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Research Article

Essential Norms of Weighted Composition Operators from the α-Bloch Space to a Weighted-Type Space on the Unit Ball

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This paper finds some lower and upper bounds for the essential norm of the weighted composition operator from α-Bloch spaces to the weighted-type space H_μ^∞ on the unit ball for the case $\alpha \geq 1$.

1. Introduction

Let $\mathbb{B} = \{z \in \mathbb{C}^n : |z| < 1\}$ be the open unit ball in $\mathbb{C}^n$, $H(\mathbb{B})$ be the class of all holomorphic functions on the unit ball, and let $H^\infty(\mathbb{B})$ be the class of all bounded holomorphic functions on $\mathbb{B}$ with the norm

$$\|f\|_\infty = \sup_{z\in\mathbb{B}} |f(z)|. \tag{1.1}$$

Let $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ be points in $\mathbb{C}^n$ and $\langle z, w\rangle = \sum_{k=1}^n z_k \overline{w}_k$. For a holomorphic function f, we denote

$$\nabla f = \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right). \tag{1.2}$$

A positive continuous function ϕ on the interval $[0,1)$ is called normal (see [1]) if there is $\delta \in [0,1)$ and a and b, $0 < a < b$ such that

$$\frac{\phi(r)}{(1-r)^a} \text{ is decreasing on } [\delta, 1) \text{ and } \lim_{r\to 1} \frac{\phi(r)}{(1-r)^a} = 0,$$

$$\frac{\phi(r)}{(1-r)^b} \text{ is increasing on } [\delta,1) \text{ and } \lim_{r\to 1}\frac{\phi(r)}{(1-r)^b} = \infty. \tag{1.3}$$

From now on, if we say that a function $\phi : \mathbb{B} \to [0,\infty)$ is normal, we will also assume that it is radial, that is, $\phi(z) = \phi(|z|),\ z \in \mathbb{B}$.

The weighted space $H_\mu^\infty = H_\mu^\infty(\mathbb{B})$ consists of all $f \in H(\mathbb{B})$ such that

$$\sup_{z\in\mathbb{B}} \mu(z)\left|f(z)\right| < \infty, \tag{1.4}$$

where μ is normal on the interval $[0,1)$. For $\mu(z) = (1-|z|^2)^\beta$, $\beta > 0$, we obtain the weighted space $H_\beta^\infty = H_\beta^\infty(\mathbb{B})$ (see, e.g., [2, 3]).

The α-Bloch space $\mathcal{B}^\alpha = \mathcal{B}^\alpha(\mathbb{B}),\ \alpha > 0$, is the space of all $f \in H(\mathbb{B})$ such that

$$b_\alpha(f) = \sup_{z\in\mathbb{B}} \left(1-|z|^2\right)^\alpha \left|\nabla f(z)\right| < \infty. \tag{1.5}$$

With the norm

$$\|f\|_{\mathcal{B}^\alpha} = \left|f(0)\right| + b_\alpha(f), \tag{1.6}$$

the space $\mathcal{B}^\alpha$ is a Banach space ([4–6]).

The little α-Bloch space $\mathcal{B}_0^\alpha = \mathcal{B}_0^\alpha(\mathbb{B})$ is the subspace of $\mathcal{B}^\alpha$ consisting of all $f \in H(\mathbb{B})$ such that

$$\lim_{|z|\to 1} \left(1-|z|^2\right)^\alpha \left|\nabla f(z)\right| = 0. \tag{1.7}$$

Let $u \in H(\mathbb{B})$ and φ be a holomorphic self-map of the unit ball. Weighted composition operator on $H(\mathbb{B})$, induced by u and φ is defined by

$$\left(uC_\varphi f\right)(z) = u(z) f\left(\varphi(z)\right), \quad z \in \mathbb{B}. \tag{1.8}$$

This operator can be regarded as a generalization of a multiplication operator and a composition operator. It is interesting to provide a function theoretic characterization when u and φ induce a bounded or compact weighted composition operator between some spaces of holomorphic functions on $\mathbb{B}$. (For some classical results in the topic see, e.g., [5]. For some recent results on this and related operators, see, e.g., [2–4, 7–25] and the references therein.)

In [18], Ohno has characterized the boundedness and compactness of weighted composition operators between H^∞ and the Bloch space $\mathcal{B}$ on the unit disk. In the setting of the unit polydisk $\mathbb{D}^n$, we have given some necessary and sufficient conditions for a weighted composition operator to be bounded or compact from $H^\infty(\mathbb{D}^n)$ to the Bloch space $\mathcal{B}(\mathbb{D}^n)$ in [12] (see, also [21]). Corresponding results for the case of the unit ball are given in [14]. Among other results, in [14], we have given some necessary and sufficient conditions for the

compactness of the operator $uC_\varphi : \mathcal{B}^\alpha(\mathbb{B}) \to H^\infty(\mathbb{B})$, which we incorporate in the following theorem.

Theorem A. *Let $\varphi = (\varphi_1, \ldots, \varphi_n)$ be a holomorphic self-map of $\mathbb{B}$ and $u \in H(\mathbb{B})$.*

(a) *If $\alpha > 1$, then the following statements are equivalent:*

(a1) *$uC_\varphi : \mathcal{B}_0^\alpha \to H^\infty$ is a compact operator,*
(a2) *$uC_\varphi : \mathcal{B}^\alpha \to H^\infty$ is a compact operator,*
(a3) *$u \in H^\infty$, and*

$$\lim_{|\varphi(z)|\to 1} \frac{|u(z)|}{\left(1 - |\varphi(z)|^2\right)^{\alpha-1}} = 0. \tag{1.9}$$

(b) *If $\alpha = 1$, then the following statements are equivalent:*

(b1) *$uC_\varphi : \mathcal{B}_0 \to H^\infty$ is a compact operator,*
(b2) *$uC_\varphi : \mathcal{B} \to H^\infty$ is a compact operator,*
(b3) *$u \in H^\infty$, and*

$$\lim_{|\varphi(z)|\to 1} |u(z)| \ln \frac{2}{1 - |\varphi(z)|^2} = 0. \tag{1.10}$$

We would also like to point out that if $\alpha \geq 1$, then the boundedness of $uC_\varphi : \mathcal{B}^\alpha \to H_\mu^\infty$ and $uC_\varphi : \mathcal{B}_0^\alpha \to H_\mu^\infty$ are equivalent (see [22] for the case $\alpha = 1$, and the proof of Theorem 3 in [14]).

The essential norm of an operator is its distance in the operator norm from the compact operators. More precisely, assume that X_1 and X_2 are Banach spaces and $A : X_1 \to X_2$ is a bounded linear operator, then the essential norm of A, denoted by $\|A\|_{e,X_1\to X_2}$, is defined as follows:

$$\|A\|_{e,X_1\to X_2} = \inf\left\{\|A + L\|_{X_1\to X_2} \,:\, L : X_1 \to X_2,\ L \text{ is compact}\right\}, \tag{1.11}$$

where $\|\cdot\|_{X_1\to X_2}$denotes the operator norm. If $X_1 = X_2$, it is simply denoted by $\|\cdot\|_e$ (see, e.g., [5, page 132]). If $A : X_1 \to X_2$ is an unbounded linear operator, then clearly $\|A\|_{e,X_1\to X_2} = \infty$.

Since the set of all compact operators is a closed subset of the set of bounded operators, it follows that an operator A is compact if and only if $\|A\|_{e,X_1\to X_2} = 0$.

Motivated by Theorem A, in this paper, we find some lower and upper bounds for the essential norm of the weighted composition operator $uC_\varphi : \mathcal{B}^\alpha(\mathbb{B})$ (or $\mathcal{B}_0^\alpha(\mathbb{B})$) $\to H_\mu^\infty(\mathbb{B})$, when $\alpha \geq 1$.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to another. The notation $a \preccurlyeq b$ means that there is a positive constant C such that $a \leq Cb$. If both $a \preccurlyeq b$ and $b \preccurlyeq a$ hold, then we say that $a \asymp b$.

2. Auxiliary results

In this section, we quote several auxiliary results which we need in the proofs of the main results in this paper. The following lemma should be folklore.

Lemma 2.1. *Let $f \in \mathcal{B}^\alpha(\mathbb{B})$, $0 < \alpha < \infty$. Then,*

$$|f(z)| \le \begin{cases} C\|f\|_{\mathcal{B}^\alpha}, & \alpha \in (0,1), \\ |f(0)| + b_1(f)\dfrac{1}{2}\ln\dfrac{1+|z|}{1-|z|}, & \alpha = 1, \\ C\dfrac{\|f\|_{\mathcal{B}^\alpha}}{(1-|z|^2)^{\alpha-1}}, & \alpha > 1, \end{cases} \tag{2.1}$$

for some $C > 0$ independent of f.

The proof of the lemma for the case $\alpha \neq 1$ can be found, for example, in [26]. The formulation of the corresponding estimate in [26], for the case $\alpha = 1$, is slightly different. In this case, Lemma 2.1 follows from the following estimate:

$$|f(z) - f(0)| = \left|\int_0^1 \langle \nabla f(tz), \overline{z}\rangle\right| \le b_1(f)\int_0^1 \frac{|z|\,dt}{1-|z|^2t^2} = b_1(f)\frac{1}{2}\ln\frac{1+|z|}{1-|z|}. \tag{2.2}$$

The next lemma can be proved in a standard way (see, e.g., the proofs of the corresponding results in [5, 27–29]).

Lemma 2.2. *Assume $\alpha > 0$, $g \in H(\mathbb{B})$, μ is normal, and φ is an analytic self-map of $\mathbb{B}$. Then, $uC_\varphi : \mathcal{B}^\alpha (\text{or } \mathcal{B}_0^\alpha) \to H_\mu^\infty$ is compact if and only if $uC_\varphi : \mathcal{B}^\alpha (\text{or } \mathcal{B}_0^\alpha) \to H_\mu^\infty$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathcal{B}^\alpha (\text{or } \mathcal{B}_0^\alpha)$ converging to zero uniformly on compacts of $\mathbb{B}$ as $k \to \infty$, one has $\|uC_\varphi f_k\|_{H_\mu^\infty} \to 0$ as $k \to \infty$.*

Lemma 2.3. *Let*

$$f_w(z) = \frac{(1-|w|^2)^\varepsilon}{(1-\langle z, w\rangle)^{\alpha+\varepsilon-1}}, \quad w \in \mathbb{B}, \tag{2.3}$$

where $\alpha > 1$ and $\varepsilon \in (0,1]$. Then,

$$\|f_w\|_{\mathcal{B}^\alpha} = (1-|w|^2)^\varepsilon + \frac{(\alpha+\varepsilon-1)(2\alpha)^\alpha |w| \left(\sqrt{\alpha^2 + |w|^2\varepsilon^2 - |w|^2\alpha^2} + \varepsilon\right)^\varepsilon}{(\alpha+\varepsilon)^\varepsilon \left(\sqrt{\alpha^2 + |w|^2\varepsilon^2 - |w|^2\alpha^2} + \alpha\right)^\alpha}. \tag{2.4}$$

Proof. We have

$$(1-|z|^2)^\alpha |\nabla f_w(z)| = (\alpha+\varepsilon-1)(1-|z|^2)^\alpha \frac{(1-|w|^2)^\varepsilon |w|}{|1-\langle z, w\rangle|^{\alpha+\varepsilon}}, \tag{2.5}$$

from which it easily follows that

$$b_\alpha(f_w) \le (\alpha+\varepsilon-1)2^{\alpha+\varepsilon}. \tag{2.6}$$

Set

$$g_s(x) = \frac{(\alpha+\varepsilon-1)s(1-s^2)^{\varepsilon}(1-x^2)^{\alpha}}{(1-sx)^{\alpha+\varepsilon}}, \quad x \in [0,1],\ s \in [0,1). \tag{2.7}$$

Then,

$$g'_s(x) = (\alpha+\varepsilon-1)s(1-s^2)^{\varepsilon}(1-x^2)^{\alpha-1}\frac{s(\alpha-\varepsilon)x^2-2\alpha x+s(\alpha+\varepsilon)}{(1-sx)^{\alpha+\varepsilon+1}}. \tag{2.8}$$

Hence, the points

$$x_{M,m} = \frac{\alpha \pm \sqrt{\alpha^2+s^2\varepsilon^2-s^2\alpha^2}}{s(\alpha-\varepsilon)} \tag{2.9}$$

are stationary for the function $g_s(x)$. Since $x_M > 1$, it follows that $g_s(x)$ attains its maximum on the interval $[0,1]$ at the point

$$x_m = \frac{\alpha - \sqrt{\alpha^2+s^2\varepsilon^2-s^2\alpha^2}}{s(\alpha-\varepsilon)} = \frac{s(\alpha+\varepsilon)}{\alpha+\sqrt{\alpha^2+s^2\varepsilon^2-s^2\alpha^2}} \in (0,1). \tag{2.10}$$

By some long but elementary calculations, it follows that

$$g_s(x_m) = \frac{(\alpha+\varepsilon-1)(2\alpha)^{\alpha}s\left(\sqrt{\alpha^2+s^2\varepsilon^2-s^2\alpha^2}+\varepsilon\right)^{\varepsilon}}{(\alpha+\varepsilon)^{\varepsilon}\left(\sqrt{\alpha^2+s^2\varepsilon^2-s^2\alpha^2}+\alpha\right)^{\alpha}}. \tag{2.11}$$

From this and since $f_w(0) = (1-|w|^2)^{\varepsilon}$, (2.4) follows. □

Remark 2.4. Note that

$$\lim_{|w|\to 1}\|f_w\|_{\mathcal{B}^\alpha} = \lim_{s\to 1-0}\left[(1-s^2)^{\varepsilon}+g_s(x_m)\right] = \frac{(\alpha+\varepsilon-1)2^{\alpha+\varepsilon}\varepsilon^{\varepsilon}\alpha^{\alpha}}{(\alpha+\varepsilon)^{\alpha+\varepsilon}}, \tag{2.12}$$

$$\lim_{\varepsilon\to 0+}\frac{(\alpha+\varepsilon-1)2^{\alpha+\varepsilon}\varepsilon^{\varepsilon}\alpha^{\alpha}}{(\alpha+\varepsilon)^{\alpha+\varepsilon}} = (\alpha-1)2^{\alpha}. \tag{2.13}$$

3. Estimates of the essential norm of $uC_\varphi : \mathcal{B}^\alpha$ (or $\mathcal{B}_0^\alpha$) $\to H_\mu^\infty$

In this section, we prove the main results in this paper. Before we formulate and prove these results, we prove another auxiliary result.

Lemma 3.1. *Assume $\alpha \in (0,\infty)$, $u \in H(\mathbb{B})$, μ is normal, φ is a holomorphic self-map of $\mathbb{B}$ such that $\|\varphi\|_\infty < 1$, and the operator $uC_\varphi : \mathcal{B}^\alpha$ (or $\mathcal{B}_0^\alpha$) $\to H_\mu^\infty$ is bounded. Then, $uC_\varphi : \mathcal{B}^\alpha$ (or $\mathcal{B}_0^\alpha$) $\to H_\mu^\infty$ is compact.*

Proof. First note that since $uC_\varphi : \mathcal{B}^\alpha$ (or $\mathcal{B}_0^\alpha$) $\to H_\mu^\infty$ is bounded and $f_0(z) \equiv 1 \in \mathcal{B}_0^\alpha \subset \mathcal{B}^\alpha$, it follows that $uC_\varphi(f_0) = u \in H_\mu^\infty$. Now, assume that $(f_k)_{k\in\mathbb{N}}$ is a bounded sequence in $\mathcal{B}^\alpha$ (or $\mathcal{B}_0^\alpha$) converging to zero on compacts of $\mathbb{B}$ as $k \to \infty$. Then, we have

$$\left\|uC_\varphi(f_k)\right\|_{H_\mu^\infty} \le \|u\|_{H_\mu^\infty} \sup_{w\in\varphi(\mathbb{B})} \left|f_k(w)\right| \longrightarrow 0, \tag{3.1}$$

as $k \to \infty$, since $\varphi(\mathbb{B})$ is contained in the ball $|w| \le \|\varphi\|_\infty$ which is a compact subset of $\mathbb{B}$, according to the assumption, $\|\varphi\|_\infty < 1$. Hence, by Lemma 2.2 , the operator $uC_\varphi : \mathcal{B}^\alpha$ (or $\mathcal{B}_0^\alpha$) $\to H_\mu^\infty$ is compact. □

Theorem 3.2. *Assume $\alpha > 1$, μ is normal, $u \in H(\mathbb{B})$, $\varphi = (\varphi_1, \ldots, \varphi_n)$ is a holomorphic self-map of $\mathbb{B}$, and $uC_\varphi : \mathcal{B}^\alpha \to H_\mu^\infty$ is bounded. Then,*

$$\begin{aligned}\frac{2^{-\alpha}}{\alpha-1}\limsup_{|\varphi(z)|\to 1}\frac{\mu(z)\left|u(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{\alpha-1}} &\le \left\|uC_\varphi\right\|_{e,\mathcal{B}_0^\alpha\to H_\mu^\infty}\\ &\le \left\|uC_\varphi\right\|_{e,\mathcal{B}^\alpha\to H_\mu^\infty}\\ &\le C\limsup_{|\varphi(z)|\to 1}\frac{\mu(z)\left|u(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{\alpha-1}},\end{aligned} \tag{3.2}$$

for some positive constant C.

Proof. Since $uC_\varphi : \mathcal{B}^\alpha \to H_\mu^\infty$ is bounded, recall that $u \in H_\mu^\infty$. If $\|\varphi\|_\infty < 1$, then, from Lemma 3.1, it follows that $uC_\varphi : \mathcal{B}^\alpha \to H_\mu^\infty$ is compact which is equivalent with $\|uC_\varphi\|_{e,\mathcal{B}^\alpha\to H_\mu^\infty} = 0$, and, consequently, $\|uC_\varphi\|_{e,\mathcal{B}_0^\alpha\to H_\mu^\infty} = 0$. On the other hand, it is clear that in this case the condition $|\varphi(z)| \to 1$ is vacuous, so that inequalities in (3.2) are vacuously satisfied.

Hence, assume $\|\varphi\|_\infty = 1$. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{B}$ such that $\lim_{k\to\infty}|\varphi(z_k)| = 1$ and $\varepsilon \in (0,1)$ be fixed. Set

$$f_k^\varepsilon(z) = \frac{\left(1-\left|\varphi(z_k)\right|^2\right)^\varepsilon}{\left(1-\langle z, \varphi(z_k)\rangle\right)^{\alpha+\varepsilon-1}}, \quad k \in \mathbb{N}. \tag{3.3}$$

By Lemma 2.3, it follows that $\sup_{k\in\mathbb{N}}\|f_k^\varepsilon\|_{\mathcal{B}^\alpha} < \infty$, moreover, it is easy to see that $f_k^\varepsilon \in \mathcal{B}_0^\alpha$ for every $k \in \mathbb{N}$ and $f_k^\varepsilon \to 0$ uniformly on compacts of $\mathbb{B}$ as $k \to \infty$. Then, by [6, Theorem 7.5], it follows that f_k^ε converges to zero weakly as $k \to \infty$. Hence, for every compact operator $L : \mathcal{B}_0^\alpha \to H_\mu^\infty$, we have $\|Lf_k^\varepsilon\|_{H_\mu^\infty} \to 0$ as $k \to \infty$.

We have

$$\left\|f_k^\varepsilon\right\|_{\mathcal{B}^\alpha}\left\|uC_\varphi + L\right\|_{\mathcal{B}_0^\alpha\to H_\mu^\infty} \ge \left\|\left(uC_\varphi + L\right)\left(f_k^\varepsilon\right)\right\|_{H_\mu^\infty} \ge \left\|uC_\varphi f_k^\varepsilon\right\|_{H_\mu^\infty} - \left\|Lf_k^\varepsilon\right\|_{H_\mu^\infty} \tag{3.4}$$

for every compact operator $L : \mathcal{B}_0^\alpha \to H_\mu^\infty$.

Letting $k \to \infty$ in (3.4) and using the definition of f_k^{ε}, we obtain

$$C(\varepsilon,\alpha)\left\|uC_\varphi + L\right\|_{\mathcal{B}_0^\alpha \to H_\mu^\infty} = \limsup_{k\to\infty}\left\|uC_\varphi f_k^{\varepsilon}\right\|_{H_\mu^\infty} \geq \limsup_{k\to\infty}\frac{\mu(z_k)\left|u(z_k)\right|}{\left(1-\left|\varphi(z_k)\right|^2\right)^{\alpha-1}}, \tag{3.5}$$

where $C(\varepsilon,\alpha)$ is the quantity in (2.12).

Taking in (3.5) the infimum over the set of all compact operators $L : \mathcal{B}_0^\alpha \to H_\mu^\infty$, then letting $\varepsilon \to 0+$ in such obtained inequality, and using (2.13), we obtain

$$2^\alpha(\alpha-1)\left\|uC_\varphi\right\|_{e,\mathcal{B}_0^\alpha \to H_\mu^\infty} \geq \limsup_{k\to\infty}\frac{\mu(z_k)\left|u(z_k)\right|}{\left(1-\left|\varphi(z_k)\right|^2\right)^{\alpha-1}}, \tag{3.6}$$

from which the first inequality in (3.2) follows.

Since the second inequality in (3.2) is obvious, we only have to prove the third one. By Lemma 3.1, we have that for each fixed $\rho \in (0,1)$ the operator $uC_{\rho\varphi} : \mathcal{B}^\alpha \to H_\mu^\infty$ is compact.

Let $\delta \in (0,1)$ be fixed, and let $(\rho_m)_{m\in\mathbb{N}}$ be a sequence of positive numbers which increasingly converges to 1, then for each fixed $m \in \mathbb{N}$, we have

$$\begin{aligned}
\left\|uC_\varphi\right\|_{e,\mathcal{B}^\alpha \to H_\mu^\infty} &\leq \left\|uC_\varphi - uC_{\rho_m\varphi}\right\|_{\mathcal{B}^\alpha \to H_\mu^\infty} \\
&= \sup_{\|f\|_{\mathcal{B}^\alpha}\leq 1}\left\|\left(uC_\varphi - uC_{\rho_m\varphi}\right)(f)\right\|_{H_\mu^\infty} \\
&= \sup_{z\in\mathbb{B}}\sup_{\|f\|_{\mathcal{B}^\alpha}\leq 1}\mu(z)\left|u(z)\left(f(\varphi(z)) - f(\rho_m\varphi(z))\right)\right| \\
&\leq \sup_{|\varphi(z)|\leq\delta}\sup_{\|f\|_{\mathcal{B}^\alpha}\leq 1}\mu(z)\left|u(z)\right|\left|f(\varphi(z)) - f(\rho_m\varphi(z))\right| \\
&\quad + \sup_{|\varphi(z)|>\delta}\sup_{\|f\|_{\mathcal{B}^\alpha}\leq 1}\mu(z)\left|u(z)\right|\left|f(\varphi(z)) - f(\rho_m\varphi(z))\right|.
\end{aligned} \tag{3.7}$$

By the mean-value theorem, we have

$$\begin{aligned}
&\sup_{|\varphi(z)|\leq\delta}\sup_{\|f\|_{\mathcal{B}^\alpha}\leq 1}\mu(z)\left|u(z)\left(f(\varphi(z)) - f(\rho_m\varphi(z))\right)\right| \\
&\quad\leq (1-\rho_m)\sup_{|\varphi(z)|\leq\delta}\sup_{\|f\|_{\mathcal{B}^\alpha}\leq 1}\mu(z)\left|u(z)\right|\left|\varphi(z)\right|\sup_{|w|\leq\delta}\left|\nabla f(w)\right| \\
&\quad\leq (1-\rho_m)\frac{\delta}{\left(1-\delta^2\right)^\alpha}\|u\|_{H_\mu^\infty}\sup_{\|f\|_{\mathcal{B}^\alpha}\leq 1}\sup_{|w|\leq\delta}\left(1-|w|^2\right)^\alpha\left|\nabla f(w)\right| \\
&\quad\leq (1-\rho_m)\frac{\delta}{\left(1-\delta^2\right)^\alpha}\|u\|_{H_\mu^\infty} \longrightarrow 0,
\end{aligned} \tag{3.8}$$

as $m \to \infty$.

Moreover, by Lemma 2.1 (case $\alpha > 1$), and known inequality

$$\|f_r\|_{\mathcal{B}^\alpha} \le \|f\|_{\mathcal{B}^\alpha}, \tag{3.9}$$

where $f_r(z) = f(rz)$, $r \in [0,1)$, we have

$$\begin{aligned}\mu(z)\left|u(z)\left(f(\varphi(z)) - f(\rho_m\varphi(z))\right)\right| &\le C\|f - f_{\rho_m}\|_{\mathcal{B}^\alpha}\frac{\mu(z)|u(z)|}{\left(1-|\varphi(z)|^2\right)^{\alpha-1}}\\ &\le 2C\|f\|_{\mathcal{B}^\alpha}\frac{\mu(z)|u(z)|}{\left(1-|\varphi(z)|^2\right)^{\alpha-1}}\end{aligned} \tag{3.10}$$

for some positive constant C. Replacing (3.10) in (3.7), letting in such obtained inequality $m \to \infty$, employing (3.8), and then letting $\delta \to 1$, the third inequality in (3.2) follows, finishing the proof of the theorem. □

Corollary 3.3. *Assume $\alpha > 1$, μ is normal, $u \in H(\mathbb{B})$, $\varphi = (\varphi_1, \ldots, \varphi_n)$ is a holomorphic self-map of $\mathbb{B}$, and the operator $uC_\varphi : \mathcal{B}^\alpha$ (or $\mathcal{B}_0^\alpha$) $\to H_\mu^\infty$ is bounded. Then, $uC_\varphi : \mathcal{B}^\alpha$ (or $\mathcal{B}_0^\alpha$) $\to H_\mu^\infty$ is compact if and only if*

$$\limsup_{|\varphi(z)|\to 1}\frac{\mu(z)|u(z)|}{\left(1-|\varphi(z)|^2\right)^{\alpha-1}} = 0. \tag{3.11}$$

Theorem 3.4. *Assume $u \in H(\mathbb{B})$, μ is normal, $\varphi = (\varphi_1, \ldots, \varphi_n)$ is a holomorphic self-map of $\mathbb{B}$, and $uC_\varphi : \mathcal{B} \to H_\mu^\infty$ is bounded. Then,*

$$\begin{aligned}C\limsup_{|\varphi(z)|\to 1}\mu(z)|u(z)|\ln\frac{1+|\varphi(z)|}{1-|\varphi(z)|} &\le \|uC_\varphi\|_{e,\mathcal{B}_0\to H_\mu^\infty}\\ &\le \|uC_\varphi\|_{e,\mathcal{B}\to H_\mu^\infty}\\ &\le \limsup_{|\varphi(z)|\to 1}\mu(z)|u(z)|\ln\frac{1+|\varphi(z)|}{1-|\varphi(z)|}\end{aligned} \tag{3.12}$$

for a positive constant C.

Proof. Clearly, $u = (uC_\varphi)(1) \in H_\mu^\infty$. If $\|\varphi\|_\infty < 1$, then the result follows as in Theorem 3.2.

Hence, assume $\|\varphi\|_\infty = 1$. We use the following family of test functions

$$h_w(z) = \left(\ln\frac{(1+|w|)^2}{1-\langle z,w\rangle}\right)^2\left(\ln\frac{1+|w|}{1-|w|}\right)^{-1}, \quad w \in \mathbb{B}. \tag{3.13}$$

We have

$$\begin{aligned}\sup_{z\in\mathbb{B}}(1-|z|^2)\left|\nabla h_w(z)\right| &= \sup_{z\in\mathbb{B}} 2\frac{(1-|z|^2)|w|}{\left|1-\langle z,w\rangle\right|}\left|\ln\frac{(1+|w|)^2}{1-\langle z,w\rangle}\right|\left(\ln\frac{1+|w|}{1-|w|}\right)^{-1} \\ &\le 4|w|\left(2\ln\frac{1+|w|}{1-|w|}+2\pi\right)\left(\ln\frac{1+|w|}{1-|w|}\right)^{-1} \le 4\left(2+\frac{2\pi}{\ln 3}\right),\end{aligned} \tag{3.14}$$

when $|w|\ge 1/2$.

From this and since $|h_w(0)|\le 4(\ln 2)^2/\ln 3$, for $|w|\ge 1/2$, we have that

$$\left\|h_w\right\|_{\mathcal{B}} \le 4\left(2+\frac{2\pi}{\ln 3}+\frac{(\ln 2)^2}{\ln 3}\right) = C_0. \tag{3.15}$$

Assume $(\varphi(z_k))_{k\in\mathbb{N}}$ is a sequence in $\mathbb{B}$ such that $|\varphi(z_k)|\to 1$ as $k\to\infty$. Note that $(h_{\varphi(z_k)})_{k\in\mathbb{N}}$ is a bounded sequence in $\mathcal{B}$ (moreover in $\mathcal{B}_0$) converging to zero uniformly on compacts of $\mathbb{B}$. Then, by [6, Theorem 7.5], it follows that $h_{\varphi(z_k)}$ converges to zero weakly as $k\to\infty$. Hence, for every compact operator $L:\mathcal{B}_0\to H_\mu^\infty$, we have

$$\lim_{k\to\infty}\left\|Lh_{\varphi(z_k)}\right\|_{H_\mu^\infty} = 0. \tag{3.16}$$

On the other hand, for every compact operator $L:\mathcal{B}_0\to H_\mu^\infty$, we have

$$\begin{aligned}\left\|h_{\varphi(z_k)}\right\|_{\mathcal{B}}\left\|uC_\varphi+L\right\|_{\mathcal{B}_0\to H_\mu^\infty} &\ge \left\|(uC_\varphi+L)h_{\varphi(z_k)}\right\|_{H_\mu^\infty} \\ &\ge \left\|uC_\varphi h_{\varphi(z_k)}\right\|_{H_\mu^\infty} - \left\|Lh_{\varphi(z_k)}\right\|_{H_\mu^\infty}.\end{aligned} \tag{3.17}$$

Using (3.15), letting $k\to\infty$ in (3.17), and applying (3.16), it follows that

$$\begin{aligned}C_0\left\|uC_\varphi+L\right\|_{\mathcal{B}_0\to H_\mu^\infty} &\ge \limsup_{k\to\infty}\left\|uC_\varphi h_{\varphi(z_k)}\right\|_{H_\mu^\infty} \\ &\ge \limsup_{k\to\infty}\mu(z_k)\left|u(z_k)h_{\varphi(z_k)}(\varphi(z_k))\right| \\ &= \limsup_{k\to\infty}\mu(z_k)\left|u(z_k)\right|\ln\frac{1+\left|\varphi(z_k)\right|}{1-\left|\varphi(z_k)\right|}.\end{aligned} \tag{3.18}$$

Taking in (3.18) the infimum over the set of all compact operators $L:\mathcal{B}_0\to H_\mu^\infty$, we obtain

$$\left\|uC_\varphi\right\|_{e,\mathcal{B}_0\to H_\mu^\infty} \ge \frac{1}{C_0}\limsup_{k\to\infty}\mu(z_k)\left|u(z_k)\right|\ln\frac{1+\left|\varphi(z_k)\right|}{1-\left|\varphi(z_k)\right|}, \tag{3.19}$$

from which the first inequality in (3.12) follows.

As in Theorem 3.2, we need only to prove the third inequality in (3.12).

Recall that for each $\rho \in (0,1)$, the operator $uC_{\rho\varphi} : \mathcal{B} \to H_\mu^\infty$ is compact. Let the sequence $(\rho_m)_{m\in\mathbb{N}}$ be as in Theorem 3.2. Note that inequality (3.7) and relationship (3.8) also hold for $\alpha = 1$. Hence, we should only estimate the quantity

$$I_m^\delta = \sup_{|\varphi(z)|>\delta} \sup_{\|f\|_{\mathcal{B}}\le 1} \mu(z)|u(z)||f(\varphi(z)) - f(\rho_m\varphi(z))|. \tag{3.20}$$

On the other hand, by Lemma 2.1 (case $\alpha = 1$) applied to the function $f - f_{\rho_m}$, which belongs to the Bloch space for each $m \in \mathbb{N}$, and inequality (3.9) with $\alpha = 1$, we have

$$\begin{aligned} \mu(z)|u(z)(f(\varphi(z)) - f(\rho_m\varphi(z)))| &\le \frac{1}{2}\|f - f_{\rho_m}\|_{\mathcal{B}}\mu(z)|u(z)| \ln\frac{1+|\varphi(z)|}{1-|\varphi(z)|} \\ &\le \|f\|_{\mathcal{B}}\mu(z)|u(z)| \ln\frac{1+|\varphi(z)|}{1-|\varphi(z)|}. \end{aligned} \tag{3.21}$$

From (3.21), by letting $m \to \infty$ in (3.7) and using (3.8) (with $\alpha = 1$), and letting $\delta \to 1$ in such obtained inequality, we obtain

$$\|uC_\varphi\|_{e,\mathcal{B}\to H_\mu^\infty} \le \limsup_{|\varphi(z)|\to 1} \mu(z)|u(z)| \ln\frac{1+|\varphi(z)|}{1-|\varphi(z)|}, \tag{3.22}$$

as desired. □

Corollary 3.5. *Assume $u \in H(\mathbb{B})$, μ is normal, $\varphi = (\varphi_1, \ldots, \varphi_n)$ is a holomorphic self-map of $\mathbb{B}$, and the operator $uC_\varphi : \mathcal{B}$ (or $\mathcal{B}_0^\alpha$) $\to H_\mu^\infty$ is bounded. Then, $uC_\varphi : \mathcal{B}$ (or $\mathcal{B}_0^\alpha$) $\to H_\mu^\infty$ is compact if and only if*

$$\limsup_{|\varphi(z)|\to 1} \mu(z)|u(z)| \ln\frac{1+|\varphi(z)|}{1-|\varphi(z)|} = 0. \tag{3.23}$$

References

[1] A. L. Shields and D. L. Williams, "Bonded projections, duality, and multipliers in spaces of analytic functions," *Transactions of the American Mathematical Society*, vol. 162, pp. 287–302, 1971.

[2] S. Stević, "Weighted composition operators between mixed norm spaces and H_α^∞ spaces in the unit ball," *Journal of Inequalities and Applications*, vol. 2007, Article ID 28629, 9 pages, 2007.

[3] X. Tang, "Weighted composition operators between Bers-type spaces and Bergman spaces," *Applied Mathematics. A Journal of Chinese Universities B*, vol. 22, no. 1, pp. 61–68, 2007.

[4] D. D. Clahane and S. Stević, "Norm equivalence and composition operators between Bloch/Lipschitz spaces of the ball," *Journal of Inequalities and Applications*, vol. 2006, Article ID 61018, 11 pages, 2006.

[5] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, Fla, USA, 1995.

[6] K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, vol. 226 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 2005.

[7] X. Fu and X. Zhu, "Weighted composition operators on some weighted spaces in the unit ball," *Abstract and Applied Analysis*, vol. 2008, Article ID 605807, 8 pages, 2008.

[8] P. Gorkin and B. D. MacCluer, "Essential norms of composition operators," *Integral Equations and Operator Theory*, vol. 48, no. 1, pp. 27–40, 2004.

[9] S. Li and S. Stević, "Composition followed by differentiation between Bloch type spaces," *Journal of Computational Analysis and Applications*, vol. 9, no. 2, pp. 195–205, 2007.
[10] S. Li and S. Stević, "Weighted composition operators from Bergman-type spaces into Bloch spaces," *Proceedings of the Indian Academy of Sciences. Mathematical Sciences*, vol. 117, no. 3, pp. 371–385, 2007.
[11] S. Li and S. Stević, "Weighted composition operators from α-Bloch space to H^∞ on the polydisc," *Numerical Functional Analysis and Optimization*, vol. 28, no. 7-8, pp. 911–925, 2007.
[12] S. Li and S. Stević, "Weighted composition operators from H^∞ to the Bloch space on the polydisc," *Abstract and Applied Analysis*, vol. 2007, Article ID 48478, 13 pages, 2007.
[13] S. Li and S. Stević, "Generalized composition operators on Zygmund spaces and Bloch type spaces," *Journal of Mathematical Analysis and Applications*, vol. 338, no. 2, pp. 1282–1295, 2008.
[14] S. Li and S. Stević, "Weighted composition operators between H^∞ and α-Bloch space in the unit ball," to appear in *Taiwanese Journal of Mathematics*.
[15] L. Luo and S.-I. Ueki, "Weighted composition operators between weighted Bergman spaces and Hardy spaces on the unit ball of $\mathbb{C}^n$," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 1, pp. 88–100, 2007.
[16] B. D. MacCluer and R. Zhao, "Essential norms of weighted composition operators between Bloch-type spaces," *The Rocky Mountain Journal of Mathematics*, vol. 33, no. 4, pp. 1437–1458, 2003.
[17] A. Montes-Rodríguez, "Weighted composition operators on weighted Banach spaces of analytic functions," *Journal of the London Mathematical Society*, vol. 61, no. 3, pp. 872–884, 2000.
[18] S. Ohno, "Weighted composition operators between H^∞ and the Bloch space," *Taiwanese Journal of Mathematics*, vol. 5, no. 3, pp. 555–563, 2001.
[19] S. Ohno, K. Stroethoff, and R. Zhao, "Weighted composition operators between Bloch-type spaces," *The Rocky Mountain Journal of Mathematics*, vol. 33, no. 1, pp. 191–215, 2003.
[20] J. Shi and L. Luo, "Composition operators on the Bloch space of several complex variables," *Acta Mathematica Sinica*, vol. 16, no. 1, pp. 85–98, 2000.
[21] S. Stević, "Composition operators between H^∞ and α-Bloch spaces on the polydisc," *Zeitschrift für Analysis und ihre Anwendungen*, vol. 25, no. 4, pp. 457–466, 2006.
[22] S. Stević, "Norm of weighted composition operators from Bloch space to H_μ^∞ on the unit ball," *Ars Combinatoria*, vol. 88, pp. 125–127, 2008.
[23] S.-I. Ueki and L. Luo, "Compact weighted composition operators and multiplication operators between Hardy spaces," *Abstract and Applied Analysis*, vol. 2008, Article ID 196498, 12 pages, 2008.
[24] X. Zhu, "Generalized weighted composition operators from Bloch type spaces to weighted Bergman spaces," *Indian Journal of Mathematics*, vol. 49, no. 2, pp. 139–150, 2007.
[25] X. Zhu, "Products of differentiation, composition and multiplication from Bergman type spaces to Bers type spaces," *Integral Transforms and Special Functions*, vol. 18, no. 3, pp. 223–231, 2007.
[26] S. Stević, "On an integral operator on the unit ball in $\mathbb{C}^n$," *Journal of Inequalities and Applications*, vol. 2005, no. 1, pp. 81–88, 2005.
[27] S. Li and S. Stević, "Riemann-Stieltjes-type integral operators on the unit ball in $\mathbb{C}^n$," *Complex Variables and Elliptic Equations*, vol. 52, no. 6, pp. 495–517, 2007.
[28] S. Stević, "Boundedness and compactness of an integral operator on a weighted space on the polydisc," *Indian Journal of Pure and Applied Mathematics*, vol. 37, no. 6, pp. 343–355, 2006.
[29] S. Stević, "Boundedness and compactness of an integral operator in a mixed norm space on the polydisk," *Sibirskiĭ Matematicheskiĭ Zhurnal*, vol. 48, no. 3, pp. 694–706, 2007.