Research Article

Pairwise Weakly Regular-Lindelöf Spaces

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We will introduce and study the pairwise weakly regular-Lindelöf bitopological spaces and obtain
some results. Furthermore, we study the pairwise weakly regular-Lindelöf subspaces and subsets,
and investigate some of their characterizations. We also show that a pairwise weakly regular-
Lindelöf property is not a hereditary property. Some counterexamples will be considered in order
to establish some of their relations.

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1. Introduction

The study of bitopological spaces was first initiated by Kelly [1] in 1963 and thereafter a
large number of papers have been done to generalize the topological concepts to bitopological
setting. In literature, there are several generalizations of the notion of Lindelöf spaces, and
these are studied separately for different reasons and purposes. In 1959, Frolík [2] introduced
the notion of weakly Lindelöf spaces and in 1996, Cammaroto and Santoro [3] studied and
gave further new results about these spaces followed by Kılıçman and Fawakhreh [4]. In the
same paper, Cammaroto and Santoro introduced the notion of weakly regular-Lindelöf spaces
by using regular covers and leave open the study of this new concept. In 2001, Fawakhreh and
Kılıçman [5] studied this new generalization of Lindelöf spaces and obtained some results.
Then, Kılıçman and Fawakhreh [6] studied subspaces of this spaces and obtained some results.

Recently, the authors studied pairwise Lindelöfness in [7] and introduced and studied
the notion of pairwise weakly Lindelöf spaces in bitopological spaces, see [8], where the
authors extended some results that were due to Cammaroto and Santoro [3], Kılıçman and
Fawakhreh [4], and Fawakhreh [9]. In [10], the authors also studied the mappings and
pairwise continuity on pairwise Lindelöf bitopological spaces. The purpose of this paper is to
define the notion of weakly regular-Lindelöf property in bitopological spaces, which we will
Throughout this paper, all spaces (X, \tau) and (X, \tau_1, \tau_2) (or simply X) are always mean topological spaces and bitopological spaces, respectively, unless explicitly stated. We always use ij- to denote the certain properties with respect to topology \tau_i and \tau_j, where i, j \in \{1, 2\} and i \neq j. By i-int(A) and i-cl(A), we will mean the interior and the closure of a subset A of X with respect to topology \tau_i, respectively. We denote by int(A) and cl(A) for the interior and the closure of a subset A of X with respect to topology \tau_i for each i = 1, 2, respectively.

If S \subseteq A \subseteq X, then i-int_A(S) and i-cl_A(S) will be used to denote the interior and closure of S with respect to topology \tau_i in the subspace A, respectively. By i-open cover of X, we mean that the cover of X by i-open sets in X; similar for the ij-regular open cover of X and so forth. We will use the notation, X is i-Lindelöf space which mean that (X, \tau_i) is a Lindelöf space, where i \in \{1, 2\}.

**Definition 2.1** (see [11, 12]). A subset S of a bitopological space (X, \tau_1, \tau_2) is said to be ij-regular open (resp., ij-regular closed) if i-int(j-cl(S)) = S (resp., i-cl(j-int(S)) = S), and S is said pairwise regular open (resp., pairwise regular closed) if it is both ij-regular open and ji-regular open (resp., ij-regular closed and ji-regular closed).

**Definition 2.2.** Let (X, \tau_1, \tau_2) be a bitopological space. A subset F of X is said to be

(i) i-open if F is open with respect to \tau_i in X, F is said open in X if it is both 1-open and 2-open in X, or equivalently, F \in \tau_1 \cap \tau_2;

(ii) i-closed if F is closed with respect \tau_i in X, F is said closed in X if it is both 1-closed and 2-closed in X, or equivalently, X \setminus F \in \tau_1 \cap \tau_2;

(iii) i-clopen if F is both i-closed and i-open in X, F is said clopen in X if it is both 1-clopen and 2-clopen in X;

(iv) ij-clopen if F is i-closed and j-open in X, F is said clopen if it is both ij-clopen and ji-clopen in X.

**Definition 2.3** (see [13]). A bitopological space (X, \tau_1, \tau_2) is said to be Lindelöf if the topological space (X, \tau_1) and (X, \tau_2) are both Lindelöf. Equivalently, (X, \tau_1, \tau_2) is Lindelöf if every i-open cover of X has a countable subcover for each i = 1, 2.
Definition 2.4 (see [1, 11]). A bitopological space \((X, \tau_1, \tau_2)\) is said to be \(ij\)-regular if for each point \(x \in X\) and for each \(i\)-open set \(V\) of \(X\) containing \(x\) there exists an \(i\)-open set \(U\) such that \(x \in U \subseteq j\text{-cl}(U) \subseteq V\), and \(X\) is said to be pairwise regular if it is both \(ij\)-regular and \(ji\)-regular.

Definition 2.5 (see [11, 14]). A bitopological space \(X\) is said to be \(ij\)-almost regular if for each \(x \in X\) and for each \(ij\)-regular open set \(V\) of \(X\) containing \(x\) there is an \(ij\)-regular open set \(U\) such that \(x \in U \subseteq j\text{-cl}(U) \subseteq V\), then \(X\) is said to be pairwise almost regular if it is both \(ij\)-almost regular and \(ji\)-almost regular.

Definition 2.6 (see [11, 12]). A bitopological space \(X\) is said to be \(ij\)-semiregular if for each \(x \in X\) and for each \(i\)-open set \(V\) of \(X\) containing \(x\) there is an \(i\)-open set \(U\) such that \(x \in U \subseteq i\text{-int}(j\text{-cl}(U)) \subseteq V\), and \(X\) is said pairwise semiregular if it is both \(ij\)-semiregular and \(ji\)-semiregular.

Definition 2.7. A bitopological space \(X\) is said to be \(ij\)-nearly Lindelöf [15] (resp., \(ij\)-almost Lindelöf [16], \(ij\)-weakly Lindelöf [8]) if for every \(i\)-open cover \(\{U_\alpha : \alpha \in \Delta\}\) of \(X\) there exists a countable subset \(\{a_n : n \in \mathbb{N}\}\) of \(\Delta\) such that

\[
X = \bigcup_{n \in \mathbb{N}} i\text{-int}(j\text{-cl}(U_{a_n})) \tag{2.1}
\]

and \(X\) is said pairwise nearly Lindelöf (resp., pairwise almost Lindelöf, pairwise weakly Lindelöf) if it is both \(ij\)-nearly Lindelöf (resp., \(ij\)-almost Lindelöf, \(ij\)-weakly Lindelöf) and \(ji\)-nearly Lindelöf (resp., \(ji\)-almost Lindelöf, \(ji\)-weakly Lindelöf).

Definition 2.8 (see [8]). A subset \(S\) of a bitopological space \(X\) is said to be \(ij\)-weakly Lindelöf relative to \(X\) if for every cover \(\{U_\alpha : \alpha \in \Delta\}\) of \(S\) by \(i\)-open subsets of \(X\) such that \(S \subseteq \bigcup_{\alpha \in \Delta} U_\alpha\) there exists a countable subset \(\{a_n : n \in \mathbb{N}\}\) of \(\Delta\) such that \(S \subseteq j\text{-cl}(\bigcup_{n \in \mathbb{N}} U_{a_n})\). \(S\) is said pairwise weakly Lindelöf relative to \(X\) if it is both \(ij\)-weakly Lindelöf relative to \(X\) and \(ji\)-weakly Lindelöf relative to \(X\).

Definition 2.9 (see [8]). A bitopological space \(X\) is said to be \(ij\)-nearly paracompact if every cover of \(X\) by \(ij\)-regular open sets admits a locally finite refinement. \(X\) is said pairwise nearly paracompact if it is both \(ij\)-nearly paracompact and \(ji\)-nearly paracompact.

3. Pairwise weakly regular-Lindelöf spaces

Definition 3.1 (see [17]). An \(i\)-open cover \(\{U_\alpha : \alpha \in \Delta\}\) of a bitopological space \(X\) is said to be \(ij\)-regular cover if for every \(\alpha \in \Delta\) there exists a nonempty \(ji\)-regular closed subset \(C_\alpha\) of \(X\) such that \(C_\alpha \subseteq U_\alpha\) and \(X = \bigcup_{\alpha \in \Delta} i\text{-int}(C_\alpha)\). \(\{U_\alpha : \alpha \in \Delta\}\) is said pairwise regular cover if it is both \(ij\)-regular cover and \(ji\)-regular cover.

Definition 3.2. A bitopological space \(X\) is said to be \(ij\)-almost regular-Lindelöf [17] (resp., \(ij\)-nearly regular-Lindelöf [18]) if for every \(ij\)-regular cover \(\{U_\alpha : \alpha \in \Delta\}\) of \(X\) there exists a countable subset \(\{a_n : n \in \mathbb{N}\}\) of \(\Delta\) such that

\[
X = \bigcup_{n \in \mathbb{N}} j\text{-cl}(U_{a_n}) \tag{3.1}
\]
then $X$ is said pairwise almost regular-Lindelöf (resp., pairwise nearly regular-Lindelöf) if it is both $ij$-almost regular-Lindelöf (resp., $ij$-nearly regular-Lindelöf) and $ji$-almost regular-Lindelöf (resp., $ji$-nearly regular-Lindelöf).

Definition 3.3. A bitopological space $X$ is said to be $ij$-weakly regular-Lindelöf if for every $ij$-regular cover $\{U_\alpha : \alpha \in \Delta\}$ of $X$, there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of $\Delta$ such that $X = \text{j-cl}(\bigcup_{n \in \mathbb{N}} U_{\alpha_n})$. $X$ is said pairwise weakly regular-Lindelöf if it is both $ij$-weakly regular-Lindelöf and $ji$-weakly regular-Lindelöf.

Obviously, every $ij$-weakly Lindelöf space is $ij$-weakly regular-Lindelöf, and every $ij$-almost regular-Lindelöf space is $ij$-weakly regular-Lindelöf.

Question 1. Is $ij$-weakly regular-Lindelöf spaces implies $ij$-weakly Lindelöf?

Question 2. Is $ij$-weakly regular-Lindelöf spaces implies $ij$-almost regular-Lindelöf?

The authors expected that the answer of these questions is no. We can answer Question 1. by some restrictions on the space with the following proposition. First of all, we need the following lemmas.

Lemma 3.4 (see [17]). Let $X$ be an $ij$-almost regular space. Then, for each $x \in X$ and for each $ij$-regular open subset $W$ of $X$ containing $x$ there exist two $ij$-regular open subsets $U$ and $V$ of $X$ such that $x \in U \subseteq j\text{-cl}(U) \subseteq V \subseteq j\text{-cl}(V) \subseteq W$.

Lemma 3.5 (see [17]). A space $X$ is $ij$-regular if and only if it is $ij$-almost regular and $ij$-semiregular.

Proposition 3.6. An $ij$-weakly regular-Lindelöf and $ij$-regular space $X$ is $ij$-weakly Lindelöf.

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be an $ij$-regular open cover of $X$. For each $x \in X$, there exists $\alpha_x \in \Delta$ such that $x \in U_{\alpha_x}$. Since $X$ is $ij$-almost regular, there exist two $ij$-regular open subsets $V_{\alpha_x}$ and $W_{\alpha_x}$ of $X$ such that $x \in V_{\alpha_x} \subseteq j\text{-cl}(V_{\alpha_x}) \subseteq W_{\alpha_x} \subseteq j\text{-cl}(W_{\alpha_x}) \subseteq U_{\alpha_x}$ by Lemma 3.4. Since for each $\alpha \in \Delta$, there exists a $ji$-regular closed set $j\text{-cl}(V_{\alpha_x})$ in $X$ such that $j\text{-cl}(V_{\alpha_x}) \subseteq W_{\alpha_x}$ and $X = \bigcup_{\alpha \in \Delta} V_{\alpha_x} = \bigcup_{\alpha \in \Delta} j\text{-int}(j\text{-cl}(V_{\alpha_x}))$, the family $\{W_{\alpha_x} : x \in X\}$ is an $ij$-regular cover of $X$. Since $X$ is $ij$-weakly regular-Lindelöf, there exists a countable set of points $\{x_n : n \in \mathbb{N}\}$ of $X$ such that $X = j\text{-cl}(\bigcup_{n \in \mathbb{N}} W_{\alpha_n}) \subseteq j\text{-cl}(\bigcup_{n \in \mathbb{N}} U_{\alpha_n})$. So, $X = j\text{-cl}(\bigcup_{n \in \mathbb{N}} U_{\alpha_n})$ and since $X$ is $ij$-semiregular, therefore $X$ is $ij$-weakly Lindelöf.

Corollary 3.7. A pairwise weakly regular-Lindelöf and pairwise regular space $X$ is pairwise weakly Lindelöf.

Proposition 3.6 implies the following corollaries.

Corollary 3.8. Let $X$ be an $ij$-regular space. Then, $X$ is $ij$-weakly regular-Lindelöf if and only if it is $ij$-weakly Lindelöf.

Corollary 3.9. Let $X$ be a pairwise regular space. Then, $X$ is pairwise weakly regular-Lindelöf if and only if it is pairwise weakly Lindelöf.

Definition 3.10 (see [8]). A bitopological space $X$ is called $ij$-weak $P$-space if for each countable family $\{U_n : n \in \mathbb{N}\}$ of $i$-open sets in $X$, we have $j\text{-cl}(\bigcup_{n \in \mathbb{N}} U_n) = \bigcup_{n \in \mathbb{N}} j\text{-cl}(U_n)$ then $X$ is called pairwise weak $P$-space if it is both $ij$-weak $P$-space and $ji$-weak $P$-space.
The following proposition shows that in $ij$-weak $P$-spaces, $ij$-almost regular-Lindelöf property equivalent to $ij$-weakly regular-Lindelöf property.

**Proposition 3.11.** Let $X$ be an $ij$-weak $P$-spaces. Then, $X$ is $ij$-almost regular-Lindelöf if and only if $X$ is $ij$-weakly regular-Lindelöf.

**Proof.** The proof follows immediately from the fact that in $ij$-weak $P$-spaces, $\bigcup_{n \in \mathbb{N}} j(\text{cl}(U_n)) = j(\text{cl}(\bigcup_{n \in \mathbb{N}} U_n))$ for any countable family $\{U_n : n \in \mathbb{N}\}$ of $i$-open sets in $X$. □

**Corollary 3.12.** Let $X$ be a pairwise weak $P$-spaces. Then, $X$ is pairwise almost regular-Lindelöf if and only if $X$ is pairwise weakly regular-Lindelöf.

If $X$ is an $ij$-almost regular space, then $X$ is $ij$-almost regular-Lindelöf if and only if it is $ij$-nearly Lindelöf (see [17]). Thus, we have the following corollary.

**Corollary 3.13.** In $ij$-almost regular and $ij$-weak $P$-spaces, $ij$-weakly regular-Lindelöf property is equivalent to $ij$-nearly Lindelöf property.

**Proof.** This is a direct consequence of Proposition 3.11 and the previous fact. □

**Corollary 3.14.** In pairwise almost regular and pairwise weak $P$-spaces, pairwise weakly regular-Lindelöf property is equivalent to pairwise nearly Lindelöf property.

**Lemma 3.15** (see [17]). An $ij$-regular and $ij$-almost regular-Lindelöf space $X$ is $i$-Lindelöf.

**Corollary 3.16.** In $ij$-regular and $ij$-weak $P$-spaces, $ij$-weakly regular-Lindelöf property is equivalent to $i$-Lindelöf property.

**Proof.** This is a direct consequence of Proposition 3.11 and Lemma 3.15. □

**Corollary 3.17.** In pairwise regular and pairwise weak $P$-spaces, pairwise weakly regular-Lindelöf property is equivalent to Lindelöf property.

**Definition 3.18** (see [8]). A subset $E$ of a bitopological space $X$ is said to be $i$-dense in $X$ or is an $i$-dense subset of $X$ if $i(\text{cl}(E)) = X$. $E$ is said dense in $X$ or is a dense subset of $X$ if it is $i$-dense in $X$ or is an $i$-dense subset of $X$ for each $i = 1, 2$.

**Definition 3.19** (see [8]). A bitopological space $X$ is said to be $i$-separable if there exists a countable $i$-dense subset of $X$. $X$ is said separable if it is $i$-separable for each $i = 1, 2$.

**Lemma 3.20** (see [8]). If the bitopological space $X$ is $j$-separable, then it is $ij$-weakly Lindelöf.

**Lemma 3.21** (see [18]). An $ij$-regular and $ij$-nearly regular-Lindelöf space $X$ is $i$-Lindelöf.

It is clear that every $ij$-nearly regular-Lindelöf is $ij$-weakly regular-Lindelöf and every $ij$-almost regular-Lindelöf space is $ij$-weakly regular-Lindelöf, but the converses are not true in general as the following example show.

**Example 3.22.** Let $\mathcal{B}$ be the collection of closed-open intervals in the real line $\mathbb{R}$:

$$\mathcal{B} = \{[a, b) : a, b \in \mathbb{R}, \ a < b\}. \quad (3.2)$$

Hence, $\mathcal{B}$ is a base for the lower limit topology $\tau_1$ on $\mathbb{R}$. Choose usual topology as topology $\tau_2$ on $\mathbb{R}$. Thus, $(\mathbb{R}, \tau_1, \tau_2)$ is a Lindelöf bitopological space (see [19]). Note that, sets of the form
Then, $X$

**Proof.** and only if other forms of $1$-open sets in Theorem 3.29.

Let $X$ be an $1$-open set such that for each $x \in \mathbb{R}$ and for each $1$-open set of the form $[a, b)$ in $\mathbb{R}$ containing $x$, there exists a $1$-open set $[a, b-e)$ with $e > 0$ such that $x \in [a, b-e) \subseteq 2\text{-cl}[a, b-e) \subseteq [a, b)$. We left to the reader to check for other forms of $1$-open sets in $\mathbb{R}$. It is clear that $\mathbb{R}$ is 2-separable since the rational numbers are a countable 2-dense subset of $\mathbb{R}$. So $(\mathbb{R} \times \mathbb{R}, \tau_1 \times \tau_1, \tau_2 \times \tau_2)$ is 12-regular and 2-separable. Thus, $\mathbb{R} \times \mathbb{R}$ is 12-weakly Lindelöf by Lemma 3.20, and so $\mathbb{R} \times \mathbb{R}$ is 12-weakly regular-Lindelöf. It is known that $\mathbb{R} \times \mathbb{R}$ is not 1-Lindelöf since the 1-closed subspace $L = \{(x, y) : y = -x\}$ is not 1-Lindelöf for it is a discrete subspace (see [19]). Since $\mathbb{R} \times \mathbb{R}$ is 12-regular, but not 1-Lindelöf, then it is neither 12-almost regular-Lindelöf nor 12-nearly regular-Lindelöf by Lemmas 3.15 and 3.21.

It is clear that every $ij$-almost Lindelöf is $ij$-weakly Lindelöf, but the converse is not true as in the following example show.

**Lemma 3.23** (see [16]). An $ij$-regular space is $ij$-almost Lindelöf if and only if it is $i$-Lindelöf.

**Example 3.24.** Let $(\mathbb{R}, \tau_1, \tau_2)$ be a bitopological space defined as in Example 3.22 above. Example 3.22 shows that $\mathbb{R} \times \mathbb{R}$ is 12-weakly Lindelöf, but not 1-Lindelöf. Since $\mathbb{R} \times \mathbb{R}$ is 12-regular, but not 1-Lindelöf, then it is nor 12-almost Lindelöf by Lemma 3.23.

**Remark 3.25.** Example 3.24 solves the open problem in [8, Question 1].

**Lemma 3.26** (see [8]). An $ij$-weakly Lindelöf, $ij$-regular, and $ij$-nearly paracompact bitopological space $X$ is $i$-Lindelöf.

**Proposition 3.27.** Let $X$ be an $ij$-regular and $ij$-nearly paracompact spaces. Then, $X$ is $i$-Lindelöf if and only if $X$ is $ij$-weakly regular-Lindelöf.

**Proof.** Let $X$ be an $ij$-regular, $ij$-nearly paracompact, and $ij$-weakly regular-Lindelöf space. Then, $X$ is $ij$-weakly Lindelöf by Proposition 3.6. So $X$ is $i$-Lindelöf by Lemma 3.26. The converse is obvious.

**Corollary 3.28.** Let $X$ be a pairwise regular and pairwise nearly paracompact spaces. Then, $X$ is Lindelöf if and only if $X$ is pairwise weakly regular-Lindelöf.

Now, we give a characterization of $ij$-weakly regular-Lindelöf spaces.

**Theorem 3.29.** A bitopological spaces $X$ is $ij$-weakly regular-Lindelöf if and only if for every family

$$
\{C_\alpha : \alpha \in \Delta\}
$$

of $i$-closed subsets of $X$ such that for each $\alpha \in \Delta$, there exists a $j$-open subset $A_\alpha$ of $X$ with $A_\alpha \supseteq C_\alpha$ and $\bigcap_{\alpha \in \Delta} j\text{-cl}(A_\alpha) = \emptyset$, there exists a countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ such that $j\text{-int}(\bigcap_{n \in \mathbb{N}} C_{\alpha_n}) = \emptyset$.

**Proof.** Let $\{C_\alpha : \alpha \in \Delta\}$ be a family of $i$-closed subsets of $X$ such that for each $\alpha \in \Delta$ there exists a $j$-open subset $A_\alpha$ of $X$ with $A_\alpha \supseteq C_\alpha$ and $\bigcap_{\alpha \in \Delta} j\text{-cl}(A_\alpha) = \emptyset$. It follows that $X = X \setminus (\bigcap_{\alpha \in \Delta} j\text{-cl}(A_\alpha)) = \bigcup_{\alpha \in \Delta} (X \setminus j\text{-cl}(A_\alpha)) = \bigcup_{\alpha \in \Delta} j\text{-int}(X \setminus A_\alpha)$. Since $C_\alpha \subseteq A_\alpha \subseteq j\text{-int}(i\text{-cl}(A_\alpha)) \subseteq i\text{-cl}(A_\alpha)$, then $X \setminus j\text{-cl}(A_\alpha) \subseteq X \setminus j\text{-int}(i\text{-cl}(A_\alpha)) \subseteq X \setminus C_\alpha$, that is, $i\text{-int}(X \setminus A_\alpha) \subseteq j\text{-cl}(j\text{-int}(X \setminus A_\alpha)) \subseteq X \setminus C_\alpha$. Therefore,

$$
X = \bigcup_{\alpha \in \Delta} j\text{-int}(X \setminus A_\alpha) \subseteq \bigcup_{\alpha \in \Delta} (X \setminus C_\alpha). \tag{3.3}
$$
So X = ∪α∈Δ(X \ C_α) and the family {X \ C_α : α ∈ Δ} is an ij-regular cover of X. Since X is
ij-weakly regular-Lindelöf, there exists a countable subfamily {X \ C_{α_n} : n ∈ N} such that
\[
X = j\text{-cl}\left(\bigcup_{n \in \mathbb{N}} X \setminus C_{α_n}\right) = j\text{-cl}\left(X \setminus \left(\bigcap_{n \in \mathbb{N}} C_{α_n}\right)\right) = X \setminus \left(j\text{-int}\left(\bigcap_{n \in \mathbb{N}} C_{α_n}\right)\right) .
\] ~ (3.4)

Therefore, j-int(∩_{n ∈ \mathbb{N}} C_{α_n}) = ∅.

Conversely, let {U_α : α ∈ Δ} be an ij-regular cover of X. By Definition 3.1, for each α ∈ Δ,
U_α is i-open set in X and there exists a ji-regular closed subset C_α of X such that C_α ⊆ U_α and
X = ∪_{α ∈ Δ} i-int(C_α). The family {X \ U_α : α ∈ Δ} of i-closed subsets of X is satisfying the
condition, for each α ∈ Δ, there exists a j-open subset X \ C_α of X such that X \ C_α ⊆ X \ U_α and
\[
\bigcap_{α ∈ Δ} i\text{-cl}(X \setminus C_α) = \bigcap_{α ∈ Δ} (X \setminus i\text{-int}(C_α)) = X \setminus \left(\bigcup_{α ∈ Δ} i\text{-int}(C_α)\right) = X \setminus X = ∅ .
\] ~ (3.5)

By hypothesis, there exists a countable subset {α_n : n ∈ N} of Δ such that j-int(∩_{n ∈ \mathbb{N}} (X \ U_{α_n})) = ∅, that is, j-int(X \ ∪_{n ∈ \mathbb{N}} U_{α_n}) = ∅. So X \ j-cl(∪_{n ∈ \mathbb{N}} U_{α_n}) = ∅ and, therefore, X = j-cl(∪_{n ∈ \mathbb{N}} U_{α_n}).

This completes the proof.

Corollary 3.30. A bitopological spaces X is pairwise weakly regular-Lindelöf if and only if for every
family {C_α : α ∈ Δ} of closed subsets of X such that for each α ∈ Δ, there exists an open subset A_α
of X with A_α ≥ C_α and ∩_{α ∈ Δ} cl(A_α) = ∅, there exists a countable subfamily {C_{α_n} : n ∈ N} such that
int(∩_{n ∈ \mathbb{N}} C_{α_n}) = ∅.

The following diagram illustrates the relationship among the generalizations of pairwise
Lindelöf spaces and the generalizations of pairwise regular-Lindelöf spaces in terms of ij-:

\[
\begin{array}{ccc}
j\text{-nearly Lindelöf} & \longrightarrow & j\text{-almost Lindelöf} & \longrightarrow & j\text{-weakly Lindelöf} \\
\downarrow & & \downarrow & & \downarrow \\
j\text{-nearly} & \longrightarrow & j\text{-almost} & \longrightarrow & j\text{-weakly} \\
\text{regular-Lindelöf} & & \text{regular-Lindelöf} & & \text{regular-Lindelöf}
\end{array}
\] ~ (3.6)

4. Pairwise weakly regular-Lindelöf subspaces and subsets

A subset S of a bitopological space X is said to be ij-weakly regular-Lindelöf (resp., pairwise
weakly regular-Lindelöf) if S is ij-weakly regular-Lindelöf (resp., pairwise weakly regular-Lindelöf) as a subspace of X, that is, S is ij-weakly regular-Lindelöf (resp., pairwise weakly
regular-Lindelöf) with respect to the inducted bitopology from the bitopology of X.

Definition 4.1 (see [17]). Let S be a subset of a bitopological space X. A cover {U_α : α ∈ Δ} of
S by i-open subsets of X such that S ⊆ ∪_{α ∈ Δ} U_α is said to be ij-regular cover of S by i-open
subsets of X if for each α ∈ Δ, there exists a nonempty ji-regular closed subset C_α of X such
that C_α ⊆ U_α and S ⊆ ∪_{α ∈ Δ} i-int(C_α). {U_α : α ∈ Δ} is said pairwise regular cover by open
subsets of X if it is both ij-regular cover of S by i-open subsets of X and ji-regular cover of S by j-open subsets of X.
Definition 4.2 (see [17]). A subset $S$ of a bitopological space $X$ is said to be $ij$-almost regular-Lindelöf relative to $X$ if for every $ij$-regular cover $\{U_{\alpha} : \alpha \in \Delta\}$ of $S$ by $i$-open subsets of $X$ there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of $\Delta$ such that $S \subseteq \bigcup_{n \in \mathbb{N}} j-cl(U_{\alpha_n})$. $S$ is said pairwise almost regular-Lindelöf relative to $X$ if it is both $ij$-almost regular-Lindelöf relative to $X$ and $ji$-almost regular-Lindelöf relative to $X$.

Definition 4.3. A subset $S$ of a bitopological space $X$ is said to be $ij$-weakly regular-Lindelöf relative to $X$ if for every $ij$-regular cover $\{U_{\alpha} : \alpha \in \Delta\}$ of $S$ by $i$-open subsets of $X$ there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of $\Delta$ such that $S \subseteq \bigcup_{n \in \mathbb{N}} j-cl(U_{\alpha_n})$. $S$ is said pairwise weakly regular-Lindelöf relative to $X$ if it is both $ij$-weakly regular-Lindelöf relative to $X$ and $ji$-weakly regular-Lindelöf relative to $X$.

Obviously, every $ij$-weakly Lindelöf relative to the space is $ij$-weakly regular-Lindelöf relative to the space and every $ij$-almost regular-Lindelöf relative to the space is $ij$-weakly regular-Lindelöf relative to the space.

Question 3. Is $ij$-weakly regular-Lindelöf relative to the space implies $ij$-weakly Lindelöf relative to the space?

Question 4. Is $ij$-weakly regular-Lindelöf relative to the space implies $ij$-almost regular-Lindelöf relative to the space?

The authors expected that the answer of both questions is no.

Theorem 4.4. A subset $S$ of a bitopological spaces $X$ is $ij$-weakly regular-Lindelöf relative to $X$ if and only if for every family $\{C_{\alpha} : \alpha \in \Delta\}$ of $i$-closed subsets of $X$ such that for each $\alpha \in \Delta$ there exists a $j$-open subset $A_{\alpha}$ of $X$ with $A_{\alpha} \supseteq C_{\alpha}$ and $(\bigcap_{\alpha \in \Delta} i-cl(A_{\alpha})) \cap S = \emptyset$ there exists a countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ such that $(j-int(\bigcap_{n \in \mathbb{N}} C_{\alpha_n})) \cap S = \emptyset$.

Proof. Let $\{C_{\alpha} : \alpha \in \Delta\}$ be a family of $i$-closed subsets of $X$ such that for each $\alpha \in \Delta$ there exists a $j$-open subset $A_{\alpha}$ of $X$ with $A_{\alpha} \supseteq C_{\alpha}$ and $(\bigcap_{\alpha \in \Delta} i-cl(A_{\alpha})) \cap S = \emptyset$. It follows that $S \subseteq X \setminus (\bigcap_{\alpha \in \Delta} i-cl(A_{\alpha})) = \bigcup_{\alpha \in \Delta} (X \setminus i-cl(A_{\alpha})) = \bigcup_{\alpha \in \Delta} i-int(X \setminus A_{\alpha})$. Since $C_{\alpha} \subseteq A_{\alpha} \subseteq j-int(i-cl(A_{\alpha})) \subseteq i-cl(A_{\alpha})$, then $X \setminus i-cl(A_{\alpha}) \subseteq X \setminus j-int(i-cl(A_{\alpha})) \subseteq X \setminus C_{\alpha}$, that is, $i-int(X \setminus A_{\alpha}) \subseteq j-cl(i-int(X \setminus A_{\alpha})) \subseteq X \setminus C_{\alpha}$. Therefore, $S \subseteq \bigcup_{\alpha \in \Delta} i-int(X \setminus A_{\alpha}) \subseteq \bigcup_{\alpha \in \Delta} i-cl(X \setminus C_{\alpha})$. So $j-cl(i-int(X \setminus A_{\alpha}))$ is a $ji$-regular closed subset of $X$ satisfying the condition of Definition 4.1. Thus, the family $\{X \setminus C_{\alpha} : \alpha \in \Delta\}$ is an $ij$-regular cover of $S$ by $i$-open subsets of $X$. Since $X$ is $ij$-weakly regular-Lindelöf relative to $X$, there exists a countable subfamily $\{X \setminus C_{\alpha_n} : n \in \mathbb{N}\}$ such that

$$S \subseteq j-cl\left(\bigcup_{n \in \mathbb{N}} (X \setminus C_{\alpha_n})\right) = j-cl\left(X \setminus \bigcap_{n \in \mathbb{N}} C_{\alpha_n}\right) = X \setminus j-int\left(\bigcap_{n \in \mathbb{N}} C_{\alpha_n}\right).$$

(4.1)

Therefore, $(j-int(\bigcap_{n \in \mathbb{N}} C_{\alpha_n})) \cap S = \emptyset$.

Conversely, let $\{U_{\alpha} : \alpha \in \Delta\}$ be an $ij$-regular cover of $S$ by $i$-open subsets of $X$. By Definition 4.1, for each $\alpha \in \Delta$, there exists a $ji$-regular closed subset $C_{\alpha}$ of $X$ such that $C_{\alpha} \subseteq U_{\alpha}$ and $S \subseteq \bigcup_{\alpha \in \Delta} i-int(C_{\alpha})$. The family $\{X \setminus U_{\alpha} : \alpha \in \Delta\}$ of $i$-closed subsets of $X$ is satisfying the condition, for each $\alpha \in \Delta$, there exists a $j$-open set $X \setminus C_{\alpha} \supseteq X \setminus U_{\alpha}$ with

$$S \subseteq \bigcup_{\alpha \in \Delta} i-int(C_{\alpha}) = X \setminus \left(\bigcap_{\alpha \in \Delta} i-int(C_{\alpha})\right) = X \setminus j-int\left(\bigcap_{\alpha \in \Delta} (X \setminus C_{\alpha})\right).$$

(4.2)
then it follows that, \((\bigcap_{\alpha \in \Delta} i-\text{cl}(X \setminus C_{\alpha})) \cap S = \emptyset\). By hypothesis, there exists a countable subset \(\{\alpha_n : n \in \mathbb{N}\}\) of \(\Delta\) such that

\[
\left( j-\text{int}\left( \bigcap_{n \in \mathbb{N}} (X \setminus U_{\alpha_n}) \right) \right) \cap S = \emptyset, \quad \text{that is,} \quad \left( j-\text{int}\left( X \setminus \bigcup_{n \in \mathbb{N}} U_{\alpha_n} \right) \right) \cap S = \emptyset. \tag{4.3}
\]

Thus we have, \((X \setminus j-\text{cl}(\bigcup_{n \in \mathbb{N}} U_{\alpha_n})) \cap S = \emptyset\) and, therefore, \(S \subseteq j-\text{cl}(\bigcup_{n \in \mathbb{N}} U_{\alpha_n})\). This completes the proof. \(\Box\)

**Corollary 4.5.** A subset \(S\) of a bitopological spaces \(X\) is pairwise weakly regular-Lindelöf relative to \(X\) if and only if for every family \(\{C_{\alpha} : \alpha \in \Delta\}\) of closed subsets of \(X\) such that for each \(\alpha \in \Delta\) there exists an open subset \(A_{\alpha}\) of \(X\) with \(A_{\alpha} \supseteq C_{\alpha}\) and \((\bigcap_{\alpha} i-\text{cl}(A_{\alpha})) \cap S = \emptyset\), there exists a countable subfamily \(\{C_{\alpha_n} : n \in \mathbb{N}\}\) such that \((\text{int}(\bigcap_{\alpha} C_{\alpha_n})) \cap S = \emptyset\).

**Proposition 4.6.** A subset \(S\) of a space \(X\) is \(ij\)-weakly regular-Lindelöf relative to \(X\) if and only if for every family \(\{U_{\alpha} : \alpha \in \Delta\}\) of \(ij\)-regular open subsets of \(X\) satisfying the conditions \(S \subseteq \bigcup_{\alpha \in \Delta} U_{\alpha}\) and for each \(\alpha \in \Delta\) there exists a nonempty \(ji\)-regular closed subset \(C_{\alpha}\) of \(X\) such that \(C_{\alpha} \subseteq U_{\alpha}\) and \(S \subseteq \bigcup_{\alpha} i-\text{int}(C_{\alpha})\), then there exists a countable subset \(\{\alpha_n : n \in \mathbb{N}\}\) of \(\Delta\) such that \(S \subseteq j-\text{cl}(\bigcup_{n \in \mathbb{N}} U_{\alpha_n})\).

**Proof.** The necessity is obvious by the Definitions 4.1 and 4.2 since every \(ij\)-regular open set in \(X\) is \(i\)-open. For the sufficiency, let \(\{U_{\alpha} : \alpha \in \Delta\}\) be a family of \(i\)-open sets in \(X\) satisfying the conditions of Definition 4.1 above. Then \(\{\text{int}(\bigcup_{\alpha} i-\text{cl}(U_{\alpha})) : \alpha \in \Delta\}\) is a family of \(ij\)-regular open sets in \(X\) satisfying the conditions of the theorem, since for each \(\alpha \in \Delta\), we have \(C_{\alpha} \subseteq U_{\alpha} \subseteq i-\text{int}(\bigcup_{\alpha} i-\text{cl}(U_{\alpha}))\). By hypothesis, there exists a countable subset \(\{\alpha_n : n \in \mathbb{N}\}\) of \(\Delta\) such that

\[
S \subseteq j-\text{cl}\left( \bigcup_{n \in \mathbb{N}} \text{int}\left( \bigcup_{\alpha_n} i-\text{cl}(U_{\alpha_n}) \right) \right) \subseteq j-\text{cl}\left( \bigcup_{n \in \mathbb{N}} \text{cl}(U_{\alpha_n}) \right), \tag{4.4}
\]

This implies that \(S\) is \(ij\)-weakly regular-Lindelöf relative to \(X\) and completes the proof. \(\Box\)

**Corollary 4.7.** A subset \(S\) of a space \(X\) is pairwise weakly regular-Lindelöf relative to \(X\) if and only if for every family \(\{U_{\alpha} : \alpha \in \Delta\}\) of pairwise regular open subsets of \(X\) satisfying the conditions \(S \subseteq \bigcup_{\alpha \in \Delta} U_{\alpha}\) and for each \(\alpha \in \Delta\) there exists a nonempty pairwise regular closed subset \(C_{\alpha}\) of \(X\) such that \(C_{\alpha} \subseteq U_{\alpha}\) and \(S \subseteq \bigcup_{\alpha} \text{int}(C_{\alpha})\), then there exists a countable subset \(\{\alpha_n : n \in \mathbb{N}\}\) of \(\Delta\) such that \(S \subseteq \text{cl}(\bigcup_{n \in \mathbb{N}} U_{\alpha_n})\).

**Proposition 4.8.** If \(\{A_k : k \in \mathbb{N}\}\) is a countable family of subsets of a space \(X\) such that each \(A_k\) is \(ij\)-weakly regular-Lindelöf relative to \(X\), then \(\bigcup_{k \in \mathbb{N}} A_k\) is \(ij\)-weakly regular-Lindelöf relative to \(X\).

**Proof.** Let \(\{U_{\alpha} : \alpha \in \Delta\}\) be an \(ij\)-regular cover of \(\bigcup_{k \in \mathbb{N}} A_k\) by \(i\)-open subsets of \(X\). Then for each \(\alpha \in \Delta\), there exists a nonempty \(ji\)-regular closed subset \(C_{\alpha}\) of \(X\) such that \(C_{\alpha} \subseteq U_{\alpha}\) and \(\bigcup_{\alpha \in \Delta} A_k \subseteq \bigcup_{\alpha \in \Delta} \text{int}(C_{\alpha})\). Let \(\Delta_k = \{\alpha \in \Delta : U_{\alpha} \cap A_k \neq \emptyset\}\), then for each \(\alpha_k \in \Delta_k \subseteq \Delta\) there exists a nonempty \(ji\)-regular closed subset \(C_{\alpha_k}\) of \(X\) such that \(C_{\alpha_k} \subseteq U_{\alpha_k}\) and \(A_k \subseteq \bigcup_{\alpha_k \in \Delta_k} \text{int}(C_{\alpha_k})\). So \(\{U_{\alpha_k} : \alpha_k \in \Delta_k\}\) is an \(ij\)-regular cover of \(A_k\) by \(i\)-open subsets of \(X\). Since \(A_k\) is \(ij\)-weakly
regular-Lindelöf relative to \( X \), there exists a countable subfamily \( \{ U_{a_n} : n \in \mathbb{N} \} \) such that \( A_k \subseteq j\text{-cl}(\bigcup_{n \in \mathbb{N}} U_{a_n}) \). But a countable union of countable sets is countable, so

\[
\bigcup_{k \in \mathbb{N}} A_k \subseteq \bigcup_{k \in \mathbb{N}} \left( j\text{-cl} \left( \bigcup_{a \in \Delta} (i_{-\text{int}}(F_a) \cap \bigcup_{n \in \mathbb{N}} U_{a_n}) \right) \right) \subseteq j\text{-cl} \left( \bigcup_{a \in \Delta} \left( i_{-\text{int}}(F_a) \cap \bigcup_{n \in \mathbb{N}} U_{a_n} \right) \right) = j\text{-cl} \left( \bigcup_{n \in \mathbb{N}} U_{a_n} \right) .
\] (4.5)

This implies that \( \bigcup \{ A_k : k \in \mathbb{N} \} \) is \( ij\)-weakly regular-Lindelöf relative to \( X \) and completes the proof. \( \square \)

**Corollary 4.9.** If \( \{ A_k : k \in \mathbb{N} \} \) is a countable family of subsets of a space \( X \) such that each \( A_k \) is pairwise weakly regular-Lindelöf relative to \( X \), then \( \bigcup \{ A_k : k \in \mathbb{N} \} \) is pairwise weakly regular-Lindelöf relative to \( X \).

**Proposition 4.10.** If \( S \) is an \( ij\)-weakly regular-Lindelöf subspace of a bitopological space \( X \), then \( S \) is \( ij\)-weakly regular-Lindelöf relative to \( X \).

**Proof.** Let \( \{ U_a : a \in \Delta \} \) be an \( ij\)-regular cover of \( S \) by \( i\)-open subsets of \( X \). Then, for each \( a \in \Delta \), there exists a nonempty \( ji\)-regular closed subset \( C_a \) of \( X \) such that \( C_a \subseteq U_a \) and \( S \subseteq \bigcup_{a \in \Delta} i_{-\text{int}}X(C_a) \). For each \( a \in \Delta \), we have \( i_{-\text{int}}X(C_a) \cap S \subseteq U_a \cap S \subseteq i\)-open sets in \( S \), and \( C_a \cap S \) is \( j\)-closed set in \( S \). Since for each \( a \in \Delta \), there exists a \( ji\)-regular closed set \( j\text{-cl}(i_{-\text{int}}X(C_a) \cap S) \) in \( S \) such that \( j\text{-cl}(j_{-\text{cl}}(i_{-\text{int}}X(C_a) \cap S)) \subseteq C_a \cap S \subseteq U_a \cap S \) and

\[
S = \left( \bigcup_{a \in \Delta} \left( i_{-\text{int}}(C_a) \cap S \right) \right) \cap S = \bigcup_{a \in \Delta} \left( i_{-\text{int}}(C_a) \cap S \right) \subseteq \bigcup_{a \in \Delta} \left( j\text{-cl}(i_{-\text{int}}(C_a) \cap S) \right),
\] (4.6)

that is, \( S = \bigcup_{a \in \Delta} i_{-\text{int}}(j\text{-cl}(i_{-\text{int}}X(C_a) \cap S)) \), then the family \( \{ U_a \cap S : a \in \Delta \} \) is an \( ij\)-regular cover of \( S \). Since \( S \) is an \( ij\)-weakly regular-Lindelöf subspace of \( X \), there exists a countable subset \( \{ a_n : n \in \mathbb{N} \} \) of \( \Delta \) such that

\[
S = j\text{-cl} \left( \bigcup_{n \in \mathbb{N}} (U_{a_n} \cap S) \right) = \left( j\text{-cl} \left( \bigcup_{n \in \mathbb{N}} (U_{a_n} \cap S) \right) \right) \cap S \subseteq j\text{-cl} \left( \bigcup_{n \in \mathbb{N}} U_{a_n} \right) .
\] (4.7)

This shows that \( S \) is \( ij\)-weakly regular-Lindelöf relative to \( X \). \( \square \)

**Corollary 4.11.** If \( S \) is a pairwise weakly regular-Lindelöf subspace of a bitopological space \( X \), then \( S \) is pairwise weakly regular-Lindelöf relative to \( X \).

**Question 5.** Is the converse of Proposition 4.10 above true?

The authors expected that the answer is no.

**Theorem 4.12.** If every \( ji\)-regular closed proper subset of a bitopological space \( X \) is \( ij\)-weakly regular-Lindelöf relative to \( X \), then \( X \) is \( ij\)-weakly regular-Lindelöf.

**Proof.** Let \( \{ U_a : a \in \Delta \} \) be an \( ij\)-regular cover of \( X \). For each \( a \in \Delta \), there exists a nonempty \( ji\)-regular closed subset \( C_a \) of \( X \) such that \( C_a \subseteq U_a \) and \( X = \bigcup_{a \in \Delta} i_{-\text{int}}(C_a) \). Fix an arbitrary \( a_0 \in \Delta \) and let \( \Delta^* = \Delta \setminus \{ a_0 \} \). Put \( K = X \setminus \left( i_{-\text{int}}(C_{a_0}) \right) \), then \( K \) is an \( ij\)-regular closed subset of \( X \) and \( K \subseteq \bigcup_{a \in \Delta^*} i_{-\text{int}}X(C_a) \). Therefore, \( \{ U_a : a \in \Delta^* \} \) is an \( ij\)-regular cover of \( K \) by \( i\)-open subsets.
of $X$ by Definition 4.1. By hypothesis, $K$ is $ij$-weakly regular-Lindelöf relative to $X$, hence there exists a countable subset $\{\alpha_n : n \in \mathbb{N}^*\}$ of $\Delta^*$ such that $K \subseteq j-\text{cl}(\bigcup_{n \in \mathbb{N}^*} U_{\alpha_n})$. So, we have

$$X = K \cup (i-\text{int}(C_{\alpha})) \subseteq K \cup (j-\text{cl}(U_{\alpha})) \subseteq \left(j-\text{cl}\left(\bigcup_{n \in \mathbb{N}^*} U_{\alpha_n}\right)\right) \cup (j-\text{cl}(U_{\alpha}))$$

(4.8)

So $X = j-\text{cl}(\bigcup_{n \in \mathbb{N}^*} U_{\alpha_n})$ and this shows that $X$ is $ij$-weakly regular-Lindelöf. 

**Corollary 4.13.** If every pairwise regular closed proper subset of a bitopological space $X$ is pairwise weakly regular-Lindelöf relative to $X$, then $X$ is pairwise weakly regular-Lindelöf.

It is very clear that Theorem 4.12 implies the following corollaries.

**Corollary 4.14.** If every $ij$-regular closed subset of a bitopological space $X$ is $ij$-weakly regular-Lindelöf relative to $X$, then $X$ is $ij$-weakly regular-Lindelöf.

**Corollary 4.15.** If every pairwise regular closed subset of a bitopological space $X$ is pairwise weakly regular-Lindelöf relative to $X$, then $X$ is pairwise weakly regular-Lindelöf.

Note that, the space $X$ in above propositions is any bitopological space. If we consider $X$ itself is an $ij$-weakly regular-Lindelöf, we have the following results.

**Theorem 4.16.** Let $X$ be an $ij$-weakly regular-Lindelöf space. If $A$ is a proper $ij$-clopen subset of $X$, then $A$ is $ij$-weakly regular-Lindelöf relative to $X$.

**Proof.** Let $\{U_a : a \in \Delta\}$ be an $ij$-regular cover of $A$ by $i$-open subsets of $X$. Hence the family $\{U_a : a \in \Delta\} \cup \{X \setminus A\}$ is an $ij$-regular cover of $X$ since $X \setminus A$ is a proper $ji$-clopen subset of $X$ is also an $ji$-regular closed subset of $X$. Since $X$ is $ij$-weakly regular-Lindelöf, there exists a countable subfamily $\{X \setminus A, U_{\alpha_1}, U_{\alpha_2}, \ldots\}$ such that

$$X = j-\text{cl}\left(\bigcup_{n \in \mathbb{N}^*} U_{\alpha_n}\right) \cup j-\text{cl}(X \setminus A) = \left(j-\text{cl}\left(\bigcup_{n \in \mathbb{N}^*} U_{\alpha_n}\right)\right) \cup (X \setminus A).$$

(4.9)

But $A$ and $X \setminus A$ are disjoint; therefore, we have $A \subseteq j-\text{cl}(\bigcup_{n \in \mathbb{N}^*} U_{\alpha_n})$. This completes the proof.

**Corollary 4.17.** Let $X$ be a pairwise weakly regular-Lindelöf space. If $A$ is a proper clopen subset of $X$, then $A$ is pairwise weakly regular-Lindelöf relative to $X$.

It is very clear that Theorem 4.16 implies the following corollary.

**Corollary 4.18.** Let $X$ be an $ij$-weakly regular-Lindelöf space. If $A$ is an $ij$-clopen subset of $X$, then $A$ is $ij$-weakly regular-Lindelöf relative to $X$.

**Corollary 4.19.** Let $X$ be a pairwise weakly regular-Lindelöf space. If $A$ is a clopen subset of $X$, then $A$ is pairwise weakly regular-Lindelöf relative to $X$.
Question 6. Is $i$-closed subspace of an $ij$-weakly regular-Lindelöf space $X$ $ij$-weakly regular-Lindelöf?

Question 7. Is $ij$-regular closed subspace of an $ij$-weakly regular-Lindelöf space $X$ $ij$-weakly regular-Lindelöf?

The authors expected that the answer of both questions is no. Observe that the condition in Theorem 4.16 that a subset should be $ij$-clopen is necessary and it is not sufficient to be only $i$-open or $ij$-regular open as example below shows. Arbitrary subspaces of $ij$-weakly regular-Lindelöf spaces need not be $ij$-weakly regular-Lindelöf nor $ij$-weakly regular-Lindelöf relative to the spaces. An $i$-open or $ij$-open subset of an $ij$-weakly regular-Lindelöf space is neither $ij$-weakly regular-Lindelöf nor $ij$-weakly regular-Lindelöf relative to the spaces as in the following example also show. We need the following lemma (see [20, page 11]).

Lemma 4.20. If $A$ is a countable subset of ordinals $\Omega$ not containing $\omega_1$, where $\omega_1$ being the first uncountable ordinal, then $\sup A < \omega_1$.

Example 4.21. Let $\Omega$ denote the set of ordinals which are less than or equal to the first uncountable ordinal number $\omega_1$, that is, $\Omega = [1, \omega_1]$. This $\Omega$ is an uncountable well-ordered set with a largest element $\omega_1$, having the property that if $\alpha \in \Omega$ with $\alpha < \omega_1$, then $\{\beta \in \Omega : \beta \leq \alpha\}$ is countable. Since $\Omega$ is a totally ordered space, it can be provided with its order topology. Let us denote this order topology by $\tau_1$. Choose discrete topology as another topology for $\Omega$ denoted by $\tau_2$. So $(\Omega, \tau_1, \tau_2)$ form a bitopological space. Now it is known that $\Omega$ is a 1-Lindelöf space [20], so it is 12-weakly Lindelöf and thus 12-weakly regular-Lindelöf. The subspace $\Omega_0 = \Omega \setminus \{\omega_1\}$ is 12-regular Lindelöf, and thus 12-weakly regular-Lindelöf. The subspace $\Omega_0$ is 1-open subspace of $\Omega$ and also 12-regular open subset of $\Omega$. Observe that $\Omega_0$ is not 12-weakly regular-Lindelöf by Corollary 3.16 since it is 12-regular and 12-weak $P$-space. Moreover, $\Omega_0$ is not 12-weakly regular-Lindelöf relative to $\Omega$. In fact, the family $\{(1, \alpha) : \alpha \in \Omega_0\}$ of 1-open sets in $\Omega$ is 12-regular cover of $\Omega_0$ by 1-open subsets of $\Omega$ because $\Omega_0 \subseteq \bigcup_{\alpha \in \Omega_0} [1, \alpha)$ and for each $\alpha \in \Omega_0$, there exists a nonempty 21-regular closed subset $[1, \alpha)$ of $\Omega$ such that $[1, \alpha) \subseteq [1, \alpha)$ and $\Omega_0 \subseteq \bigcup_{\alpha \in \Omega_0} [1, \alpha) = \bigcup_{\alpha \in \Omega_0} 1\text{-int}([1, \alpha])$. But the family $\{(1, \alpha) : \alpha \in \Omega_0\}$ has no countable subfamily $\{(1, \alpha_n) : n \in \mathbb{N}\}$ such that $\Omega_0 \subseteq 2\text{-cl}(\bigcup_{n \in \mathbb{N}} [1, \alpha_n)) = \bigcup_{n \in \mathbb{N}} [1, \alpha_n)$. For if $\{(1, \alpha_1), (1, \alpha_2), \ldots\}$ satisfy the condition: 2-closures of unions of its elements cover $\Omega_0$, then $\sup \{\alpha_1, \alpha_2, \ldots\} = \omega_1$ which is impossible by Lemma 4.20.

So we can conclude that an $ij$-weakly regular-Lindelöf property is not hereditary property and, therefore, pairwise weakly regular-Lindelöf property is not so.

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