Research Article

Inclusion Properties for Certain Subclasses of Analytic Functions Defined by a Linear Operator

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The purpose of the present paper is to investigate some inclusion properties of certain subclasses of analytic functions associated with a family of linear operators, which are defined by means of the Hadamard product or convolution. Some integral preserving properties are also considered.

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1. Introduction

Let \( \mathcal{A} \) denote the class of functions of the form

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k
\]

which are analytic in the open unit disk \( \mathbb{U} = \{ z \in \mathbb{C} : |z| < 1 \} \). If \( f \) and \( g \) are analytic in \( \mathbb{U} \), we say that \( f \) is subordinate to \( g \), written \( f \prec g \) or \( f(z) \prec g(z) \) if there exists an analytic function \( w \) in \( \mathbb{U} \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) for \( z \in \mathbb{U} \) such that \( f(z) = g(w(z)) \). We denote by \( S^*, \mathcal{K}, \) and \( \mathcal{C} \) the subclasses of \( \mathcal{A} \) consisting of all analytic functions which are, respectively, starlike, convex, and close-to-convex in \( \mathbb{U} \).

Let \( \mathcal{N} \) be the class of all functions \( \phi \) which are analytic and univalent in \( \mathbb{U} \) and for which \( \phi(\mathbb{U}) \) is convex with \( \phi(0) = 1 \) and \( \text{Re}\{\phi(z)\} > 0 \) for \( z \in \mathbb{U} \).

Making use of the principle of subordination between analytic functions, many authors investigated the subclasses \( S^*(\phi), \mathcal{K}(\phi), \) and \( \mathcal{C}(\phi, \psi) \) of the class \( \mathcal{A} \) for \( \phi, \psi \in \mathcal{N} \) (cf. [1, 2]), which are defined by

\[
S^*(\phi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} < \phi(z) \text{ in } \mathbb{U} \right\},
\]
Furthermore, we note that classes \( \phi, \psi \) function space of analytic and univalent functions in was introduced and studied by Carlson and Sha

\[ \mathcal{K}(\phi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} < \phi(z) \text{ in } U \right\}, \]

\[ C(\phi, \psi) := \left\{ f \in \mathcal{A} : \exists g \in S^*(\phi) \text{ s.t. } \frac{zf'(z)}{g(z)} < \psi(z) \text{ in } U \right\}. \]

\[ \mathcal{S}^* \left( \frac{1 + Az}{1 + Bz} \right) = \mathcal{S}^*[A, B] \quad (-1 \leq B < A \leq 1), \]

\[ \mathcal{K} \left( \frac{1 + Az}{1 + Bz} \right) = \mathcal{K}[A, B] \quad (-1 \leq B < A \leq 1). \]

We now define the function \( h(a, c)(z) \) by

\[ h(a, c)(z) := \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+1}, \quad (z \in \mathbb{U}; a \in \mathbb{R}; c \in \mathbb{R} \setminus \mathbb{Z}_0^\infty; \mathbb{Z}_0^\infty := \{0, -1, -2, \ldots\}), \]

where \((v)_k\) is the Pochhammer symbol (or the shifted factorial) defined (in terms of the Gamma function) by

\[ (v)_k := \frac{\Gamma(v + k)}{\Gamma(v)} = \begin{cases} 1 & \text{if } k = 0, \ v \in \mathbb{C} \setminus \{0\}, \\ v(v+1) \cdots (v+k-1) & \text{if } k \in \mathbb{N} := \{1, 2, \ldots\}, \ v \in \mathbb{C}. \end{cases} \]

We also denote by \( L(a, c) : \mathcal{A} \to \mathcal{A} \) the operator defined by

\[ L(a, c) f(z) = h(a, c)(z) * f(z) \quad (z \in \mathbb{U}; f \in \mathcal{A}), \]

where the symbol \((*)\) stands for the Hadamard product (or convolution). Then it is easily observed from definitions (1.4) and (1.6) that \( L(2, 1) f(z) = zf'(z) \) and

\[ z(L(a, c) f(z))' = aL(a+1, c) f(z) - (a-1)L(a, c) f(z). \]

Furthermore, we note that \( L(n+1, 1) f(z) = D^n f(z) \quad (n > -1) \), where the symbol \( D^n \) denotes the familiar Ruscheweyh derivative \([5]\) (also, see \([6]\)) for \( n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). The operator \( L(a, c) \) was introduced and studied by Carlson and Shaffer \([7]\) which has been used widely on the space of analytic and univalent functions in \( \mathbb{U} \) (see also \([8]\)).

By using the operator \( L(a, c) \), we introduce the following classes of analytic functions for \( \phi, \psi \in \mathcal{A}, a \in \mathbb{R} \) and \( c \in \mathbb{R} \setminus \mathbb{Z}_0^\infty \):

\[ \mathcal{S}_{a,c}(\phi) := \left\{ f \in \mathcal{A} : L(a, c) f(z) \in \mathcal{S}^*(\phi) \right\}, \]

\[ \mathcal{K}_{a,c}(\phi) := \left\{ f \in \mathcal{A} : L(a, c) f(z) \in \mathcal{K}(\phi) \right\}, \]

\[ C_{a,c}(\phi, \psi) := \left\{ f \in \mathcal{A} : L(a, c) f(z) \in C(\phi, \psi) \right\}. \]
We also note that
\[ f(z) \in \mathcal{K}_{a,c}(\phi) \iff z f'(z) \in S_{a,c}(\phi). \] (1.9)

In particular, we set
\[ S_{a,c} \left( \frac{1 + Az}{1 + Bz} \right) = S_{a,c} [A, B] \quad ( -1 \leq B < A \leq 1 ), \] (1.10)
\[ \mathcal{K}_{a,c} \left( \frac{1 + Az}{1 + Bz} \right) = \mathcal{K}_{a,c} [A, B] \quad ( -1 \leq B < A \leq 1 ). \]

In this paper, we investigate several inclusion properties of the classes \( S_{a,c}(\phi), \mathcal{K}_{a,c}(\phi), \) and \( C_{a,c}(\phi,q) \). The integral preserving properties in connection with the operator \( L(a,c) \) are also considered. Furthermore, relevant connections of the results presented here with those obtained in earlier works are pointed out.

2. Inclusion properties involving the operator \( L(a,c) \)

The following lemmas will be required in our investigation.

Lemma 2.1 (see [9, pages 60-61]). Let \( a_2 \geq a_1 > 0 \). If \( a_2 \geq 2 \) or \( a_1 + a_2 \geq 3 \), then the function \( h(a_1, a_2)(z) \) defined by (1.4) belongs to the class \( \mathcal{K} \).

Lemma 2.2 (see [10]). Let \( f \in \mathcal{K} \) and \( g \in S^* \). Then for every analytic function \( Q \) in \( \mathbb{U} \),
\[ \frac{(f*Qg)}{(f*g)}(\mathbb{U}) \subset \overline{co}(Q(\mathbb{U})), \] (2.1)
where \( \overline{co}(Q(\mathbb{U})) \) denote the closed convex hull of \( Q(\mathbb{U}) \).

Theorem 2.3. Let \( a_2 \geq a_1 > 0 \), \( c \in \mathbb{R} \setminus \mathbb{Z}_0^- \), and \( \phi \in \mathcal{N} \). If \( a_2 \geq 2 \) or \( a_1 + a_2 \geq 3 \), then
\[ S_{a_2,c}(\phi) \subset S_{a_1,c}(\phi). \] (2.2)

Proof. Let \( f \in S_{a_2,c}(\phi) \). Then there exists an analytic function \( \varphi \) in \( \mathbb{U} \) with \(|\varphi(z)| < 1 \) (\( z \in \mathbb{U} \)) and \( \varphi(0) = 0 \) such that
\[ \frac{z (L(a_2,c) f(z))'}{L(a_2,c) f(z)} = \phi(\varphi(z)) \quad (z \in \mathbb{U}). \] (2.3)

By using (1.6) and (2.3), we have
\[
\frac{z (L(a_1,c) f(z))'}{L(a_1,c) f(z)} = \frac{z (h(a_1,c) (z) * f(z))'}{h(a_1,c) (z) * f(z)}
= \frac{z (h(a_2,c) (z) * h(a_1,a_2) (z) * f(z))'}{h(a_2,c) (z) * h(a_1,a_2) (z) * f(z)}
= \frac{h(a_1,a_2)(z) * z (L(a_2,c) f(z))'}{h(a_1,a_2)(z) * L(a_2,c) f(z)}
= \frac{h(a_1,a_2)(z) * \phi(\varphi(z)) L(a_2,c) f(z)}{h(a_1,a_2)(z) * L(a_2,c) f(z)}. \] (2.4)
It follows from (2.3) and Lemma 2.1 that \( L(a_2, c)f(z) \in S^* \) and \( h(a_1, a_2)(z) \in \mathcal{K} \), respectively. Then by applying Lemma 2.2 to (2.4), we obtain

\[
\frac{\{h(a_1, a_2)(z)\ast \phi(w)L(a_2, c)f\}}{\{h(a_1, a_2)(z)\ast L(a_2, c)f\}} (U) \subset \overline{\text{co}}(\phi(U)) \subset \phi(U),
\]

(2.5)
since \( \phi \) is convex univalent. Therefore, from the definition of subordination and (2.5), we have

\[
\frac{z(L(a_1, c)f(z))'}{L(a_1, c)f(z)} < \phi(z) \quad (z \in U),
\]

(2.6)

or, equivalently, \( f \in S_{a,c}^*(\phi) \), which completes the proof of Theorem 2.3.

\[\Box\]

**Theorem 2.4.** Let \( a \in \mathbb{R}, c_2 \geq c_1 > 0 \) and \( \phi \in \mathcal{N} \). If \( c_2 \geq 2 \) or \( c_1 + c_2 \geq 3 \), then

\[
S_{a,c_1}(\phi) \subset S_{a,c_2}(\phi).
\]

(2.7)

**Proof** (\( f \in S_{a,c_1}(\phi) \)). Using a similar argument as in the proof of Theorem 2.3, we obtain

\[
\frac{z(L(a, c_2)f(z))'}{L(a, c_2)f(z)} = \frac{h(a_1, a_2)(z)\ast \phi(w(z))L(a, c_2)f(z)}{h(a_1, a_2)(z)\ast L(a, c_2)f(z)},
\]

(2.8)

where \( w \) is an analytic function in \( U \) with \(|w(z)| < 1 \ (z \in U) \) and \( w(0) = 0 \). Applying Lemma 2.1 and the fact that \( L(a, c_1)f(z) \in S^* \), we see that

\[
\frac{\{h(a_1, a_2)\ast h(w)L(a_2, c)f\}}{\{h(a_1, a_2)\ast L(a_2, c)f\}} (U) \subset \overline{\text{co}}(\phi(U)) \subset \phi(U),
\]

(2.9)
since \( \phi \) is convex univalent. Thus the proof of Theorem 2.3 is completed.

\[\Box\]

**Corollary 2.5.** Let \( a_2 \geq a_1 > 0, c_2 \geq c_1 > 0 \), and \( \phi \in \mathcal{N} \). If \( a_2 \geq \min\{2, 3-a_1\} \) and \( c_2 \geq \min\{2, 3-c_1\} \), then

\[
S_{a_2,c_1}(\phi) \subset S_{a_2,c_2}(\phi) \subset S_{a_1,c_2}(\phi).
\]

(2.10)

**Theorem 2.6.** Let \( a_2 \geq a_1 > 0, c_2 \geq c_1 > 0 \) and \( \phi \in \mathcal{N} \). If \( a_2 \geq \min\{2, 3-a_1\} \) and \( c_2 \geq \min\{2, 3-c_1\} \), then

\[
\mathcal{K}_{a_2,c_1}(\phi) \subset \mathcal{K}_{a_2,c_2}(\phi) \subset \mathcal{K}_{a_1,c_2}(\phi).
\]

(2.11)
Proof. Applying (1.9) and Corollary 2.5, we observe that

\[ f(z) \in \mathcal{K}_{a_2,c_1}(\phi) \iff L(a_2,c_1)f(z) \in \mathcal{K}(\phi) \]
\[ \iff z(L(a_2,c_1)f(z))' \in S^*(\phi) \]
\[ \iff L(a_2,c_1)(zf'(z)) \in S^*(\phi) \]
\[ \iff zf'(z) \in S_{a_2,c_1}(\phi) \]
\[ \iff zf'(z) \in S_{a_2,c_1}(\phi) \]
\[ \iff z(L(a_2,c_2)f(z))' \in S^*(\phi) \]
\[ \iff L(a_2,c_2)f(z) \in \mathcal{K}(\phi) \]
\[ \iff f(z) \in \mathcal{K}_{a_2,c_1}(\phi), \]

which evidently proves Theorem 2.6. \( \square \)

Taking \( \phi(z) = (1 + Az)/(1 + Bz) \) \((-1 \leq B < A \leq 1; z \in \mathbb{U}) \) in Corollary 2.5 and Theorem 2.6, we have the following corollary.

**Corollary 2.7.** Let \( a_2 \geq a_1 > 0 \) and \( c_2 \geq c_1 > 0 \). If \( a_2 \geq \min\{2,3-a_1\} \) and \( c_2 \geq \min\{2,3-c_1\} \), then

\[ S_{a_2,c_1}[A,B] \subset S_{a_2,c_2}[A,B] \subset S_{a_1,c_2}[A,B] \quad (-1 \leq B < A \leq 1), \]
\[ \mathcal{K}_{a_2,c_1}[A,B] \subset \mathcal{K}_{a_2,c_2}[A,B] \subset \mathcal{K}_{a_1,c_2}[A,B] \quad (-1 \leq B < A \leq 1). \] (2.13)

To prove the theorems below, we need the following lemma.

**Lemma 2.8.** Let \( \phi \in \mathcal{K} \). If \( f \in \mathcal{K} \) and \( q \in S^*(\phi) \), then \( f*q \in S^*(\phi) \).

**Proof.** Let \( q \in S^*(\phi) \). Then

\[ zq'(z) = q(z)\phi'(\omega(z)) \quad (z \in \mathbb{U}), \] (2.14)

where \( \omega \) is an analytic function in \( \mathbb{U} \) with \( |\omega(z)| < 1 \) \((z \in \mathbb{U})\) and \( \omega(0) = 0 \). Thus we have

\[ \frac{z(f(z)*q(z))'}{f(z)*q(z)} = \frac{f(z)*q(z)}{f(z)*q(z)} = \frac{f(z)*\phi(\omega(z))q(z)}{f(z)*q(z)} \quad (z \in \mathbb{U}). \] (2.15)

By using similar arguments to those used in the proof of Theorem 2.3, we conclude that (2.15) is subordinated to \( \phi \) in \( \mathbb{U} \) and so \( f*q \in S^*(\phi) \). \( \square \)
Theorem 2.9. Let $a_2 \geq a_1 > 0$, $c_2 \geq c_1 > 0$ and $\phi, \psi \in \mathcal{N}$. If $a_2 \geq \min\{2, 3 - a_1\}$ and $c_2 \geq \min\{2, 3 - c_1\}$, then
\[
\mathcal{C}_{a_2, c_1}(\phi, \psi) \subset \mathcal{C}_{a_2, c_2}(\phi, \psi) \subset \mathcal{C}_{a_1, c_2}(\phi, \psi).
\] (2.16)

Proof. First of all, we show that
\[
\mathcal{C}_{a_2, c_1}(\phi, \psi) \subset \mathcal{C}_{a_2, c_2}(\phi, \psi).
\] (2.17)

Let $f \in \mathcal{C}_{a_2, c_1}(\phi, \psi)$. Then there exists a function $q_2 \in \mathcal{S}(\phi)$ such that
\[
\frac{z(L(a_2, c_1)f(z))'}{q_2(z)} < \psi(z) \quad (z \in \mathbb{U}).
\] (2.18)

From (2.18), we obtain
\[
z(L(a_2, c_1)f(z))' = \psi(w(z)) \quad (z \in \mathbb{U}),
\] (2.19)

where $w$ is an analytic function in $\mathbb{U}$ with $|w(z)| < 1$ (z \in \mathbb{U}) and $w(0) = 0$. By virtue of Lemmas 2.1 and 2.8, we see that $h(a_1, a_2)(z) \ast q_2(z) = q_1(z)$ belongs to $\mathcal{S}(\phi)$. Then we have
\[
\frac{z(L(a_2, c_2)f(z))'}{q_1(z)} = \frac{h(c_1, c_2)(z) \ast z(L(a_2, c_1)f(z))'}{h(c_1, c_2)(z) \ast q_2(z)} = \frac{h(c_1, c_2)(z) \ast \psi(w(z))q_2(z)}{h(c_1, c_2)(z) \ast q_2(z)} < \psi(z) \quad (z \in \mathbb{U}),
\] (2.20)

which implies that $f \in \mathcal{C}_{a_2, c_2}(\phi, \psi)$.

Moreover, the proof of the second part is similar to that of the first part and so we omit the details involved. \qed

3. Inclusion properties involving various operators

The next theorem shows that the classes $\mathcal{S}_{a,c}(\phi)$, $\mathcal{K}_{a,c}(\phi)$, and $\mathcal{C}_{a,c}(\phi, \psi)$ are invariant under convolution with convex functions.

Theorem 3.1. Let $a > 0$, $c \in \mathbb{R} \setminus \mathbb{Z}_0^-$, $\phi, \psi \in \mathcal{N}$ and let $g \in \mathcal{K}$. Then

(i) $f \in \mathcal{S}_{a,c}(\phi) \Rightarrow g \ast f \in \mathcal{S}_{a,c}(\phi)$,
(ii) $f \in \mathcal{K}_{a,c}(\phi) \Rightarrow g \ast f \in \mathcal{K}_{a,c}(\phi)$,
(iii) $f \in \mathcal{C}_{a,c}(\phi, \psi) \Rightarrow g \ast f \in \mathcal{C}_{a,c}(\phi, \psi)$.

Proof. (i) Let $f \in \mathcal{S}_{a,c}(\phi)$. Then we have
\[
\frac{z(L(a,c)(g \ast f)(z))'}{L(a,c)(g \ast f)(z)} = \frac{g(z) \ast z(L(a,c)f(z))'}{g(z) \ast L(a,c)f(z)}.
\] (3.1)

By using the same techniques as in the proof of Theorem 2.3, we obtain (i).
Nak Eun Cho

(ii) Let \( f \in \mathcal{K}_{a,c}(\phi) \). Then, by (1.9), \( zf'(z) \in S_{a,c}(\phi) \) and hence from (i), \( g(z)*zf'(z) \in S_{a,c}(\phi) \). Since

\[
g(z)*zf'(z) = z(g*f)'(z),
\]

we have (ii) applying (1.9) once again.

(iii) Let \( f \in \mathcal{C}_{a,c}(\phi, \psi) \). Then there exists a function \( q \in S^*(\phi) \) such that

\[
z(L(a,c)f(z))' = q(w(z))q(z) \quad (z \in \mathbb{U}),
\]

where \( w \) is an analytic function in \( \mathbb{U} \) with \( |w(z)| < 1 \) \( (z \in \mathbb{U}) \) and \( w(0) = 0 \). From Lemma 2.8, we have that \( g*q \in S^*(\phi) \). Since

\[
\frac{z(L(a,c)(g*f)(z))'}{(g*q)(z)} = \frac{g(z)*zf'(z)}{g(z)*q(z)} = \frac{g(z)*qw(z)q(z)}{g(z)*q(z)} < q(z) \quad (z \in \mathbb{U}),
\]

we obtain (iii). \( \Box \)

Now we consider the following operators [5, 11] defined by

\[
\Psi_1(z) = \sum_{k=1}^{\infty} \frac{1 + c}{k+c} z^k \quad \text{(Re}\{c\} \geq 0; \ z \in \mathbb{U}),
\]

\[
\Psi_2(z) = \frac{1}{1-x} \log \left[ \frac{1-xz}{1-z} \right] \quad \text{(log} 1 = 0; \ |x| \leq 1, \ x \neq 1; \ z \in \mathbb{U}).
\]

It is well known ([12], see also [5]) that the operators \( \Psi_1 \) and \( \Psi_2 \) are convex univalent in \( \mathbb{U} \). Therefore, we have the following result, which can be obtained from Theorem 3.1 immediately.

**Corollary 3.2.** Let \( a > 0 \), \( c \in \mathbb{R} \setminus \mathbb{Z}^- \), \( \phi, \psi \in \mathcal{A} \) and let \( \Psi_i \) \( (i = 1, 2) \) be defined by (3.5). Then

(i) \( f \in S_{a,c}(\phi) \) \( \Rightarrow \Psi_if \in S_{a,c}(\phi) \),

(ii) \( f \in \mathcal{K}_{a,c}(\phi) \) \( \Rightarrow \Psi_if \in \mathcal{K}_{a,c}(\phi) \),

(iii) \( f \in \mathcal{C}_{a,c}(\phi, \psi) \) \( \Rightarrow \Psi_if \in \mathcal{C}_{a,c}(\phi, \psi) \).

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