Research Article

Isomorphisms and Derivations in Lie C*-Algebras

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We investigate isomorphisms between C*-algebras, Lie C*-algebras, and JC*-algebras, and derivations on C*-algebras, Lie C*-algebras, and JC*-algebras associated with the Cauchy–Jensen functional equation 2f((x + y/2) + z) = f(x) + f(y) + 2f(z).

1. Introduction and preliminaries

Ulam [1] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems, containing the stability problem of homomorphisms. Hyers [2] proved the stability problem of additive mappings in Banach spaces. Rassias [3] provided a generalization of Hyers’ theorem which allows the Cauchy difference to be unbounded:

Let f : E → E' be a mapping from a normed vector space E into a Banach space E' subject to the inequality

\[ \| f(x + y) - f(x) - f(y) \| \leq \epsilon (\| x \|^p + \| y \|^p) \]  \hspace{1cm} (1.1)

for all x, y ∈ E, where ε and p are constants with ε > 0 and p < 1. The inequality (1.1) that was introduced by Rassias [3] provided a lot of influence in the development of a generalization of the Hyers-Ulam stability concept. This new concept is known as Hyers-Ulam-Rassias stability of functional equations. Gavruta [4] provided a further generalization of Th. M. Rassias’ theorem. Several mathematicians have contributed works on these subjects (see [4–14]).

Rassias [15] provided an alternative generalization of Hyers’ stability theorem which allows the Cauchy difference to be unbounded, as follows.
Theorem 1.1. Let $f : E \to E'$ be a mapping from a normed vector space $E$ into a Banach space $E'$ subject to the inequality

$$
\| f(x + y) - f(x) - f(y) \| \leq \epsilon \|x\|^p \|y\|^p
$$

(1.2)

for all $x, y \in E$, where $\epsilon$ and $p$ are constants with $\epsilon > 0$ and $0 \leq p < 1/2$. Then the limit

$$
L(x) = \lim_{n \to \infty} \frac{f(2^nx)}{2^n}
$$

(1.3)

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$
\| f(x) - L(x) \| \leq \frac{\epsilon}{2 - 4p} \|x\|^{2p}
$$

(1.4)

for all $x \in E$. If $p < 0$, then inequality (1.2) holds for $x, y \neq 0$, and (1.4) for $x \neq 0$. If $p > 1/2$, then inequality (1.2) holds for all $x, y \in E$, and the limit

$$
A(x) = \lim_{n \to \infty} 2^n f \left( \frac{x}{2^n} \right)
$$

(1.5)

exists for all $x \in E$ and $A : E \to E'$ is the unique additive mapping which satisfies

$$
\| f(x) - A(x) \| \leq \frac{\epsilon}{4^p - 2} \|x\|^{2p}
$$

(1.6)

for all $x \in E$.

In 1982–1994, a generalization of this result was established by J. M. Rassias with a weaker (unbounded) condition controlled by (or involving) a product of different powers of norms. However, there was a singular case. Then for this singularity, a counterexample was given by Găvruta [16]. The above-mentioned stability involving a product of different powers of norms is called Ulam-Găvruta-Rassias stability by Sibaha et al. [17] and Ravi and Arunkumar[18]. This stability is called Hyers-Ulam-Rassias stability involving a product of different powers of norms by Park [10]. Note that both Ulam stabilities specifically called: “Ulam-Găvruta-Rassias stability of mappings” and “Hyers-Ulam-Rassias stability of mappings involving a product of powers of norms are identical in meaning stability notions. Besides Euler-Lagrange quadratic mappings were introduced by Rassias [19], motivated from the pertinent algebraic quadratic equation. Thus he introduced and investigated the relative quadratic functional equation [20, 21]. In addition, he generalized and investigated the general pertinent Euler-Lagrange quadratic mappings [22]. Analogous quadratic mappings were introduced and investigated by the same author [23, 24]. Therefore, this introduction of Euler-Lagrange mappings and equations in functional equations and inequalities provided an interesting cornerstone in analysis, because this kind of Euler-Lagrange-Rassias mappings (resp., Euler-Lagrange-Rassias equations) is of particular interest in probability theory and stochastic analysis by marrying these fields of research results to functional equations and inequalities via the introduction of new Euler-Lagrange-Rassias quadratic weighted means and Euler-Lagrange-Rassias fundamental mean equations [21, 22, 25]. For further research developments in
stability of functional equations, the readers are referred to the works of Park [6–13], Rassias [15, 19–24, 26–36], J. M. Rassias and M. J. Rassias [25, 37–39], Rassias [40–43], Skof [44], and the references cited therein.

Gilányi [45] showed that if $f$ satisfies the functional inequality
\[
\|2f(x) + 2f(y) - f(x - y)\| \leq \|f(x + y)\|, \tag{1.7}
\]
then $f$ satisfies the Jordan-von Neumann functional inequality
\[
2f(x) + 2f(y) = f(x + y) + f(x - y) \tag{1.8}
\]

Jordan observed that $L(\mathcal{H})$ is a (nonassociative) algebra via the anticommutator product $x \circ y := (xy + yx)/2$. A commutative algebra $X$ with product $x \circ y$ is called a Jordan algebra. A Jordan $C^*$-subalgebra of a $C^*$-algebra, endowed with the anticommutator product, is called a $JC^*$-algebra. A $C^*$-algebra $\mathcal{C}$, endowed with the Lie product $[x, y] = (xy - yx)/2$ on $\mathcal{C}$, is called a Lie $C^*$-algebra (see [6, 7, 13]).

This paper is organized as follows. In Section 2, we investigate isomorphisms and derivations in $C^*$-algebras associated with the Cauchy-Jensen functional equation. In Section 3, we investigate isomorphisms and derivations in $JC^*$-algebras associated with the Cauchy-Jensen functional equation. In Section 4, we investigate isomorphisms and derivations in $JC^*$-algebras associated with the Cauchy-Jensen functional equation.

2. Isomorphisms and derivations in $C^*$-algebras

Throughout this section, assume that $A$ is a $C^*$-algebra with norm $\| \cdot \|_A$, and that $B$ is a $C^*$-algebra with norm $\| \cdot \|_B$.

**Lemma 2.1** [11]. Let $f : A \to B$ be a mapping such that
\[
\|f(x) + f(y) + 2f(z)\|_B \leq \bigg\|2f \left( \frac{x+y}{2} + z \right) \bigg\|_B \tag{2.1}
\]
for all $x, y, z \in A$. Then $f$ is Cauchy additive, that is, $f(x + y) = f(x) + f(y)$.

In this section, we investigate $C^*$-algebra isomorphisms between $C^*$-algebras and linear derivations on $C^*$-algebras associated with the Cauchy-Jensen functional equation.

**Theorem 2.2.** Let $r > 1$ and $\theta$ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping such that
\[
\|\mu f(x) + f(y) + 2f(z)\|_B \leq \bigg\|2f \left( \frac{\mu x + y}{2} + z \right) \bigg\|_B, \tag{2.2}
\]
\[
\|f(xy) - f(x)f(y)\|_B \leq \theta(\|x\|_B^2 r + \|y\|_B^2 r), \tag{2.3}
\]
\[
\|f(x^*) - f(x)^*\|_B \leq \theta(\|x\|_A^r + \|x\|_A^r) \tag{2.4}
\]
for all \( \mu \in \mathbb{T}^1 := \{ \lambda \in \mathbb{C} \mid |\lambda| = 1 \} \) and all \( x, y, z \in A \). Then the mapping \( f : A \to B \) is a \( C^* \)-algebra isomorphism.

**Proof.** Let \( \mu = 1 \) in (2.2). By Lemma 2.1, the mapping \( f : A \to B \) is Cauchy additive. So \( f(0) = 0 \) and \( f(x) = \lim_{n \to \infty} 2^n f(x/2^n) \) for all \( x \in A \).

Letting \( y = -\mu x \) and \( z = 0 \), we get

\[
||\mu f(x) + f(-\mu x)||_B \leq ||2f(0)||_B = 0
\]  
(2.5)

for all \( x \in A \) and all \( \mu \in \mathbb{T}^1 \). So

\[
\mu f(x) - f(\mu x) = \mu f(x) + f(-\mu x) = 0
\]  
(2.6)

for all \( x \in A \) and all \( \mu \in \mathbb{T}^1 \). Hence \( f(\mu x) = \mu f(x) \) for all \( x \in A \) and all \( \mu \in \mathbb{T}^1 \). By the same reasoning as in the proof of [8, Theorem 2.1], the mapping \( f : A \to B \) is \( \mathbb{C} \)-linear.

It follows from (2.3) that

\[
||f(xy) - f(x)f(y)||_B = \lim_{n \to \infty} 4^n \left[ \left| f\left(\frac{xy}{2^n \cdot 2^n}\right) - f\left(\frac{x}{2^n}\right) f\left(\frac{y}{2^n}\right) \right|_B \right]
\]  
(2.7)

\[
\leq \lim_{n \to \infty} 4^n \theta \left( \|x\|_A^{2r} + \|y\|_A^{2r} \right) = 0
\]

for all \( x, y \in A \). Thus

\[
f(xy) = f(x)f(y)
\]  
(2.8)

for all \( x, y \in A \).

It follows from (2.4) that

\[
||f(x^*) - f(x)^*||_B = \lim_{n \to \infty} 2^n \left[ \left| f\left(\frac{x^*}{2^n}\right) - f\left(\frac{x}{2^n}\right)^* \right|_B \right]
\]  
(2.9)

\[
\leq \lim_{n \to \infty} \frac{2^n \theta}{2nr} \left( \|x\|_A^r + \|x\|_A^r \right) = 0
\]

for all \( x \in A \). Thus

\[
f(x^*) = f(x)^*
\]  
(2.10)

for all \( x \in A \). Hence the bijective mapping \( f : A \to B \) is a \( C^* \)-algebra isomorphism. \( \square \)
Theorem 2.3. Let $r < 1$ and $\theta$ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping satisfying (2.2), (2.3), and (2.4). Then the mapping $f : A \to B$ is a $C^*$-algebra isomorphism.

Proof. The proof is similar to the proof of Theorem 2.2.

\[ \| f(xy) - f(x)f(y) \|_A \leq \theta (\| x \|_A^r + \| y \|_A^r) \] (2.11)

for all $x, y \in A$. Then the mapping $f : A \to A$ is a linear derivation.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to A$ is $\mathbb{C}$-linear.

It follows from (2.11) that
\[ \| f(xy) - f(x)y - xf(y) \|_A = \lim_{n \to \infty} 4^n \left\| f \left( \frac{xy}{4^n} \right) - f \left( \frac{x}{2^n} \right) \frac{y}{2^n} - f \left( \frac{y}{2^n} \right) \right\|_A \]
\[ \leq \lim_{n \to \infty} \frac{4^n \theta}{4^{nr}} (\| x \|_A^r + \| y \|_A^r) = 0 \] (2.12)

for all $x, y \in A$. So
\[ f(xy) = f(x)y + xf(y) \] (2.13)

for all $x, y \in A$. Thus the mapping $f : A \to A$ is a linear derivation.

\[ \| f(x^*) - f(x)^* \|_B \leq \theta \cdot \| x \|_A^r \cdot \| x \|_A^r \] (2.14)

for all $\mu \in \mathbb{T}$ and all $x, y \in A$. Then the mapping $f : A \to B$ is a $C^*$-algebra isomorphism.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to B$ is $\mathbb{C}$-linear.

It follows from (2.14) that
\[ \| f(xy) - f(x)f(y) \|_B = \lim_{n \to \infty} 4^n \left\| f \left( \frac{xy}{2^n} \cdot 2^n \right) - f \left( \frac{x}{2^n} \right) f \left( \frac{y}{2^n} \right) \right\|_B \]
\[ \leq \lim_{n \to \infty} \frac{4^n \theta}{4^{nr}} \cdot \| x \|_A^r \cdot \| y \|_A^r = 0 \] (2.16)
for all \( x, y \in A \). Thus

\[
 f(xy) = f(x)f(y)
\]  

for all \( x, y \in A \).

It follows from (2.15) that

\[
 \| f(x^*) - f(x)^* \|_B = \lim_{n \to \infty} 2^n \left\| f\left( \frac{x^*}{2^n} \right) - f\left( \frac{x}{2^n} \right)^* \right\|_B
\]

\[
 \leq \lim_{n \to \infty} \frac{2^n \theta}{2^{nr}} \cdot \| x \|_A^{r/2} \cdot \| y \|_A^{r/2} = 0
\]

for all \( x \in A \). Thus

\[
 f(x^*) = f(x)^*
\]

for all \( x \in A \). Hence the bijective mapping \( f : A \to B \) is a \( C^* \)-algebra isomorphism.

**Theorem 2.7.** Let \( r < 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : A \to B \) be a bijective mapping satisfying (2.2), (2.14), and (2.15). Then the mapping \( f : A \to B \) is a \( C^* \)-algebra isomorphism.

**Proof.** The proof is similar to the proofs of Theorems 2.2 and 2.6.

**Theorem 2.8.** Let \( r > 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : A \to A \) be a mapping satisfying (2.2) such that

\[
 \| f(xy) - f(x)y - xf(y) \|_A \leq \theta \cdot \| x \|_A^r \cdot \| y \|_A^r
\]

for all \( x, y \in A \). Then the mapping \( f : A \to A \) is a linear derivation.

**Proof.** By the same reasoning as in the proof of Theorem 2.2, the mapping \( f : A \to A \) is \( C \)-linear.

It follows from (2.20) that

\[
 \| f(xy) - f(x)y - xf(y) \|_A = \lim_{n \to \infty} 4^n \left\| f\left( \frac{xy}{4^n} \right) - f\left( \frac{x}{2^n} \right) \frac{y}{2^n} - \frac{x}{2^n} f\left( \frac{y}{2^n} \right) \right\|_A
\]

\[
 \leq \lim_{n \to \infty} \frac{4^n \theta}{4^{nr}} \cdot \| x \|_A^r \cdot \| y \|_A^r = 0
\]

for all \( x, y \in A \). So

\[
 f(xy) = f(x)y + xf(y)
\]

for all \( x, y \in A \). Thus the mapping \( f : A \to A \) is a linear derivation.

**Theorem 2.9.** Let \( r < 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : A \to A \) be a mapping satisfying (2.2) and (2.20). Then the mapping \( f : A \to A \) is a linear derivation.

**Proof.** The proof is similar to the proofs of Theorems 2.2 and 2.8.
### 3. Isomorphisms and derivations in Lie $C^*$-algebras

Throughout this section, assume that $A$ is a Lie $C^*$-algebra with norm $\| \cdot \|_A$, and that $B$ is a Lie $C^*$-algebra with norm $\| \cdot \|_B$.

**Definition 3.1** [6, 7, 13]. A bijective $\mathbb{C}$-linear mapping $H : A \to B$ is called a **Lie $C^*$-algebra isomorphism** if $H : A \to B$ satisfies

$$H([x,y]) = [H(x),H(y)]$$

(3.1)

for all $x, y \in A$.

**Definition 3.2** [6, 7, 13]. A $\mathbb{C}$-linear mapping $D : A \to A$ is called a **Lie derivation** if $D : A \to A$ satisfies

$$D([x,y]) = [Dx,y] + [x,Dy]$$

(3.2)

for all $x, y \in A$.

In this section, we investigate Lie $C^*$-algebra isomorphisms between Lie $C^*$-algebras and Lie derivations on Lie $C^*$-algebras associated with the Cauchy-Jensen functional equation.

**Theorem 3.3.** Let $r > 1$ and $\theta$ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping satisfying (2.2) such that

$$\|f([x,y]) - [f(x),f(y)]\|_B \leq \theta(\|x\|_A^{2r} + \|y\|_A^{2r})$$

(3.3)

for all $x, y \in A$. Then the mapping $f : A \to B$ is a Lie $C^*$-algebra isomorphism.

**Proof.** By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to B$ is $\mathbb{C}$-linear. It follows from (3.3) that

$$\|f([x,y]) - [f(x),f(y)]\|_B = \lim_{n \to \infty} 4^n \|f\left(\frac{x}{2^n},\frac{y}{2^n}\right) - \left[f\left(\frac{x}{2^n}\right),f\left(\frac{y}{2^n}\right)\right]\|_B$$

(3.4)

$$\leq \lim_{n \to \infty} \frac{4^n \theta}{4^{nr}} (\|x\|_A^{2r} + \|y\|_A^{2r}) = 0$$

for all $x, y \in A$. Thus

$$f([x,y]) = [f(x),f(y)]$$

(3.5)

for all $x, y \in A$. Hence the bijective mapping $f : A \to B$ is a Lie $C^*$-algebra isomorphism, as desired. □

**Theorem 3.4.** Let $r < 1$ and $\theta$ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping satisfying (2.2) and (3.3). Then the mapping $f : A \to B$ is a Lie $C^*$-algebra isomorphism.

**Proof.** The proof is similar to the proofs of Theorems 2.2 and 3.3. □
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Theorem 3.5. Let \( r > 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : A \to A \) be a mapping satisfying (2.2) such that

\[
\|f([x,y]) - [f(x), y] - [x, f(y)]\|_A \leq \theta(\|x\|_A^r + \|y\|_A^r)
\]

(3.6)

for all \( x, y \in A \). Then the mapping \( f : A \to A \) is a Lie derivation.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping \( f : A \to A \) is \( \mathbb{C} \)-linear.

It follows from (3.6) that

\[
\|f([x,y]) - [f(x), y] - [x, f(y)]\|_A
\]

\[
= \lim_{n \to \infty} 4^n \|f\left(\frac{[x,y]}{4^n}\right) - \left[ f\left(\frac{x}{2^n}\right), \frac{y}{2^n}\right] - \left[ \frac{x}{2^n}, f\left(\frac{y}{2^n}\right)\right]\|_A
\]

(3.7)

\[
\leq \lim_{n \to \infty} 4^n \theta 4^{nr} (\|x\|_A^r + \|y\|_A^r) = 0
\]

(3.8)

for all \( x, y \in A \). So

\[
f([x,y]) = [f(x), y] + [x, f(y)]
\]

(3.9)

for all \( x, y \in A \). Thus the mapping \( f : A \to A \) is a Lie derivation.

Theorem 3.6. Let \( r < 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : A \to A \) be a mapping satisfying (2.2) and (3.6). Then the mapping \( f : A \to A \) is a Lie derivation.

Proof. The proof is similar to the proofs of Theorems 2.2 and 3.5.

Theorem 3.7. Let \( r > 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : A \to B \) be a bijective mapping satisfying (2.2) such that

\[
\|f([x,y]) - [f(x), f(y)]\|_B \leq \theta \cdot \|x\|_A^r \cdot \|y\|_A^r
\]

(3.10)

for all \( x, y \in A \). Then the mapping \( f : A \to B \) is a Lie C*-algebra isomorphism.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping \( f : A \to B \) is \( \mathbb{C} \)-linear.

It follows from (3.10) that

\[
\|f([x,y]) - [f(x), f(y)]\|_B = \lim_{n \to \infty} 4^n \|f\left(\frac{[x,y]}{2^n \cdot 2^n}\right) - \left[ f\left(\frac{x}{2^n}\right), f\left(\frac{y}{2^n}\right)\right]\|_B
\]

(3.11)

\[
\leq \lim_{n \to \infty} 4^n \theta 4^{nr} \cdot \|x\|_A^r \cdot \|y\|_A^r = 0
\]

for all \( x, y \in A \). Thus

\[
f([x,y]) = [f(x), f(y)]
\]

(3.12)

for all \( x, y \in A \). Hence the bijective mapping \( f : A \to B \) is a Lie C*-algebra isomorphism, as desired.
Theorem 3.8. Let $r < 1$ and $\theta$ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping satisfying (2.2) and (3.9). Then the mapping $f : A \to B$ is a Lie $C^*$-algebra isomorphism.

Proof. The proof is similar to the proofs of Theorems 2.2, 2.6, and 3.7. □

Theorem 3.9. Let $r > 1$ and $\theta$ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying (2.2) such that

$$\|f([x,y]) - [f(x),y] - [x,f(y)]\|_A \leq \theta \cdot \|x\|_A^r \cdot \|y\|_A^r$$

(3.12)

for all $x, y \in A$. Then the mapping $f : A \to A$ is a Lie derivation.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to A$ is $C$-linear.

It follows from (3.12) that

$$\|f([x,y]) - [f(x),y] - [x,f(y)]\|_A = \lim_{n \to \infty} 4^n \|f\left(\frac{x}{4^n}, \frac{y}{4^n}\right) - \left(\frac{x}{2^n}, \frac{y}{2^n}\right)\|_A$$

(3.13)

$$\leq \lim_{n \to \infty} \frac{4^n\theta}{4^{nr}} \cdot \|x\|_A^r \cdot \|y\|_A^r = 0$$

for all $x, y \in A$. So

$$f([x,y]) = [f(x),y] + [x,f(y)]$$

(3.14)

for all $x, y \in A$. Thus the mapping $f : A \to A$ is a Lie derivation. □

Theorem 3.10. Let $r < 1$ and $\theta$ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying (2.2) and (3.12). Then the mapping $f : A \to A$ is a Lie derivation.

Proof. The proof is similar to the proofs of Theorems 2.2, 2.8, and 3.9. □

4. Isomorphisms and derivations in $JC^*$-algebras

Throughout this section, assume that $A$ is a $JC^*$-algebra with norm $\|\cdot\|_A$, and that $B$ is a $JC^*$-algebra with norm $\|\cdot\|_B$.

Definition 4.1 [7, 13]. A bijective $C$-linear mapping $H : A \to B$ is called a $JC^*$-algebra isomorphism if $H : A \to B$ satisfies

$$H(x \circ y) = H(x) \circ H(y)$$

(4.1)

for all $x, y \in A$.

Definition 4.2 [7, 13]. A $C$-linear mapping $D : A \to A$ is called a Jordan derivation if $D : A \to A$ satisfies

$$D(x \circ y) = Dx \circ y + x \circ Dy$$

(4.2)

for all $x, y \in A$. 
In this section, we investigate $JC^*$-algebra isomorphisms between $JC^*$-algebras and Jordan derivations on $JC^*$-algebras associated with the Cauchy-Jensen functional equation.

**Theorem 4.3.** Let $r > 1$ and $\theta$ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping satisfying (2.2) such that

$$
||f(x \circ y) - f(x) \circ f(y)||_B \leq \theta(||x||_A^{2r} + ||y||_A^{2r})
$$

(4.3)

for all $x, y \in A$. Then the mapping $f : A \to B$ is a $JC^*$-algebra isomorphism.

**Proof.** By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to B$ is $C$-linear.

It follows from (4.3) that

$$
||f(x \circ y) - f(x) \circ f(y)||_B = \lim_{n \to \infty} 4^n ||f\left(\frac{x \circ y}{2^n}\right) - f\left(\frac{x}{2^n}\right) \circ f\left(\frac{y}{2^n}\right)||_B
$$

(4.4)

$$
\leq \lim_{n \to \infty} \frac{4^n \theta}{4^{nr}} (||x||_A^{2r} + ||y||_A^{2r}) = 0
$$

(4.5)

for all $x, y \in A$. Thus

$$
f(x \circ y) = f(x) \circ f(y)
$$

(4.6)

for all $x, y \in A$. Hence the bijective mapping $f : A \to B$ is a $JC^*$-algebra isomorphism, as desired. □

**Theorem 4.4.** Let $r < 1$ and $\theta$ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping satisfying (2.2) and (4.3). Then the mapping $f : A \to B$ is a $JC^*$-algebra isomorphism.

**Proof.** The proof is similar to the proofs of Theorems 2.2 and 4.3. □

**Theorem 4.5.** Let $r > 1$ and $\theta$ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying (2.2) such that

$$
||f(x \circ y) - f(x) \circ y - x \circ f(y)||_A \leq \theta(||x||_A^{2r} + ||y||_A^{2r})
$$

(4.6)

for all $x, y \in A$. Then the mapping $f : A \to A$ is a Jordan derivation.

**Proof.** By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to A$ is $C$-linear.

It follows from (4.6) that

$$
||f(x \circ y) - f(x) \circ y - x \circ f(y)||_A = \lim_{n \to \infty} 4^n \left||f\left(\frac{x \circ y}{4^n}\right) - f\left(\frac{x}{2^n}\right) \circ f\left(\frac{y}{2^n}\right)\right||_A
$$

(4.7)
for all $x, y \in A$. So

$$f(x \circ y) = f(x) \circ y + x \circ f(y)$$ (4.8)

for all $x, y \in A$. Thus the mapping $f : A \to A$ is a Jordan derivation. □

**Theorem 4.6.** Let $r < 1$ and $\theta$ be positive real numbers, and let $f : A \to A$ be a mapping satisfying (2.2) and (4.6). Then the mapping $f : A \to A$ is a Jordan derivation.

**Proof.** The proof is similar to the proofs of Theorems 2.2 and 4.5. □

**Theorem 4.7.** Let $r > 1$ and $\theta$ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping satisfying (2.2) such that

$$\|f(x \circ y) - f(x) \circ f(y)\|_B \leq \theta \cdot \|x\|_A^r \cdot \|y\|_A^r$$ (4.9)

for all $x, y \in A$. Then the mapping $f : A \to B$ is a JC*-algebra isomorphism.

**Proof.** By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to B$ is $\mathbb{C}$-linear.

It follows from (4.9) that

$$\|f(x \circ y) - f(x) \circ f(y)\|_B = \lim_{n \to \infty} 4^n \|f\left(\frac{x \circ y}{2^n} \cdot 2^n\right) - f\left(\frac{x}{2^n}\right) \circ f\left(\frac{y}{2^n}\right)\|_B$$

$$\leq \lim_{n \to \infty} \frac{4^n \theta}{4^nr} \cdot \|x\|_A^r \cdot \|y\|_A^r = 0$$ (4.10)

for all $x, y \in A$. Thus

$$f(x \circ y) = f(x) \circ f(y)$$ (4.11)

for all $x, y \in A$. Hence the bijective mapping $f : A \to B$ is a JC*-algebra isomorphism, as desired. □

**Theorem 4.8.** Let $r < 1$ and $\theta$ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping satisfying (2.2) and (4.9). Then the mapping $f : A \to B$ is a JC*-algebra isomorphism.

**Proof.** The proof is similar to the proofs of Theorems 2.2, 2.6, and 4.7. □

**Theorem 4.9.** Let $r > 1$ and $\theta$ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying (2.2) such that

$$\|f(x \circ y) - f(x) \circ y - x \circ f(y)\|_A \leq \theta \cdot \|x\|_A^r \cdot \|y\|_A^r$$ (4.12)

for all $x, y \in A$. Then the mapping $f : A \to A$ is a Jordan derivation.

**Proof.** By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to A$ is $\mathbb{C}$-linear.
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It follows from (4.6) that

\[
\| f(x \circ y) - f(x) \circ y - x \circ f(y) \|_A = \lim_{n \to \infty} 4^n \left\| f \left( \frac{x \circ y}{4^n} \right) - f \left( \frac{x}{2^n} \right) \circ \frac{y}{2^n} - \frac{x}{2^n} \circ f \left( \frac{y}{2^n} \right) \right\|_A
\]
\[
\leq \lim_{n \to \infty} \frac{4^n \theta}{4^{nr}} \cdot \| x \|_A^r \cdot \| y \|_A^2 = 0
\]

(4.13)

for all \( x, y \in A \). So

\[
f(x \circ y) = f(x) \circ y + x \circ f(y)
\]

(4.14)

for all \( x, y \in A \). Thus the mapping \( f : A \to A \) is a Jordan derivation.

Theorem 4.10. Let \( r < 1 \) and \( \theta \) be positive real numbers, and let \( f : A \to A \) be a mapping satisfying (2.2) and (4.12). Then the mapping \( f : A \to A \) is a Jordan derivation.

Proof. The proof is similar to the proofs of Theorems 2.2, 2.8, and 4.9.

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