Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2008, Article ID 136592, 11 pages
doi:10.1155/2008/136592

Research Article

Functional Inequalities Associated with Additive Mappings

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Received 13 May 2008; Revised 25 June 2008; Accepted 1 August 2008

Recommended by John Rassias

The functional inequality $\|f(x)+2f(y)+2f(z)\| \le \|2f(x/2+y+z)\| + \phi\,(x,y,z)\,(x,y,z \in G)$ is investigated, where G is a group divisible by 2, $f : G \to X$ and $\phi : G^3 \to [0,\infty)$ are mappings, and X is a Banach space. The main result of the paper states that the assumptions above together with (1) $\phi(2x,-x,0) = 0 = \phi(0,x,-x)\,(x \in G)$ and (2) $\lim_{n\to\infty}(1/2^n)\phi(2^{n+1}x, 2^n y, 2^n z) = 0$, or $\lim_{n\to\infty} 2^n \phi(x/2^{n-1}, y/2^n, z/2^n) = 0\ (x,y,z \in G)$, imply that f is additive. In addition, some stability theorems are proved.

1. Introduction and preliminaries

The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. The study of stability problems had been formulated by Ulam [1] during a talk in 1940: *under what condition does there exist a homomorphism near an approximate homomorphism?* In the following year 1941, Hyers [2] had answered affirmatively the question of Ulam for Banach spaces, which states that *if $\varepsilon \ge 0$ and $f : \mathcal{X} \to \mathcal{Y}$ is a mapping with $\mathcal{X}$ a normed space, $\mathcal{Y}$ a Banach space such that*

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon \tag{1.1}$$

for all $x, y \in \mathcal{X}$, then there exists a unique additive mapping $L : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - L(x)\| \le \varepsilon \tag{1.2}$$

for all $x \in \mathcal{X}$. Then, Aoki [3] in 1950 and Rassias [4] in 1978 proved the following generalization of Hyers' theorem [2] by considering the case when the inequality (1.1) is unbounded.

Proposition 1.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping from a normed space $\mathcal{X}$ into a Banach space $\mathcal{Y}$ subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon\left(\|x\|^p + \|y\|^p\right) \tag{1.3}$$

for all $x, y \in \mathcal{X}$, where ε and p are constants with $\varepsilon \ge 0$ and $p < 1$. Then, there exists a unique additive mapping $L : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - L(x)\| \le \frac{2\varepsilon}{2 - 2^p}\|x\|^p \tag{1.4}$$

for all $x \in \mathcal{X}$. If $p < 0$, then inequality (1.3) *holds for $x, y \neq 0$ and* (1.4) *for $x \neq 0$.*

Following the techniques of the proof of the corollary of Hyers [2], we observed that Hyers introduced (in 1941) Hyers continuity condition about the continuity of the mapping $f(tx)$ in $t \in \mathbb{R}$ for each fixed x, and then he proved homogenouity of degree one and, therefore, the famous linearity. This condition has been assumed further till now through the complete Hyers direct method in order to prove linearity for generalized Hyers-Ulam stability problem forms (cf., [5]).

In 1991, Gajda [6] provided an affirmative answer to Rassias' question whether his theorem can be extended for values of p greater than one. However, it was shown by Gajda [6] as well as by Rassias and Šemrl [7] that one cannot prove a theorem similar to [4] when $p = 1$. On the other hand, Rassias [8–10] generalized Hyers' stability result by presenting a weaker condition controlled by (or involving) a product of different powers of norms (from the right-hand side of assumed conditions) as follows.

Proposition 1.2. *Suppose that there exist constants $\varepsilon \ge 0$ and $p, q \in \mathbb{R}$ such that $r = p + q \neq 1$, and $f : \mathcal{X} \to \mathcal{Y}$ is a mapping with $\mathcal{X}$ a normed space, $\mathcal{Y}$is a Banach space such that the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon\|x\|^p\|y\|^q \tag{1.5}$$

holds for all $x, y \in \mathcal{X}$. Then, there exists a unique additive mapping $L : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - L(x)\| \le \frac{\varepsilon}{|2 - 2^r|}\|x\|^r \tag{1.6}$$

for all $x \in \mathcal{X}$.

Since then, more generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings have been investigated in [11–32].

Recently, Roh and Shin [33] proved that if $f : X \to Y$ is a mapping from a normed space X into a Banach space Y satisfying the inequality

$$\|f(x) + 2f(y) + 2f(z)\| \le \left\|2f\left(\frac{x}{2} + y + z\right)\right\| + \varepsilon\|x\|^r\|y\|^r\|z\|^r \tag{1.7}$$

for all $x, y, z \in X$ and some $\varepsilon \ge 0$, then it is additive. In addition, they investigated the stability in Banach spaces.

In this paper, we will consider a mapping on a group instead of a normed space which satisfies the following inequality:

$$\|f(x) + 2f(y) + 2f(z)\| \le \left\|2f\left(\frac{x}{2} + y + z\right)\right\| + \phi(x, y, z) \tag{1.8}$$

for all $x, y, z \in G$, where G is a group, $f : G \to X$ and $\phi : G^3 \to [0, \infty)$ are mappings, and X is a Banach space.

Inequalities of the type above and some corresponding equations were examined by several authors [34–36]. In [35], it has been proved that if G is a (not necessarily 2-divisible) group, X is an inner product space and the mapping $f : G \to X$ satisfies

$$\|f(x) + f(y)\| \leq \|f(x + y)\| \tag{1.9}$$

for all $x, y \in G$, then it is additive. By replacing $x = 0$ in (1.8), we obtain

$$\|f(0) + 2f(y) + 2f(z)\| \leq \|2f(0 + y + z)\| + 0 \tag{1.10}$$

for all $y, z \in G$, which implies the inequality (1.9). Therefore, Theorem 2.1 in this paper is a special case of the result cited above for mappings which maps into Hilbert spaces. However, Theorem 2.1 is also valid for Banach spaces, while the result in [35] is not. (A counter example was constructed in [34]).

2. Main results

Theorem 2.1. *Let $(G, +)$ be a 2-divisible group and $(X, \|\cdot\|)$ a Banach space. Assume that a mapping $\phi : G^3 \to [0, \infty)$ satisfies the assumptions*

(1) $\phi(2x, -x, 0) = 0 = \phi(0, x, -x)$ $(x \in G)$,

(2) $\lim_{n\to\infty}(1/2^n)\phi(2^{n+1}x, 2^n y, 2^n z) = 0$ *or* $\lim_{n\to\infty} 2^n \phi(x/2^{n-1}, y/2^n, z/2^n) = 0$ $(x, y, z \in G)$,

and that the mapping $f : G \to X$ satisfies the inequality (1.8). *Then, f is additive.*

Proof. By letting $x = y = z = 0$ in (1.8), we get $f(0) = 0$; and by letting $x = 2x$, $y = -x$ and $z = 0$ in (1.8), we have

$$f(2x) = -2f(-x) \tag{2.1}$$

for all $x \in G$. Also, by letting $x = 0$ and $z = -y$ in (1.8), we obtain

$$f(-y) = -f(y) \tag{2.2}$$

for all $y \in G$.

Next, we are in the position to show that f is additive. We will consider two different cases for second assumption of ϕ.

Case 1. Assume $\lim_{n\to\infty} 2^n \phi(x/2^{n-1}, y/2^n, z/2^n) = 0$ for all $x, y, z \in G$. We get by (2.1) and (2.2)

$$f(2x) = 2f(x), \quad f(x) = 2f\left(\frac{x}{2}\right) = 4f\left(\frac{x}{4}\right) = \cdots = 2^n f\left(\frac{x}{2^n}\right) \tag{2.3}$$

for all positive integer n and all $x \in G$. Therefore, we can define $f(x) := \lim_{n\to\infty} 2^n f(x/2^n)$ for all $x \in G$. Due to (1.8), (2.1), and (2.2), we obtain

$$\begin{aligned} \|f(x) + f(y) - f(x + y)\| &= \lim_{n\to\infty} 2^n \left\| f\left(\frac{x}{2^n}\right) + f\left(\frac{y}{2^n}\right) + f\left(\frac{-x-y}{2^n}\right) \right\| \\ &\leq \lim_{n\to\infty} 2^n \left\| f\left(\frac{x}{2^n}\right) + f\left(\frac{y}{2^n}\right) + \frac{1}{2} f\left(\frac{-x-y}{2^{n-1}}\right) \right\| \\ &\leq \lim_{n\to\infty} 2^{n-1} \phi\left(\frac{-x-y}{2^{n-1}}, \frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \end{aligned} \tag{2.4}$$

for all $x, y \in G$. Thus, $f(x + y) = f(x) + f(y)$.

Case 2. Assume $\lim_{n\to\infty}(1/2^n)\phi(2^{n+1}x, 2^n y, 2^n z) = 0$ for all $x, y, z \in G$. We get by (2.1) and (2.2)

$$f(2x) = 2f(x), \quad f(x) = \frac{1}{2}f(2x) = \frac{1}{4}f(4x) = \cdots = \frac{1}{2^n}f(2^n x) \tag{2.5}$$

for all positive integer n and all $x \in G$. Therefore, we can define $f(x) := \lim_{n\to\infty}(1/2^n)f(2^n x)$ for all $x \in G$. Due to (1.8), (2.1), and (2.2), we obtain

$$\begin{aligned} \|f(x) + f(y) - f(x+y)\| &= \lim_{n\to\infty}\frac{1}{2^n}\|f(2^n x) + f(2^n y) + f(2^n(-x-y))\| \\ &\le \lim_{n\to\infty}\frac{1}{2^{n+1}}\phi(2^{n+1}(-x-y), 2^n x, 2^n y) = 0 \end{aligned} \tag{2.6}$$

for all $x, y \in G$. Thus, $f(x+y) = f(x) + f(y)$. □

Next, we will study the generalized Hyers-Ulam stability of functional inequality (1.8).

Theorem 2.2. *Let $(G,+)$ be a 2-divisible abelian group and $(X, \|\cdot\|)$ a Banach space. Assume that a mapping $\phi : G^3 \to [0,\infty)$ satisfies the assumptions*

(1) $\rho(x) = \sum_{j=0}^{\infty} 2^j[\phi(x/2^j, -x/2^{j+1}, 0) + \phi(0, x/2^{j+1}, -x/2^{j+1})] < \infty$ $(x \in G)$,

(2) $\lim_{n\to\infty} 2^n\phi((-x-y)/2^{n-1}, x/2^n, y/2^n) = 0$ $(x, y \in G)$,

and the inequality (1.8). *Then, there exists a unique additive mapping $L : G \to X$ such that*

$$\|L(x) - f(x)\| \le \rho(x) \tag{2.7}$$

for all $x \in G$.

Proof. Letting $x = y = z = 0$ in (1.8), we get $\|f(0)\| \le (1/3)\phi(0,0,0)$. By assumption, we should have $\phi(0,0,0) = 0$, since $\lim_{n\to\infty} 2^n\phi(0,0,0) = 0$. Hence, $f(0) = 0$. So, by letting $y = -x/2$ and $z = 0$ in (1.8), we have

$$\left\|f(x) + 2f\left(-\frac{x}{2}\right)\right\| \le \phi\left(x, -\frac{x}{2}, 0\right) \tag{2.8}$$

for all $x \in G$. Letting $x = 0$, $y = x$, and $z = -x$ in (1.8), we obtain

$$\|f(x) + f(-x)\| \le \frac{1}{2}\phi(0, x, -x) \tag{2.9}$$

for all $x \in G$. Therefore, we have

$$\begin{aligned} \left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| &\le \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\le \sum_{j=l}^{m-1}\left[\left\|2^j f\left(\frac{x}{2^j}\right) + 2^{j+1} f\left(\frac{-x}{2^{j+1}}\right)\right\| + \left\|2^{j+1} f\left(\frac{-x}{2^{j+1}}\right) + 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|\right] \\ &\le \sum_{j=l}^{m-1}\left[2^j\phi\left(\frac{x}{2^j}, -\frac{x}{2^{j+1}}, 0\right) + 2^j\phi\left(0, \frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}}\right)\right] \end{aligned} \tag{2.10}$$

for all nonnegative integers m and l with $m > l$ and all $x \in G$. It means that the sequence $\{2^n f(x/2^n)\}$ is a Cauchy sequence. Since X is complete, the sequence $\{2^n f(x/2^n)\}$ converges. So we can define a mapping $L : G \to X$ by $L(x) := \lim_{n\to\infty} 2^n f(x/2^n)$ for all $x \in G$. Moreover, by letting $l = 0$ and passing $m \to \infty$, we get (2.7).

Now, we claim that the mapping L is additive. We note by (2.9) that

$$\left\|L(x) + L(-x)\right\| = \lim_{n\to\infty} 2^n \left\| f\left(\frac{x}{2^n}\right) + f\left(-\frac{x}{2^n}\right)\right\| \le \lim_{n\to\infty} 2^{n-1}\phi\left(0, \frac{x}{2^n}, -\frac{x}{2^n}\right) = 0. \tag{2.11}$$

So we have $L(-x) = -L(x)$. By (1.8), (2.8), and (2.9), we have

$$\begin{aligned}
\left\|L(x) + L(y) - L(x+y)\right\| &= \lim_{n\to\infty} 2^n \left\| f\left(\frac{x}{2^n}\right) + f\left(\frac{y}{2^n}\right) + f\left(\frac{-x-y}{2^n}\right)\right\| \\
&\le \lim_{n\to\infty} 2^n \left\| \frac{1}{2} f\left(\frac{-x-y}{2^{n-1}}\right) + f\left(\frac{x}{2^n}\right) + f\left(\frac{y}{2^n}\right)\right\| \\
&\quad + \lim_{n\to\infty} 2^n \left\| f\left(\frac{-x-y}{2^n}\right) + \frac{1}{2} f\left(\frac{x+y}{2^{n-1}}\right)\right\| \\
&\quad + \lim_{n\to\infty} 2^{n-1} \left\| f\left(\frac{-x-y}{2^{n-1}}\right) + f\left(\frac{x+y}{2^{n-1}}\right)\right\| \\
&\le \lim_{n\to\infty} 2^{n-1}\phi\left(\frac{-x-y}{2^{n-1}}, \frac{x}{2^n}, \frac{y}{2^n}\right) + \lim_{n\to\infty} 2^{n-1}\phi\left(\frac{x+y}{2^{n-1}}, \frac{-x-y}{2^n}, 0\right) \\
&\quad + \lim_{n\to\infty} 2^{n-2}\phi\left(0, \frac{-x-y}{2^{n-1}}, \frac{x+y}{2^{n-1}}\right) = 0
\end{aligned} \tag{2.12}$$

for all $x, y \in G$.

Now, to prove uniqueness of the mapping L, let us assume that $T : G \to X$ is an additive mapping satisfying (2.7). Then, we obtain due to (2.7) that

$$\begin{aligned}
\left\|L(x) - T(x)\right\| &= 2^n \left\| L\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right\| \\
&\le 2^n \left[\left\| L\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\| + \left\| T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|\right] \\
&\le 2^{n+1}\rho\left(\frac{x}{2^n}\right) \longrightarrow 0
\end{aligned} \tag{2.13}$$

for all $x \in G$, as $n \to \infty$. □

Theorem 2.3. *Let $(G, +)$ be a 2-divisible abelian group and $(X, \|\cdot\|)$ a Banach space. Assume that a mapping $\phi : G^3 \to [0, \infty)$ satisfies the assumptions*

(1) $\sum_{j=0}^{\infty} (1/2^j)[\phi(-2^{j+1}x, 2^j x, 0) + (1/2)\phi(0, 2^{j+1}x, -2^{j+1}x)] < \infty$ $(x \in G)$,

(2) $\lim_{n\to\infty} (1/2^n)\phi(2^{n+1}(-x-y), 2^n x, 2^n y) = 0$ $(x, y \in G)$,

and the inequality (1.8). *Then, there exists a unique additive mapping $L : G \to X$ such that*

$$\left\|L(x) - f(x)\right\| \le \eta(x) \tag{2.14}$$

for all $x \in G$, where

$$\eta(x) = \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} \left[\phi\left(-2^{j+1}x, 2^j x, 0\right) + \frac{1}{2}\phi\left(0, 2^{j+1}x, -2^{j+1}x\right) + \frac{11}{6}\phi(0, 0, 0)\right]. \tag{2.15}$$

Proof. By letting $x = y = z = 0$ in (1.8), we get

$$\|f(0)\| \le \frac{1}{3}\phi(0,0,0). \tag{2.16}$$

We also have, by letting $y = -x/2$ and $z = 0$ in (1.8), that

$$\left\|\frac{1}{2}f(x) + f\left(-\frac{x}{2}\right)\right\| \le \frac{1}{2}\phi\left(x, -\frac{x}{2}, 0\right) + \frac{2}{3}\phi(0,0,0) \tag{2.17}$$

for all $x \in G$. Next, by letting $x = 0$, $y = x$, and $z = -x$ in (1.8), we obtain

$$\|f(x) + f(-x)\| \le \frac{1}{2}\phi(0,x,-x) + \frac{1}{2}\phi(0,0,0) \tag{2.18}$$

for all $x \in G$. Therefore, we have

$$\begin{aligned}
&\left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\| \\
&\quad\le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f(2^j x) - \frac{1}{2^{j+1}}f(2^{j+1}x)\right\| \\
&\quad\le \sum_{j=l}^{m-1}\left[\left\|\frac{1}{2^j}f(2^j x) + \frac{1}{2^{j+1}}f(-2^{j+1}x)\right\| + \left\|\frac{1}{2^{j+1}}f(2^{j+1}x) + \frac{1}{2^{j+1}}f(-2^{j+1}x)\right\|\right] \\
&\quad\le \sum_{j=l}^{m-1}\left[\frac{1}{2^{j+1}}\phi(-2^{j+1}x, 2^j x, 0) + \frac{1}{2^{j+2}}\phi(0, 2^{j+1}x, -2^{j+1}x) + \frac{1}{2^{j+1}}\frac{11}{6}\phi(0,0,0)\right]
\end{aligned} \tag{2.19}$$

for all nonnegative integers m and l with $m > l$ and all $x \in G$. It follows that the sequence $\{(1/2^n)f(2^n x)\}$ is a Cauchy and so it is convergent since X is complete. So one can define a mapping $L : G \to X$ by $L(x) := \lim_{n\to\infty}(1/2^n)f(2^n x)$ for all $x \in G$. By letting $l = 0$ and taking the limit $m \to \infty$, we arrive at (2.14).

Now, we claim that the mapping L is additive. By (2.18), one notes

$$\|L(x) + L(-x)\| = \lim_{n\to\infty}\frac{1}{2^n}\left\|f(2^n x) + f(-2^n x)\right\| \le \lim_{n\to\infty}\frac{1}{2^{n+1}}\left[\phi(0, 2^n x, -2^n x) + \phi(0,0,0)\right] = 0. \tag{2.20}$$

So we have $L(-x) = -L(x)$. By (1.8), (2.17), and (2.18), we obtain

$$\begin{aligned}
\|L(x) + L(y) - L(x+y)\| &= \lim_{n\to\infty}\frac{1}{2^n}\|f(2^n x) + f(2^n y) + f(2^n(-x-y))\| \\
&\le \lim_{n\to\infty}\frac{1}{2^n}\left\|\frac{1}{2}f(2^{n+1}(-x-y)) + f(2^n x) + f(2^n y)\right\| \\
&\quad + \lim_{n\to\infty}\frac{1}{2^n}\left\|f(2^n(-x-y)) + \frac{1}{2}f(2^{n+1}(x+y))\right\| \\
&\quad + \lim_{n\to\infty}\frac{1}{2^{n+1}}\|f(2^{n+1}(-x-y)) + f(2^{n+1}(x+y))\| \\
&\le \lim_{n\to\infty}\frac{1}{2^{n+1}}\left[\phi(2^{n+1}(-x-y), 2^n x, 2^n y) + \frac{2}{3}\phi(0,0,0)\right] \\
&\quad + \lim_{n\to\infty}\frac{1}{2^{n+1}}\left[\phi(2^{n+1}(x+y), 2^n(-x-y), 0) + \frac{4}{3}\phi(0,0,0)\right] \\
&\quad + \lim_{n\to\infty}\frac{1}{2^{n+2}}\left[\phi(0, 2^{n+1}(-x-y), 2^{n+1}(x+y)) + \phi(0,0,0)\right] = 0
\end{aligned} \tag{2.21}$$

for all $x, y \in G$.

Now, to show uniqueness of the mapping L, let us assume that $T : G \to X$ is another additive mapping satisfying (2.14). Then, by (2.14), and assumptions of ϕ, we have

$$\begin{aligned}\|L(x) - T(x)\| &= \frac{1}{2^n}\|L(2^n x) - T(2^n x)\| \\ &\le \frac{1}{2^n}\left[\|L(2^n x) - f(2^n x)\| + \|T(2^n x) - f(2^n x)\|\right] \\ &\le \frac{1}{2^{n-1}}\eta(2^n x) \longrightarrow 0\end{aligned} \tag{2.22}$$

for all $x \in G$, as $n \to \infty$. □

With the help of Theorems 2.2 and 2.3, we obtain the following corollaries.

Corollary 2.4. *Suppose that $f : E \to X$ is a mapping from a normed space E into a Banach space X subject to the inequality*

$$\|f(x) + 2f(y) + 2f(z)\| \le \left\|2f\left(\frac{x}{2} + y + z\right)\right\| + \varepsilon\left(\|x\|^p + \|y\|^p + \|z\|^p\right) \tag{2.23}$$

for all $x, y, z \in E$, where ε and p are constants with $\varepsilon \ge 0$ and $p > 1$. Then, there exists a unique additive mapping $L : E \to X$ such that

$$\|L(x) - f(x)\| \le \frac{2^p + 3}{2^p - 2}\varepsilon\|x\|^p \tag{2.24}$$

for all $x \in E$.

Corollary 2.5. *Suppose that $f : E \to X$ is a mapping from a normed space E into a Banach space X subject to the inequality* (2.23) *for all $x, y, z \in E$, where ε and p are constants with $\varepsilon \ge 0$ and $p < 1$. Then, there exists a unique additive mapping $L : E \to X$ such that*

$$\|L(x) - f(x)\| \le \frac{2^{p+2} + 2}{2 - 2^p}\varepsilon\|x\|^p \tag{2.25}$$

for all $x \in E$.

Theorem 2.6. *Let $(G, +)$ be a 2-divisible abelian group and $(X, \|\cdot\|)$ a Banach space. Assume that a mapping $\phi : G^3 \to [0, \infty)$ satisfies the assumptions*

(1) $\sum_{j=0}^{\infty} 2^j[\phi(x/2^j, -x/2^{j+1}, 0) + \phi(-x/2^j, x/2^{j+1}, 0)] < \infty$ $(x \in G)$,

(2) $\lim_{n\to\infty} 2^n\phi(-x - y/2^{n-1}, x/2^n, y/2^n) = 0$ $(x, y \in G)$,

and the inequality (1.8). *Then, there exists a unique additive mapping $L : G \to X$ such that*

$$\left\|L(x) - \frac{f(x) - f(-x)}{2}\right\| \le \gamma(x) \tag{2.26}$$

for all $x \in G$, where

$$\gamma(x) = \sum_{j=0}^{\infty} 2^{j-1}\left[\phi\left(\frac{x}{2^j}, -\frac{x}{2^{j+1}}, 0\right) + \phi\left(-\frac{x}{2^j}, \frac{x}{2^{j+1}}, 0\right)\right]. \tag{2.27}$$

Proof. Letting $x = y = z = 0$ in (1.8), we get $\|f(0)\| \leq (1/3)\phi(0,0,0)$. By assumption, we should have $\phi(0,0,0) = 0$, since $\lim_{n\to\infty} 2^n \phi(0,0,0) = 0$. Hence, we obtain $f(0) = 0$. By letting $y = -x/2$ and $z = 0$ in (1.8), we have

$$\left\| f(x) + 2f\left(-\frac{x}{2}\right) \right\| \leq \phi\left(x, -\frac{x}{2}, 0\right) \tag{2.28}$$

for all $x \in G$. Let $g(x) = (f(x) - f(-x))/2$. Then, we obtain

$$\begin{aligned} \|2g(x) - g(2x)\| &\leq \left\| f(x) + \frac{f(-2x)}{2} \right\| + \left\| f(-x) + \frac{f(2x)}{2} \right\| \\ &\leq \frac{1}{2}\left[\phi(-2x, x, 0) + \phi(2x, -x, 0)\right] \end{aligned} \tag{2.29}$$

for all $x \in G$. Hence, for all nonnegative integers m and l with $m > l$ and all $x \in G$,

$$\begin{aligned} \left\| 2^l g\left(\frac{x}{2^l}\right) - 2^m g\left(\frac{x}{2^m}\right) \right\| &\leq \sum_{j=l}^{m-1} \left\| 2^j g\left(\frac{x}{2^j}\right) - 2^{j+1} g\left(\frac{x}{2^{j+1}}\right) \right\| \\ &\leq \sum_{j=l}^{m-1} 2^{j-1} \left[\phi\left(-\frac{x}{2^j}, \frac{x}{2^{j+1}}, 0\right) + \phi\left(\frac{x}{2^j}, -\frac{x}{2^{j+1}} 0\right)\right]. \end{aligned} \tag{2.30}$$

So the sequence $\{2^n g(x/2^n)\}$ is a Cauchy sequence. Due to the completeness of X, this sequence is convergent. Let $L : G \to X$ be a mapping defined by $L(x) := \lim_{n\to\infty} 2^n g(x/2^n)$ for all $x \in G$. Letting $l = 0$ and sending $m \to \infty$, we get (2.26).

Next, we claim that the mapping L is additive. We first note that $L(-x) = -L(x)$ because $g(-x) = -g(x)$. So, by (1.8) and (2.28), we obtain

$$\begin{aligned} \|L(x) + L(y) - L(x+y)\| &= \lim_{n\to\infty} 2^n \left\| g\left(\frac{x}{2^n}\right) + g\left(\frac{y}{2^n}\right) + g\left(\frac{-x-y}{2^n}\right) \right\| \\ &\leq \lim_{n\to\infty} 2^n \left[\left\| \frac{f(x/2^n)}{2} + \frac{f(y/2^n)}{2} + \frac{f(-x-y/2^{n-1})}{4} \right\| \right. \\ &\qquad \left. + \left\| \frac{f(-x/2^n)}{2} + \frac{f(-y/2^n)}{2} + \frac{f(x+y/2^{n-1})}{4} \right\| \right] \\ &\quad + \lim_{n\to\infty} 2^n \left[\left\| \frac{f(-x-y/2^n)}{2} + \frac{f(x+y/2^{n-1})}{4} \right\| \right. \\ &\qquad \left. + \left\| \frac{f(x+y/2^n)}{2} + \frac{f(-x-y/2^{n-1})}{4} \right\| \right] \\ &\leq \lim_{n\to\infty} 2^{n-2} \left[\phi\left(\frac{-x-y}{2^{n-1}}, \frac{x}{2^n}, \frac{y}{2^n}\right) + \phi\left(\frac{x+y}{2^{n-1}}, \frac{-x}{2^n}, \frac{-y}{2^n}\right) \right] \\ &\quad + \lim_{n\to\infty} 2^{n-2} \left[\phi\left(\frac{x+y}{2^{n-1}}, \frac{-x-y}{2^n}, 0\right) + \phi\left(\frac{-x-y}{2^{n-1}}, \frac{x+y}{2^n}, 0\right) \right] = 0 \end{aligned} \tag{2.31}$$

for all $x, y \in G$. So we have $L(x+y) = L(x) + L(y)$.

The proof of uniqueness for L is similar to the proof of Theorem 2.2. □

Theorem 2.7. *Let $(G,+)$ be a 2-divisible abelian group and $(X,\|\cdot\|)$ a Banach space. Assume that a mapping $\phi : G^3 \to [0,\infty)$ satisfies the assumptions*

(1) $\sum_{j=0}^{\infty}(1/2^j)[\phi(-2^{j+1}x,2^jx,0)+\phi(2^{j+1}x,-2^jx,0)]<\infty$ $(x\in G)$,

(2) $\lim_{n\to\infty}(1/2^n)\phi(2^{n+1}(-x-y),2^nx,2^ny)=0$ $(x,y\in G)$,

and the inequality (1.8). *Then, there exists a unique additive mapping $L : G \to X$ such that*

$$\left\|L(x)-\frac{f(x)-f(-x)}{2}\right\|\le\delta(x) \tag{2.32}$$

for all $x\in G$, where

$$\delta(x)=\sum_{j=0}^{\infty}\frac{1}{2^{j+2}}\left[\phi\left(-2^{j+1}x,2^jx,0\right)+\phi(2^{j+1}x,-2^jx,0)+\frac{8}{3}\phi(0,0,0)\right]. \tag{2.33}$$

Proof. Due to (2.17), we have

$$\left\|\frac{1}{2}f(x)+f\left(-\frac{x}{2}\right)\right\|\le\frac{1}{2}\phi\left(x,-\frac{x}{2},0\right)+\frac{2}{3}\phi(0,0,0) \tag{2.34}$$

for all $x\in G$. So, we obtain

$$\begin{aligned}\left\|\frac{1}{2^l}g(2^lx)-\frac{1}{2^m}g(2^mx)\right\| &\le\sum_{j=l}^{m-1}\left\|\frac{1}{2^j}g(2^jx)-\frac{1}{2^{j+1}}g(2^{j+1}x)\right\|\\ &\le\sum_{j=l}^{m-1}\frac{1}{2^{j+2}}\left[\phi\left(-2^{j+1}x,2^jx,0\right)+\phi\left(2^{j+1}x,-2^jx,0\right)+\frac{8}{3}\phi(0,0,0)\right]\end{aligned} \tag{2.35}$$

for all nonnegative integers m and l with $m>l$ and all $x\in G$. This means that the sequence $\{(1/2^n)g(2^nx)\}$ is a Cauchy sequence. Since X is complete, the sequence $\{(1/2^n)g(2^nx)\}$ converges. Thus, we may define a mapping $L : G \to X$ by $L(x) := \lim_{n\to\infty}(1/2^n)g(2^nx)$ for all $x\in G$. Letting $l=0$ and passing the limit $m\to\infty$, we get (2.32).

Now, we claim that the mapping L is additive. By (2.16), (1.8), and (2.34), we have

$$\begin{aligned}&\|L(x)+L(y)-L(x+y)\|\\ &\quad=\lim_{n\to\infty}\frac{1}{2^n}\|g(2^nx)+g(2^ny)+g(2^n(-x-y))\|\\ &\quad\le\lim_{n\to\infty}\frac{1}{2^{n+2}}\left[\phi\left(2^{n+1}(-x-y),2^nx,2^ny\right)+\phi\left(2^{n+1}(x+y),-2^nx,-2^ny\right)\right]\\ &\qquad+\lim_{n\to\infty}\frac{1}{2^{n+2}}\left[\phi\left(2^{n+1}(x+y),2^n(-x-y),0\right)+\phi\left(2^{n+1}(-x-y),2^n(x+y),0\right)\right]\\ &\qquad+\lim_{n\to\infty}\frac{1}{2^n}\phi(0,0,0)=0\end{aligned} \tag{2.36}$$

for all $x,y\in G$.

The proof of uniqueness for L is similar to the proof of Theorem 2.3. □

Acknowledgments

The authors would like to thank referees for their valuable comments regarding a previous version of this paper. This work was supported by the Korea Research Foundation Grant funded by the Korean Government (KRF-2008-531-C00008).

References

[1] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Tracts in Pure and Applied Mathematics, no. 8, Interscience Publishers, New York, NY, USA, 1960.
[2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, no. 4, pp. 222–224, 1941.
[3] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64–66, 1950.
[4] Th. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.
[5] L. Maligranda, "A result of Tosio Aoki about a generalization of Hyers-Ulam stability of additive functions—a question of priority," *Aequationes Mathematicae*, vol. 75, no. 3, pp. 289–296, 2008.
[6] Z. Gajda, "On stability of additive mappings," *International Journal of Mathematics and Mathematical Sciences*, vol. 14, no. 3, pp. 431–434, 1991.
[7] Th. M. Rassias and P. Šemrl, "On the behavior of mappings which do not satisfy Hyers-Ulam stability," *Proceedings of the American Mathematical Society*, vol. 114, no. 4, pp. 989–993, 1992.
[8] J. M. Rassias, "On approximation of approximately linear mappings by linear mappings," *Journal of Functional Analysis*, vol. 46, no. 1, pp. 126–130, 1982.
[9] J. M. Rassias, "On approximation of approximately linear mappings by linear mappings," *Bulletin des Sciences Mathématiques*, vol. 108, no. 4, pp. 445–446, 1984.
[10] J. M. Rassias, "Solution of a problem of Ulam," *Journal of Approximation Theory*, vol. 57, no. 3, pp. 268–273, 1989.
[11] D. G. Bourgin, "Classes of transformations and bordering transformations," *Bulletin of the American Mathematical Society*, vol. 57, pp. 223–237, 1951.
[12] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, vol. 34 of *Progress in Nonlinear Differential Equations and Their Applications*, Birkh"auser, Boston, Mass, USA, 1998.
[13] K.-W. Jun and H.-M. Kim, "Stability problem of Ulam for generalized forms of Cauchy functional equation," *Journal of Mathematical Analysis and Applications*, vol. 312, no. 2, pp. 535–547, 2005.
[14] K.-W. Jun and Y.-H. Lee, "A generalization of the Hyers-Ulam-Rassias stability of the Pexiderized quadratic equations," *Journal of Mathematical Analysis and Applications*, vol. 297, no. 1, pp. 70–86, 2004.
[15] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, Fla, USA, 2001.
[16] S.-M. Jung, "Hyers-Ulam stability of a system of first order linear differential equations with constant coefficients," *Journal of Mathematical Analysis and Applications*, vol. 320, no. 2, pp. 549–561, 2006.
[17] C.-G. Park, "Homomorphisms between Poisson JC^*-algebras," *Bulletin of the Brazilian Mathematical Society*, vol. 36, no. 1, pp. 79–97, 2005.
[18] C. Park, "Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebras," *Bulletin des Sciences Mathématiques*, vol. 132, no. 2, pp. 87–96, 2008.
[19] C.-G. Park and Th. M. Rassias, "Hyers-Ulam stability of a generalized Apollonius type quadratic mapping," *Journal of Mathematical Analysis and Applications*, vol. 322, no. 1, pp. 371–381, 2006.
[20] W. Fechner, "Stability of a functional inequality associated with the Jordan-von Neumann functional equation," *Aequationes Mathematicae*, vol. 71, no. 1-2, pp. 149–161, 2006.
[21] S.-M. Jung and J. M. Rassias, "A fixed point approach to the stability of a functional equation of the spiral of Theodorus," *Fixed Point Theory and Applications*, vol. 2008, Article ID 945010, 7 pages, 2008.
[22] H.-M. Kim and J. M. Rassias, "Generalization of Ulam stability problem for Euler-Lagrange quadratic mappings," *Journal of Mathematical Analysis and Applications*, vol. 336, no. 1, pp. 277–296, 2007.
[23] Y.-S. Lee and S.-Y. Chung, "Stability of an Euler-Lagrange-Rassias equation in the spaces of generalized functions," *Applied Mathematics Letters*, vol. 21, no. 7, pp. 694–700, 2008.

[24] P. Nakmahachalasint, "On the generalized Ulam-Gavruta-Rassias stability of mixed-type linear and Euler-Lagrange-Rassias functional equations," *International Journal of Mathematics and Mathematical Sciences*, vol. 2007, Article ID 63239, 10 pages, 2007.
[25] C.-G. Park, "Stability of an Euler-Lagrange-Rassias type additive mapping," *International Journal of Applied Mathematics & Statistics*, vol. 7, no. Fe07, pp. 101–111, 2007.
[26] A. Pietrzyk, "Stability of the Euler-Lagrange-Rassias functional equation," *Demonstratio Mathematica*, vol. 39, no. 3, pp. 523–530, 2006.
[27] J. M. Rassias, "On a new approximation of approximately linear mappings by linear mappings," *Discussiones Mathematicae*, vol. 7, pp. 193–196, 1985.
[28] J. M. Rassias, "On the stability of the Euler-Lagrange functional equation," *Chinese Journal of Mathematics*, vol. 20, no. 2, pp. 185–190, 1992.
[29] J. M. Rassias, "Alternative contraction principle and alternative Jensen and Jensen type mappings," *International Journal of Applied Mathematics & Statistics*, vol. 4, no. M06, pp. 1–10, 2006.
[30] J. M. Rassias and M. J. Rassias, "Refined Ulam stability for Euler-Lagrange type mappings in Hilbert spaces," *International Journal of Applied Mathematics & Statistics*, vol. 7, no. Fe07, pp. 126–132, 2007.
[31] J. Tabor, "Stability of the Fischer-Muszély functional equation," *Publicationes Mathematicae Debrecen*, vol. 62, no. 1-2, pp. 205–211, 2003.
[32] J. Tabor and J. Tabor, "19. remark, (solution of the 7. Problem by K. Nikodem), report of meeting," *Aequationes Mathematicae*, vol. 61, no. 3, pp. 307–309, 2001.
[33] J. Roh and H. J. Shin, "Approximation of Cauchy additive mappings," *Bulletin of the Korean Mathematical Society*, vol. 44, no. 4, pp. 851–860, 2007.
[34] P. Fischer and Gy. Muszély, "On some new generalizations of the functional equation of Cauchy," *Canadian Mathematical Bulletin*, vol. 10, pp. 197–205, 1967.
[35] Gy. Maksa and P. Volkmann, "Characterization of group homomorphisms having values in an inner product space," *Publicationes Mathematicae Debrecen*, vol. 56, no. 1-2, pp. 197–200, 2000.
[36] P. Volkmann, "Pour une fonction réelle f l'inéquation $|f(x) + f(y)| \leq |f(x + y)|$ et l'équation de Cauchy sont équivalentes," in *Proceedings of the 23rd International Symposium on Functional Equations*, University of Waterloo, Gargnano, Italy, June 1985.