Research Article

Modulus of Convexity, the Coefficient $R(1, X)$, and Normal Structure in Banach Spaces

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Let $\delta_X(\epsilon)$ and $R(1, X)$ be the modulus of convexity and the Domínguez-Benavides coefficient, respectively. According to these two geometric parameters, we obtain a sufficient condition for normal structure, that is, a Banach space X has normal structure if $2\delta_X(1 + \epsilon) > \max\{(R(1, x) - 1)\epsilon, 1 - (1 - \epsilon/R(1, X) - 1)\}$ for some $\epsilon \in [0, 1]$ which generalizes the known result by Gao and Prus.

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1. Introduction

Let X be a Banach space. Throughout the paper, denote by S_X, B_X the unit sphere and unit ball of X , respectively. Recall that a Banach space X is said to have normal structure (resp., weak, normal, structure) if for every closed bounded (resp., weakly compact) convex subset C of X with $\text{diam } C > 0$, there exists $x \in C$ such that $\sup\{\|x - y\| : y \in C\} < \text{diam } C$, where $\text{diam } C = \sup\{\|x - y\| : x, y \in C\}$. For a reflexive Banach space, the normal structure and weak normal structure are the same. Recently a good deal of investigations have focused on finding the sufficient conditions with various geometrical constants for a Banach space to have normal structure (see, e.g., [1–5]). The geometric condition sufficient for normal structure in terms of the modulus of convexity is given by Goebel [6], who proved that X has normal structure provided that $\delta_X(1) > 0$. Here the function $\delta_X(\epsilon) : [0, 2] \rightarrow [0, 1]$, defined by Clarkson [7] as

$$\delta_X(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : x, y \in B_X, \|x - y\| \geq \epsilon \right\}, \quad (1.1)$$

is called the modulus of convexity of X . Later Gao and Prus generalized the above results as the following (see [2, 8]).

Theorem 1.1. *A Banach space X has normal structure provided that $\delta_X(1 + \epsilon) > \epsilon/2$ for some $\epsilon \in [0, 1]$.*

In this paper, we obtain a class of Banach spaces with normal structure, which involves the coefficient $R(1, X)$. This coefficient is defined by Domínguez Benavides [9] as

$$R(1, X) = \sup \left\{ \liminf_{n \rightarrow \infty} \|x + x_n\| \right\}, \quad (1.2)$$

where the supremum is taken over all $x \in X$ with $\|x\| \leq 1$ and all weakly null sequence (x_n) in B_X such that

$$D[(x_n)] := \limsup_{n \rightarrow \infty} \left(\limsup_{m \rightarrow \infty} \|x_n - x_m\| \right) \leq 1. \quad (1.3)$$

Obviously, $1 \leq R(1, X) \leq 2$.

2. Main results

Let us begin this section with a sufficient condition for a Banach space X having weak normal structure and the idea in the following proof is due to [5, Lemma 5].

Lemma 2.1. *Let X be a Banach space for which B_{X^*} is w^* -sequentially compact. If X does not have weak normal structure, then for any $\eta > 0$, there exist $x_1, x_2 \in S_X$ and $f_1, f_2 \in S_{X^*}$, such that*

- (1) $|||x_1 - x_2|| - 1| < \eta$;
- (2) $|f_i(x_j)| < \eta$ for $i \neq j$ and $f_i(x_i) = 1$, $i, j = 1, 2$;
- (3) $\|x_1 + x_2\| \leq R(1, X)(1 + \eta)$.

Proof. Assume that X does not have weak normal structure. It is well known that (see, e.g., [10]) there exists a sequence $\{x_n\}$ in X satisfying

- (1) x_n is weakly convergent to 0;
- (2) $\text{diam}(\{x_n\}_{n=1}^\infty) = 1 = \lim_n \|x_n - x\|$ for all $x \in \text{clco}\{x_n\}_{n=1}^\infty$.

Since B_{X^*} is w^* -sequentially compact, we can find $\{f_n\}$ in S_{X^*} satisfying

- (3) $f_n(x_n) = \|x_n\|$ for all n ;
- (4) $f_n \xrightarrow{w^*} f$ for some $f \in B_{X^*}$.

Let $\eta \in (0, 1)$ sufficiently small and $\epsilon = \eta/3$. Then, by the properties of the sequence (x_n) , we can choose $n_1 \in \mathbb{N}$ such that

$$|f(x_{n_1})| < \frac{\epsilon}{2}, \quad 1 - \epsilon \leq \|x_{n_1}\| \leq 1. \quad (2.1)$$

Note that the sequence $\{x_n\}$ is weakly null and verifies $D[\{x_n\}] = 1$. It follows from the definition of $R(1, X)$ that

$$\liminf_n \|x_n + x_{n_1}\| \leq R(1, X). \quad (2.2)$$

The rest of the proof is similar to that of [5, Lemma 5]. □

Theorem 2.2. *A Banach space X has normal structure provided that $\delta_X(1 + \epsilon) > f(\epsilon)$ for some $\epsilon \in [0, 1]$, where the function $f(\epsilon)$ is defined as*

$$f(\epsilon) := \begin{cases} (R(1, X) - 1) \frac{\epsilon}{2}, & 0 \leq \epsilon \leq \frac{1}{R(1, X)}, \\ \frac{1}{2} \left(1 - \frac{1 - \epsilon}{R(1, X) - 1} \right), & \frac{1}{R(1, X)} < \epsilon \leq 1. \end{cases} \quad (2.3)$$

Proof. Observe that X is uniformly nonsquare [11] and then reflexive. Therefore normal structure and weak normal structure coincide.

Assume first that X fails to have weak normal structure. Fix $\eta > 0$ sufficiently small and $\epsilon \in [0, 1]$. It follows that there exist $x_1, x_2 \in S_X$ and $f_1, f_2 \in S_{X^*}$, satisfying the condition in Lemma 2.1. Next, denote by $R := R(1, X)$ and consider two cases for $\epsilon \in [0, 1]$.

Case 1 ($\epsilon \in [0, 1/R]$). Now let us put

$$x = \frac{x_1 - x_2}{1 + \eta}, \quad y = \frac{(1 - (R - 1)\epsilon)x_1 + \epsilon x_2}{1 + \eta}, \quad (2.4)$$

and so $x \in B_X$,

$$\|y\| = \left\| \frac{\epsilon}{1 + \eta} (x_1 + x_2) + \frac{1 - R\epsilon}{1 + \eta} x_1 \right\| \leq R\epsilon + (1 - R\epsilon) = 1, \quad (2.5)$$

and also that

$$\begin{aligned} \|x - y\| &= \left\| \frac{(R - 1)\epsilon}{1 + \eta} x_1 - \frac{1 + \epsilon}{1 + \eta} x_2 \right\| \geq \frac{1 + \epsilon}{1 + \eta} f_2(x_2) - \frac{(R - 1)\epsilon}{1 + \eta} f_2(x_1) \geq \frac{1 + \epsilon - \eta}{1 + \eta}, \\ \|x + y\| &= \left\| \frac{(2 - (R - 1))\epsilon}{1 + \eta} x_1 - \frac{1 - \epsilon}{1 + \eta} x_2 \right\| \geq \frac{(2 - (R - 1))\epsilon}{1 + \eta} f_1(x_1) - \frac{1 - \epsilon}{1 + \eta} f_1(x_2) \\ &\geq \left(1 - \frac{2\eta}{1 + \eta} \right) (2 - (R - 1)\epsilon). \end{aligned} \quad (2.6)$$

By the definition of modulus of convexity,

$$\left(1 - \frac{2\eta}{1 + \eta} \right) (2 - (R - 1)\epsilon) \leq \|x + y\| \leq 2(1 - \delta_X(\|x - y\|)) \leq 2 \left(1 - \delta_X \left(\frac{1 + \epsilon - \eta}{1 + \eta} \right) \right), \quad (2.7)$$

or equivalently,

$$(2 - (R - 1)\epsilon) \leq 2 \left(1 - \delta_X \left(\frac{1 + \epsilon - \eta}{1 + \eta} \right) \right) \left(1 + \frac{2\eta}{1 - \eta} \right). \quad (2.8)$$

Letting $\eta \rightarrow 0$, we have

$$2\delta_X(1 + \epsilon) \leq (R - 1)\epsilon, \quad (2.9)$$

which contradicts our hypothesis.

Case 2 ($\epsilon \in (1/R, 1]$). In this case $R > 1$, otherwise $\epsilon > 1$. Let

$$x' = \frac{x_2 - x_1}{1 + \eta}, \quad y' = \frac{(1 - (R - 1)\epsilon')x_1 + \epsilon'x_2}{1 + \eta}, \quad (2.10)$$

where $\epsilon' = 1 - (R - 1)\epsilon \in [0, 1/R]$. It follows from Case 1 that $x, y \in B_X$,

$$\|x - y\| \geq \left(1 - \frac{2\eta}{1 + \eta}\right) \left(2 - (R - 1)\epsilon'\right), \quad \|x + y\| \geq \frac{1 + \epsilon' - \eta}{1 + \eta}. \quad (2.11)$$

This implies that

$$\delta_X(2 - (R - 1)\epsilon') \leq \frac{1}{2}(1 - \epsilon'), \quad (2.12)$$

which is equivalent to

$$\delta_X(1 + \epsilon) \leq \frac{1}{2} \left(1 - \frac{1 - \epsilon}{R - 1}\right). \quad (2.13)$$

This is a contradiction. □

Remark 2.3. (1) It is readily seen that $f(\epsilon) \leq \epsilon/2$ for any $\epsilon \in [0, 1]$ and Theorem 2.2 is therefore a generalization of Theorem 1.1. Moreover this generalization is strict whenever X is the space with $R(1, X) < 2$.

(2) Consider the space $X = \mathbb{R}^2$ with the norm $\|(x, y)\| := \max(|x|, |y|, |x - y|)$. It is known that $\delta_X(\epsilon) = \max\{0, (\epsilon - 1)/2\}$ [8] and $R(1, X) = 1$, then X has normal structure from Theorem 2.2, but lies out of the scope of Theorem 1.1.

Corollary 2.4. *Let X be a Banach space with $R(1, X) = 1$ and $\delta_X(1 + \epsilon) > 0$ for some $\epsilon \in [0, 1]$, then X has normal structure.*

Corollary 2.5. *If X is a Banach space with*

$$\delta_X\left(1 + \frac{1}{R(1, X)}\right) > \frac{1}{2}\left(1 - \frac{1}{R(1, X)}\right), \quad (2.14)$$

then X has normal structure.

Remark 2.6. Corollary 2.5 is equivalent to [4, Corollary 24].

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