q-Genocchi Numbers and Polynomials Associated with Fermionic $p$-Adic Invariant Integrals on $\mathbb{Z}_p$

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The main purpose of this paper is to present a systemic study of some families of multiple Genocchi numbers and polynomials. In particular, by using the fermionic $p$-adic invariant integral on $\mathbb{Z}_p$, we construct $p$-adic Genocchi numbers and polynomials of higher order. Finally, we derive the following interesting formula:

$$G^{(k)}_{n+k,q}(x) = 2^k k! \left(\sum_{i=0}^n \sum_{l=0}^\infty \delta_{l,x} (l+1)^n \right),$$

where $G^{(k)}_{n+k,q}(x)$ are the $q$-Genocchi polynomials of order $k$.

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1. Introduction

Let $p$ be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ will, respectively, denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, the complex number field and the $p$-adic completion of the algebraic closure of $\mathbb{Q}_p$. Let $v_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$. When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|1 - q|_p < 1$, see [1–6].

In $\mathbb{C}$, the ordinary Euler polynomials are defined as

$$e^t e^{nt} = \sum_{n=0}^\infty E_n(x) \frac{t^n}{n!}, \quad (|t| < x). \quad (1.1)$$

In the case $x = 0$, $E_n(0) = E_n$ are called Euler numbers, see [1–13]. Let $\delta_{0,n}$ be the Kronecker symbol. From (1.1) we derive the following relation:

$$E_0 = 1, \quad (E + 1)^n + E_n = 2\delta_{0,n}, \quad n \in \mathbb{N}, \quad (1.2)$$
(cf. [7–13]). Here, we use the technique method notation by replacing $E^n$ by $E_n$ ($n \geq 0$), symbolically. The first few are 1, $-1/2$, 0, $1/4$, , and $E_{2k} = 0$ for $k = 1, 2, \ldots$. A sequence consisting of the Genocchi numbers $G_n$ satisfies the following relations:

$$\frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \quad (|t| < \pi), \quad (1.3)$$

see [11, 12]. It satisfies $G_1 = 1$, $G_3 = G_5 = G_7 = \cdots = G_{2k+1} = 0$, $k = 1, 2, 3, \ldots$, and even coefficients are given by

$$G_{2n} = 2(1 - 2^{2n})B_{2n} = 2nE_{2n-1}(0), \quad (1.4)$$

where $B_n$ is Bernoulli numbers. The first few Genocchi numbers for even integers are $-1, 1, -3, 17, -155, 2073, \ldots$ The first few prime Genocchi numbers are $-3$ and 17, which occur at $n = 6$ and 8. There are no others with $n < 10^5$. We now define the Genocchi polynomials as follows:

$$\frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \quad (|t| < \pi). \quad (1.5)$$

Thus, we note that

$$G_n(x) = \sum_{l=0}^{n} \binom{n}{l} G_l x^{n-l}. \quad (1.6)$$

In this paper, we use the following notations: $[x]_q = (1 - q^x)/(1 - q)$ and $[x]_{-q} = (1 + (-q)^x)/(1 + q)$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the $p$-adic $q$-deformed fermionic integral on $\mathbb{Z}_p$ is defined as

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[pN]_q} \sum_{x=0}^{pN-1} f(x)(-q)^x, \quad (1.7)$$

see [1–4]. The fermionic $p$-adic invariant integral on $\mathbb{Z}_p$ can be obtained as $q \to 1$. That is,

$$I_1(f) = \lim_{q \to 1} I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x). \quad (1.8)$$

From (1.8), we easily derive the following integral equation related to fermionic invariant $p$-adic integral on $\mathbb{Z}_p$:

$$I_1(f_1) + I_1(f) = 2f(0), \quad (1.9)$$

where $f_1(x) = f(x + 1)$, see [5].

The purpose of this paper is to present a systemic study of some families of multiple Genocchi numbers and polynomials by using the fermionic multivariate $p$-adic invariant integral on $\mathbb{Z}_p$. In addition, we will investigate some interesting identities related to Genocchi numbers and polynomials.
2. Genocchi numbers associated with fermionic $p$-adic invariant integral on $\mathbb{Z}_p$

From (1.9) we can derive

$$t \int_{\mathbb{Z}_p} e^{x t} d\mu_{-1}(x) = \frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}. \tag{2.1}$$

$$t \int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \tag{2.2}$$

where $G_n(x)$ are Genocchi polynomials. It is easy to check that

$$t \int_{\mathbb{Z}_p} e^{(2x+1)t} d\mu_{-1}(x) = \frac{2t}{e^t + e^{-t}} = t \sech t = \frac{1}{2} \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n-1} \left( \frac{n}{l} \right) 2^l G_l \right) \frac{t^n}{n!}. \tag{2.3}$$

By comparing the coefficient on both sides in (2.1), we easily see that

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(x) = \frac{G_{n+1}(x)}{n+1}. \tag{2.4}$$

Therefore, we obtain the following proposition.

**Proposition 2.1.** For $k \in \mathbb{Z}_+$,

(i) $\int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(x) = G_{n+1}(x)/(n+1)$ (Witt’s formula for Genocchi polynomials);

(ii) $\int_{\mathbb{Z}_p} e^{(2x+1)t} d\mu_{-1}(x) = (1/(n+1))((1/2)\sum_{l=0}^{n+1} \left( \frac{n}{l} \right) 2^l G_l)$, where $\left( \frac{n}{l} \right) = (n(n-1)\cdots(n-l+1))/l!$.

Let $\mathfrak{O}_p = \{ x \in \mathbb{C}_p \mid |x|_p \leq 1 \}$ be the integer ring of $\mathbb{C}_p$. We note that $i = (-1)^{1/2} \in \mathfrak{O}_p$. By using Taylor expansion, we see that

$$e^{ix} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}. \tag{2.5}$$

In the $p$-adic number field, sin $x$ and cos $x$ are defined as

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots. \tag{2.6}$$

From (2.4) and (2.5), we derive

$$e^{ix} = \cos x + i \sin x. \tag{2.7}$$

This is equivalent to

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}. \tag{2.8}$$
By (2.7), we easily see that
\[
\sec t = \frac{2}{e^{it} + e^{-it}} = \int_{\mathbb{Z}_p} e^{(2x+1)it} \, d\mu_{-1}(x) = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (2x+1)^n \, d\mu_{-1}(x) \frac{i^n t^n}{n!}.
\]

(2.8)

It is not difficult to show that \(\int_{\mathbb{Z}_p} (2x+1)^{2n+1} \, d\mu_{-1}(x) = 0\) for \(n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}\). From (2.8), we note that
\[
\sec t = \sum_{n=0}^{\infty} t^n \int_{\mathbb{Z}_p} (2x+1)^n \, d\mu_{-1}(x) \frac{i^n t^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \int_{\mathbb{Z}_p} (2x+1)^{2n} \, d\mu_{-1}(x) \frac{t^{2n}}{(2n)!}.
\]

(2.9)

Thus, we have
\[
t \sec t = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \left( \sum_{l=0}^{2n+1} \binom{2n+1}{l} (2l+1) \right) 2^l G_l \frac{t^{2n+1}}{(2n+1)!}.
\]

(2.10)

Now we consider the fermionic multivariate \(p\)-adic invariant integral on \(\mathbb{Z}_p\) as follows:
\[
t^k \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x_1+x_2+\cdots+x_k)t} \, d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) = \frac{t^k}{(e^t+1) \cdots (e^t+1)} = \sum_{n=0}^{\infty} G_n^{(k)} \frac{t^n}{n!}.
\]

(2.11)

where \(G_n^{(k)}\) are the \(n\)th Genocchi number of order \(k\). By comparing the coefficient on both sides in (2.11), we see that \(G_0^{(k)} = G_1^{(k)} = \cdots = G_n^{(k)} = 0\), and
\[
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1+x_2+\cdots+x_k)^n \, d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) (n+k)_k = G_n^{(k)} (n+k),
\]

(2.12)

where \((n+k)_k\) is the Jordan factor which is defined by \((n+k)_k = (n+k) \cdots (n+1)\). Thus, we note that
\[
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1+x_2+\cdots+x_k)^n \, d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) = \frac{G_n^{(k)}}{\binom{n+k}{n(n+k)} n!},
\]

(2.13)

for \(k \in \mathbb{N}, \ n \in \mathbb{Z}_+\).  

**Theorem 2.2.** For \(n \in \mathbb{Z}_+, \ k \in \mathbb{N},\)
\[
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1+x_2+\cdots+x_k)^n \, d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) = \frac{G_n^{(k)}}{\binom{n+k}{n(n+k)} n!}.
\]

(2.14)
The multinomial coefficient is well known as
\[
(x_1 + x_2 + \cdots + x_k)^n = \sum_{l_1 + \cdots + l_k = n} \binom{n}{l_1, \ldots, l_k} x_1^{l_1} x_2^{l_2} \cdots x_k^{l_k}.
\] (2.15)

Therefore, we obtain the following corollary.

**Corollary 2.3.** For \( n \in \mathbb{Z}_+ \), \( k \in \mathbb{N} \),
\[
\sum_{l_1 + \cdots + l_k = n} \binom{n}{l_1, \ldots, l_k} \frac{G_{l_1+1}}{l_1+1} \left(\frac{G_{l_2+1}}{l_2+1}\right) \cdots \left(\frac{G_{l_k+1}}{l_k+1}\right) = \frac{G_{n+k}^{(k)}}{(n+k)!}.
\] (2.16)

For \( q \in \mathbb{C}_p \) with \( |1 - q| < 1 \), it is not difficult to show that
\[
t \int_{\mathbb{Z}_p} q^x e^{xt} d\mu_{-1}(x) = \frac{2t}{qe^t + 1}.
\] (2.17)

Now, we define the \( q \)-extension of the Genocchi numbers as follows:
\[
\frac{2t}{qe^t + 1} = \sum_{n=0}^{\infty} G_{n,q} \frac{t^n}{n!}.
\] (2.18)

By (2.17) and (2.18), we easily see that
\[
\frac{G_{n+1,q}}{n+1} = \int_{\mathbb{Z}_p} q^x x^n d\mu_{-1}(x).
\] (2.19)

With the same motivation to construct the Genocchi polynomials of higher order, we can consider the \( q \)-extension of higher-order Genocchi numbers as follows:
\[
t^k \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{x_1 + 2x_2 + \cdots + (k-1)x_k} e^{(x_1 + x_2 + \cdots + x_k)t} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k)
\]
\[
= \frac{t^{k-2}}{(q^t + 1)(q^t e^t + 1) \cdots (q^{k-1} e^t + 1)} e^{xt} \sum_{n=0}^{\infty} G_{n,q}^{(k)} \frac{t^n}{n!},
\] (2.20)

where \( G_{n,q}^{(k)} \) are the \( q \)-Genocchi polynomials of order \( k \). The basic \( q \)-natural numbers are defined as
\[
[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \cdots + q^{n-1}, \quad n \in \mathbb{N}.
\] (2.21)

The \( q \)-factorial of \( n \) is defined as
\[
[n]_q! = [n]_q [n-1]_q \cdots [2]_q [1]_q = (1 + q + \cdots + q^{n-1}) \cdots (1 + q) \cdot 1.
\] (2.22)
The $q$-binomial coefficient is also defined as
\[
\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} = \frac{[n]_q [n-1]_q \cdots [n-k+1]_q}{[k]_q!}.
\] (2.23)

Note that $\lim_{q \to 1} \binom{n}{k}_q = \binom{n}{k} = (n(n-1) \cdots (n-k+1))/k!$. The $q$-binomial coefficient satisfies the following recursion formula:
\[
\binom{n+1}{k}_q = \binom{n}{k}_q + q^k \binom{n}{k-1}_q = q^{n-k} \binom{n}{k-1}_q + \binom{n}{k}_q.
\] (2.24)

From this recursion formula, we can derive
\[
\binom{n}{k}_q = \sum_{d_0+d_1+\cdots+d_k=k-1} q^{(d_0+1)d_1+\cdots+k d_k}.
\] (2.25)

The $q$-binomial expansion is given by
\[
\prod_{i=1}^n (a + bq^{i-1}) = \sum_{k=0}^{\infty} \binom{n}{k}_q q^k a^{n-k} b^k,
\] (2.26)
\[
\prod_{i=1}^n (1 - bq^{i-1})^{-1} = \sum_{k=0}^{\infty} \binom{n+k-1}{k}_q b^k.
\]

By (2.20) and (2.26), we see that
\[
\int_{Z_q} \cdots \int_{Z_q} q^{x_1^2 + x_2 + \cdots + (k-1)x_k} e^{(x_1 + x_2 + \cdots + x_k)/n} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k)
\]
\[
= t^k 2^k \prod_{i=1}^k (1 - (-q)^{-1} e^{t})^{-1} e^{x t}
\]
\[
= t^k 2^k \sum_{l=0}^{\infty} \binom{k+l-1}{l}_q (-1)^l e^{(l+x)t}
\]
\[
= t^k \sum_{n=0}^{\infty} 2^k \sum_{l=0}^{\infty} \binom{k+l-1}{l}_q (-1)^l (e^{x})^n \frac{t^n}{n!}
\] (2.27)

Therefore, we obtain the following theorem.

**Theorem 2.4.** For $n \in \mathbb{Z}_+$, $k \in \mathbb{N}$, we have
\[
\int_{Z_q} \cdots \int_{Z_q} q^{x_1^2 + x_2 + \cdots + (k-1)x_k} (x_1 + x_2 + \cdots + x_k)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k)
\]
\[
= 2^k \sum_{l=0}^{\infty} \binom{k+l-1}{l}_q (-1)^l (e^{x})^n.
\] (2.28)
By (2.20), it is not difficult to show that
\[
G_{n+k,q}^{(k)}(x) = k! \binom{n+k}{k} \sum_{k=0}^{\infty} \sum_{d_0+d_1+\ldots+d_k=n-k} q^{d_0+d_1+\ldots+d_k} (-1)^i (l+x)^n.
\]
Therefore, we obtain the following corollary.

**Corollary 2.5.** For \( n \in \mathbb{Z}_+ \), \( k \in \mathbb{N} \),
\[
G_{n+k,q}^{(k)}(x) = 2^k k! \binom{n+k}{k} \sum_{l=0}^{\infty} \binom{k+l-1}{l} (-1)^i (l+x)^n.
\]

**Corollary 2.6.** For \( n \in \mathbb{Z}_+ \), \( k \in \mathbb{N} \),
\[
G_{n+k,q}^{(k)}(x) = 2^k k! \binom{n+k}{k} \sum_{l=0}^{\infty} \sum_{d_0+d_1+\ldots+d_k=n-k} q^{d_0+d_1+\ldots+d_k} (-1)^i (l+x)^n.
\]

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### References


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