Research Article

A New q -Analogue of Bernoulli Polynomials Associated with p -Adic q -Integrals

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We will study a new q -analogue of Bernoulli polynomials associated with p -adic q -integrals. Furthermore, we examine the Hurwitz-type q -zeta functions, replacing p -adic rational integers x with a q -analogue $[x]_q$ for a p -adic number q with $|q - 1|_p < 1$, which interpolate q -analogue of Bernoulli polynomials.

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1. Introduction

Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ will, respectively, represent the ring of p -adic rational integers, the field of p -adic rational numbers, the complex number field, and the p -adic completion of the algebraic closure of $\mathbb{Q}_p$. The p -adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p = 1/p$ and q is a p -adic number in $\mathbb{C}_p$ with $|q - 1|_p < 1$. We use the notation

$$[x]_q = \frac{1 - q^x}{1 - q} \quad (1.1)$$

(cf. [1–13]) for all $x \in \mathbb{Z}_p$. Hence, $\lim_{q \rightarrow 1} |x|_q = x$. For a fixed odd positive integer d with $(p, d) = 1$, let

$$\begin{aligned} X &= X_d = \varprojlim_n \frac{\mathbb{Z}}{dp^n \mathbb{Z}}, & X_1 &= \mathbb{Z}_p, \\ X^* &= \bigcup_{\substack{0 < a < dp \\ (a, p) = 1}} (a + dp\mathbb{Z}_p), \\ a + dp^n \mathbb{Z}_p &= \{x \in X \mid x \equiv a \pmod{dp^n}\}, \end{aligned} \quad (1.2)$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^n$. For any $n \in \mathbb{N}$,

$$\mu_q(a + dp^n \mathbb{Z}_p) = \frac{q^a}{[dp^n]_q} \quad (1.3)$$

is known to be a distribution on X (cf. [1–13]).

We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by $f \in UD(\mathbb{Z}_p)$, if the difference quotients

$$F_f(x, y) = \frac{f(x) - f(y)}{x - y} \quad (1.4)$$

have a limit $l = f'(a)$ as $(x, y) \rightarrow (a, a)$ (cf. [2, 6, 7]). The p -adic q -integral of a function $f \in UD(\mathbb{Z}_p)$ was defined as

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x. \quad (1.5)$$

By using p -adic q -integrals on $\mathbb{Z}_p$, it is well known that

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (x + s)^n d\mu_1(s) \frac{t^n}{n!}, \quad (1.6)$$

where $\mu_1(x + p^n \mathbb{Z}_p) = 1/p^n$. Then we note that the Bernoulli polynomials $B_n(x)$ were defined as

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \quad (1.7)$$

From (1.6) and (1.7), we have

$$B_n(x) = \int_{\mathbb{Z}_p} (x + s)^n d\mu_1(s) \quad (1.8)$$

for all $n \in \mathbb{N} \cup \{0\}$. We note that $[0]_q = (1 - q^0)/(1 - q) = 0$.

In Section 2, we study a q -analogue of Bernoulli polynomials associated with p -adic q -integrals—simply, we say q -Bernoulli polynomials. In Section 3, we examine Hurwitz-type q -zeta functions, replacing p -adic rational integers x with a q -analogue $[x]_q$ for a p -adic number q with $|q - 1|_p < 1$, which interpolate q -analogue of Bernoulli polynomials.

2. A new q -analogue of Bernoulli polynomials

In this section, from the view of (1.8), we can define a new q -analogue of Bernoulli polynomials as follows:

$$\beta_n^q(x) = \int_{\mathbb{Z}_p} ([x]_q + [s]_q)^n d\mu_q(s). \quad (2.1)$$

We note that $\beta_n^q = \beta_n^q(0)$ are called the q -Bernoulli numbers. Then we find some properties of q -Bernoulli numbers and polynomials as follows.

Theorem 2.1. For $n \in \mathbb{N} \cup \{0\}$, one has

$$\beta_n^q = \frac{1}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{[l+1]_q}. \quad (2.2)$$

Proof. From (1.5) with $x = 0$, we can find the following:

$$\begin{aligned} \beta_n^q &= \int_{\mathbb{Z}_p} [s]_q^n d\mu_q(s) \\ &= \lim_{N \rightarrow \infty} \sum_{j=0}^{p^N-1} [j]_q^n \frac{q^j}{[p^N]_q} \\ &= \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{(1-q)^{n-1}} \lim_{N \rightarrow \infty} \sum_{j=0}^{p^N-1} q^{j(l+1)} \frac{1}{1-q^{p^N}} \\ &= \frac{1}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{[l+1]_q}. \end{aligned} \quad (2.3)$$

□

Theorem 2.2. For $n \in \mathbb{N} \cup \{0\}$ and d being an odd positive integer with $(p, d) = 1$, one has

$$\beta_n^q(x) = [d]_q^{n-1} \sum_{l=0}^n \binom{n}{l} \beta_l^{q^{d^d}} \sum_{i=0}^{d-1} q^{i(l+1)} \left(\left[\frac{x}{d} \right]_{q^d} + \left[\frac{i}{d} \right]_{q^d} \right)^{n-l}. \quad (2.4)$$

Proof. From (1.5), we can derive (2.4) as follows:

$$\begin{aligned} \beta_n^q(x) &= \int_{\mathbb{Z}_p} ([x]_q + [s]_q)^n d\mu_q(s) \\ &= \lim_{N \rightarrow \infty} \frac{1}{[dp^N]_q} \sum_{a=0}^{dp^N-1} ([x]_q + [a]_q)^n q^a \\ &= \lim_{N \rightarrow \infty} \frac{1-q}{1-q^{dp^N}} \sum_{i=0}^{d-1} \sum_{k=0}^{p^N-1} ([x]_q + [i+dk]_q)^n q^{i+dk} \end{aligned}$$

$$\begin{aligned}
&= \lim_{N \rightarrow \infty} \frac{1}{[d]_q} \frac{1}{[p^N]_{q^d}} \sum_{i=0}^{d-1} q^i \sum_{k=0}^{p^N-1} [d]_q^n \left(\left[\frac{x}{d} \right]_{q^d} + \left[\frac{i}{d} \right]_{q^d} + q^i [k]_{q^d} \right)^n (q^d)^k \\
&= \lim_{N \rightarrow \infty} \frac{1}{[d]_q} \frac{1}{[p^N]_{q^d}} \sum_{i=0}^{d-1} q^i \sum_{k=0}^{p^N-1} \sum_{l=0}^n \binom{n}{l} \left(\left[\frac{x}{d} \right]_{q^d} + \left[\frac{i}{d} \right]_{q^d} \right)^{n-l} (q^i [k]_{q^d})^l (q^d)^k \\
&= [d]_q^{n-1} \sum_{i=0}^{d-1} \sum_{l=0}^n \binom{n}{l} q^{i(l+1)} \left(\left[\frac{x}{d} \right]_{q^d} + \left[\frac{i}{d} \right]_{q^d} \right)^{n-l} \lim_{N \rightarrow \infty} \frac{1}{[p^N]_{q^d}} \sum_{k=0}^{p^N-1} [k]_{q^d}^l (q^d)^k \\
&= [d]_q^{n-1} \sum_{i=0}^{d-1} \sum_{l=0}^n \binom{n}{l} q^{i(l+1)} \left(\left[\frac{x}{d} \right]_{q^d} + \left[\frac{i}{d} \right]_{q^d} \right)^{n-l} \int_{\mathbb{Z}_p} [s]_{q^d}^l d\mu_{q^d}(s) \\
&= [d]_q^{n-1} \sum_{l=0}^n \binom{n}{l} \beta_l^{q^d} \sum_{i=0}^{d-1} q^{i(l+1)} \left(\left[\frac{x}{d} \right]_{q^d} + \left[\frac{i}{d} \right]_{q^d} \right)^{n-l},
\end{aligned} \tag{2.5}$$

since $a = i + dk$ and

$$([x]_q + [i + dk]_q)^n = [d]_q^n \left(\left[\frac{x}{d} \right]_{q^d} + \left[\frac{i}{d} \right]_{q^d} + q^i [k]_{q^d} \right)^n \tag{2.6}$$

for $a = 0, 1, \dots, dp^N - 1$, $i = 0, 1, \dots, d - 1$, and $k = 0, 1, \dots, p^N - 1$.

Let $G^q(x, t)$ be the generating function of q -Bernoulli polynomials as follows:

$$G^q(x, t) = \sum_{n=0}^{\infty} \beta_n^q(x) \frac{t^n}{n!}. \tag{2.7}$$

From (2.2) and (2.7), we can obtain the following theorem. □

Theorem 2.3. *Let $G^q(x, t)$ be as in the above generating function. Then, one has*

$$G^q(x, t) = (1 - q) \sum_{m=0}^{\infty} q^m e^{([x]_q + [m]_q)t}. \tag{2.8}$$

Proof. By using (2.2) and (2.7), we can derive (2.8) as follows:

$$\begin{aligned}
G^q(x, t) &= e^{([x]_q + \beta^q)t} = e^{[x]_q t} e^{\beta^q t} = e^{[x]_q t} \sum_{n=0}^{\infty} \beta_n^q \frac{t^n}{n!} \\
&= e^{[x]_q t} \sum_{n=0}^{\infty} \frac{1}{(1 - q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{[l + 1]_q} \frac{t^n}{n!} \\
&= e^{[x]_q t} \sum_{n=0}^{\infty} \frac{1}{(1 - q)^{n-1}} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{1 - q^{l+1}} \frac{t^n}{n!} \\
&= e^{[x]_q t} \sum_{n=0}^{\infty} \frac{1}{(1 - q)^{n-1}} \sum_{l=0}^n \binom{n}{l} (-1)^l \sum_{m=0}^{\infty} q^{(l+1)m} \frac{t^n}{n!}
\end{aligned}$$

$$\begin{aligned}
&= e^{[x]_q t} (1-q) \sum_{m=0}^{\infty} q^m \sum_{n=0}^{\infty} \frac{1}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{lm} \frac{t^n}{n!} \\
&= e^{[x]_q t} (1-q) \sum_{m=0}^{\infty} q^m \sum_{n=0}^{\infty} [m]_q^n \frac{t^n}{n!} \\
&= (1-q) \sum_{m=0}^{\infty} q^m e^{([x]_q + [m]_q)t}.
\end{aligned} \tag{2.9}$$

□

3. A new formula for Hurwitz-type q -zeta functions

In this section, we consider the generating functions $F(t, x)$ which interpolate the q -Bernoulli polynomials $\beta_n^{*q}(x)$ as follows:

$$F(t, x) = \sum_{m=0}^{\infty} q^m e^{([x]_q + [m]_q)t} = \sum_{m=0}^{\infty} \beta_n^{*q}(x) \frac{t^m}{m!}. \tag{3.1}$$

From (3.1), we directly obtain the following theorem.

Theorem 3.1. *For each $k \in \mathbb{N} \cup \{0\}$, one has*

$$\beta_k^{*q}(x) = \sum_{m=0}^{\infty} q^m ([x]_q + [m]_q)^k. \tag{3.2}$$

Proof. By the k th differentiation on both sides of (3.1), we can derive (3.2) as follows:

$$\beta_k^{*q}(x) = \frac{d^k}{dt^k} F(x, t)|_{t=0} = \sum_{m=0}^{\infty} q^m ([x]_q + [m]_q)^k. \tag{3.3}$$

We remark that

$$-\frac{\beta_k^{*q}(x)}{k} = \frac{1}{k} \sum_{m=0}^{\infty} q^m ([x]_q + [m]_q)^k \tag{3.4}$$

for $k \in \mathbb{N}$. From (3.2), we derive a q -extension of Hurwitz-type zeta function as follows: for $s \in \mathbb{C}$ with $\Re(s) > 1$ and $\Re(x) > 0$, we define

$$\zeta^q(s, x) = \frac{1}{1-s} \sum_{m=0}^{\infty} \frac{q^m}{([x]_q + [m]_q)^s}. \tag{3.5}$$

Note that the functions $\zeta^q(s, x)$ are analytic on $\Re(s) > 1$ and they have simple pole at $s = 1$. From (3.2), (3.4), and (3.5), we can see that Hurwitz-type q -zeta functions interpolate q -Bernoulli polynomials as follows. □

Theorem 3.2. For each $k \in \mathbb{N}$, one has

$$\zeta^q(1-k, x) = -\frac{\beta_k^{*q}(x)}{k}. \quad (3.6)$$

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