned} \quad (2.34)$$

Taking $\limsup$ of both sides, we have

$$\begin{aligned} \limsup_n (V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u)) &\leq \frac{\lambda}{\lambda - 1} \limsup_n (V_{[\lambda n]}^{(1)}(\Delta u) - V_n^{(1)}(\Delta u)) \\ &\quad + \limsup_n \max_{n+1 \leq k \leq [\lambda n]} \sum_{j=n+1}^k \frac{M_j}{j} + C_1 \log \lambda. \end{aligned} \quad (2.35)$$

Since the first term on the right-hand side of the inequality above vanishes,

$$\limsup_n (V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u)) \leq \limsup_n \max_{n+1 \leq k \leq [\lambda n]} \sum_{j=n+1}^k \frac{M_j}{j} + C_1 \log \lambda. \quad (2.36)$$

Taking the limit of both sides as $\lambda \rightarrow 1^+$, we obtain

$$\limsup_n (V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u)) \leq 0. \quad (2.37)$$

In a similar way from Lemma 2.4(ii), we have

$$\liminf_n (V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u)) \geq 0. \quad (2.38)$$

From (2.37) and (2.38), we have

$$\lim_n V_n^{(0)}(\Delta u) = \lim_n V_n^{(1)}(\Delta u). \quad (2.39)$$

Since

$$\sigma_n(u) = u_0 + \sum_{k=1}^n \frac{V_k^{(0)}(\Delta u)}{k}, \quad (2.40)$$

from identity (1.2), we can write (u_n) as

$$u_n = V_n^{(0)}(\Delta u) + \sum_{k=1}^n \frac{V_k^{(0)}(\Delta u)}{k} + u_0. \quad (2.41)$$

Thus, $u = (u_n)$ is slowly oscillating. $\square$

It should be noted that if we take $m = 2$ in Corollary 2.7, we get Č. V. Stanojević and V. B. Stanojević's result.

COROLLARY 2.8. *For the real sequence $u = (u_n)$, let there exist a nonnegative sequence $M = (M_n)$ such that $(\sum_{k=1}^n (M_k/k))$ is slowly oscillating and condition (2.2) is satisfied. If $A(\sigma(u))$ exists, then $\lim_n u_n = A(\sigma(u))$.*

Proof. Existence of the limit $A(\sigma(u))$ implies that $A(V^{(1)}(\Delta u)) = 0$. By Corollary 2.7, we have $V_n^{(0)}(\Delta u) = o(1)$ and hence $A(V^{(0)}(\Delta u)) = 0$. From (1.2), it follows that $A(u) = 0$. By Tauber's second theorem [5], $\lim_n u_n = A(\sigma(u))$. $\square$

COROLLARY 2.9. *For the real sequence $u = (u_n)$, let there exist a nonnegative sequence $M = (M_n)$ such that $(\sum_{k=1}^n (M_k/k))$ is slowly oscillating and condition (2.2) is satisfied. If (1.5) exist, then $\lim_n u_n = A(u)$.*

Proof. Since existence of $A(u)$ implies that of $A(\sigma(u))$, proof follows from Corollary 2.8. $\square$

Corollary 2.9 with $m = 3$ follows from Corollary 2.9 with $m = 2$. Indeed, we have for a sequence $u = (u_n)$,

$$\begin{aligned} \omega_n^{(3)}(u) &= (n\Delta)_3 V_n^{(2)}(\Delta u) = (n\Delta)_2 (n\Delta V_n^{(2)}(\Delta u)) \\ &= (n\Delta)_2 (V_n^{(1)}(\Delta u) - V_n^{(2)}(\Delta u)) \\ &= (n\Delta)_2 (V_n^{(0)}(\Delta V^{(1)}(\Delta u))). \end{aligned} \quad (2.42)$$

We note that for a sequence $u = (u_n)$,

$$V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u) = V_n^{(0)}(\Delta V^{(0)}(\Delta u)). \quad (2.43)$$

Taking the arithmetic means of both sides of (2.43), we have

$$V_n^{(1)}(\Delta u) - V_n^{(2)}(\Delta u) = V_n^{(1)}(\Delta V^{(0)}(\Delta u)). \quad (2.44)$$

Using (1.2), the identity (2.44) can be expressed as

$$V_n^{(1)}(\Delta u) - V_n^{(2)}(\Delta u) = V_n^{(0)}(\Delta V^{(1)}(\Delta u)). \quad (2.45)$$

From (2.44) and (2.45), we have

$$V_n^{(1)}(\Delta V^{(0)}(\Delta u)) = V_n^{(0)}(\Delta V^{(1)}(\Delta u)). \quad (2.46)$$

We have, by (2.45) and (2.42),

$$\omega_n^{(3)}(u) = (n\Delta)_2(V_n^{(1)}(\Delta V^{(0)}(\Delta u))). \quad (2.47)$$

Existence of the limit $A(u)$ implies that $A(V^{(0)}(\Delta u)) = 0$. By Corollary 2.9 with $m = 2$, we obtain that $V_n^{(0)}(\Delta u) = o(1)$. By Tauber's second theorem [5], $\lim_n u_n = A(u)$.

3. Proof of Theorem 2.1

Proof. From the condition (2.1), it follows that $\sigma_n(M) = O(1)$. Taking the arithmetic mean of both sides of (2.2), we obtain

$$\sigma_n(\omega^{(m)}(u)) = (n\Delta)_m V_n^{(m)}(\Delta u) \geq -\sigma_n(M) \geq -C_0 \quad (3.1)$$

for some constant C_0 . By the existence of the limit (1.7),

$$\lim_{x \rightarrow 1^-} (1-x) \sum_{n=0}^{\infty} (n\Delta)_{m-1} V_n^{(m)}(\Delta u) x^n = 0. \quad (3.2)$$

Since

$$n\Delta((n\Delta)_{m-1} V_n^{(m)}(\Delta u)) \geq -C_0, \quad (3.3)$$

by Hardy and Littlewood's theorem [1],

$$(n\Delta)_{m-1} V_n^{(m)}(\Delta u) = o(1). \quad (3.4)$$

From

$$n\Delta((n\Delta)_{m-1} V_n^{(m)}(\Delta u)) = (n\Delta)_{m-1} V_n^{(m-1)}(\Delta u) - (n\Delta)_{m-1} V_n^{(m)}(\Delta u) \geq -C_0 \quad (3.5)$$

and (3.4), it follows that

$$(n\Delta)_{m-1} V_n^{(m-1)}(\Delta u) \geq -C_1 \quad (3.6)$$

for some constant C_1 . The existence of the limit (1.7) implies that

$$\lim_{x \rightarrow 1^-} (1-x) \sum_{n=0}^{\infty} (n\Delta)_{m-2} V_n^{(m-1)}(\Delta u) x^n = 0. \quad (3.7)$$

Since

$$n\Delta((n\Delta)_{m-2}V_n^{(m-1)}(\Delta u)) \geq -C_1, \quad (3.8)$$

again by Hardy and Littlewood's theorem [1],

$$(n\Delta)_{m-2}V_n^{(m-1)}(\Delta u) = o(1). \quad (3.9)$$

Continuing in this way, in $(m-2)$ th step we get

$$(n\Delta)_2V_n^{(2)}(\Delta u) \geq -C_{m-3} \quad (3.10)$$

for some constant C_{m-3} . Since

$$\lim_{x \rightarrow 1^-} (1-x) \sum_{n=0}^{\infty} n\Delta V_n^{(2)}(\Delta u) x^n = 0, \quad (3.11)$$

we get

$$n\Delta V_n^{(2)}(\Delta u) = o(1). \quad (3.12)$$

From

$$n\Delta(n\Delta V_n^{(2)}(\Delta u)) = n\Delta V_n^{(1)}(\Delta u) - n\Delta V_n^{(2)}(\Delta u) \geq -C_{m-3} \quad (3.13)$$

and (3.12) we get

$$n\Delta V_n^{(1)}(\Delta u) \geq -C_{m-2} \quad (3.14)$$

for some constant C_{m-2} . By the existence of the limit (1.7), we obtain that $(V_n^{(1)}(\Delta u))$ converges to $A(V^{(1)}(\Delta u))$. From Lemma 2.6, convergence of $(V_n^{(1)}(\Delta u))$, and condition (2.2), it follows that

$$n\Delta V_n^{(0)}(\Delta u) \geq -(M_n + C) \quad (3.15)$$

for some constant C . Applying Lemma 2.4(i) and (ii) to $(V_n^{(0)}(\Delta u))$, we have $V_n^{(0)}(\Delta u) = O(1)$. Thus, $u = (u_n)$ is moderately oscillating. $\square$

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