Research Article

On the $q$-Extension of Apostol-Euler Numbers and Polynomials

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Recently, Choi et al. (2008) have studied the $q$-extensions of the Apostol-Bernoulli and the Apostol-Euler polynomials of order $n$ and multiple Hurwitz zeta function. In this paper, we define Apostol’s type $q$-Euler numbers $E_{n,q}$ and $q$-Euler polynomials $E_{n,q}(x)$. We obtain the generating functions of $E_{n,q}$ and $E_{n,q}(x)$, respectively. We also have the distribution relation for Apostol’s type $q$-Euler polynomials. Finally, we obtain $q$-zeta function associated with Apostol’s type $q$-Euler numbers and Hurwitz’s type $q$-zeta function associated with Apostol’s type $q$-Euler polynomials for negative integers.

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1. Introduction

Let $p$ be a fixed odd prime. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}$ and $\mathbb{C}_p$ will, respectively, denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, the complex number field, and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then one assumes $|q - 1|_p < 1$. We also use the notations

$$ [x]_q = \frac{1 - q^x}{1 - q}, \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q} \quad \forall x \in \mathbb{Z}_p \quad (1.1) $$

For a fixed odd positive integer $d$ with $(p,d) = 1$, let

$$ X = X_d = \lim_{N \to \infty} \frac{Z_d}{dp^N}, \quad X_1 = \mathbb{Z}_p, $$
\[ X^* = \bigcup_{0 \leq a < dp} (a + dp\mathbb{Z}_p), \]
\[ a + dp^N\mathbb{Z}_p = \{ x \in X \mid x \equiv a \pmod{dp^N} \}, \]

where \( a \in \mathbb{Z} \) lies in \( 0 \leq a < dp^N \). The distribution is defined by

\[ \mu_q(a + dp^N\mathbb{Z}_p) = \frac{q^a}{[dp^N]_q}. \]

We say that \( f \) is a uniformly differentiable function at a point \( a \in \mathbb{Z}_p \) and denote this property by \( f \in \text{UD}(\mathbb{Z}_p) \), if the difference quotients \( F_f(x, y) = (f(x) - f(y))/(x - y) \) have a limit \( l = f'(a) \) as \( (x, y) \rightarrow (a, a) \). For \( f \in \text{UD}(\mathbb{Z}_p) \), the \( p \)-adic invariant \( q \)-integral is defined as

\[ I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x. \]  

The fermionic \( p \)-adic \( q \)-measures on \( \mathbb{Z}_p \) are defined as

\[ \mu_{-q}(a + dp^N\mathbb{Z}_p) = \frac{(-q)^a}{[dp^N]_{-q}}, \]

and the fermionic \( p \)-adic invariant \( q \)-integral on \( \mathbb{Z}_p \) is defined as

\[ I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x \]

for \( f \in \text{UD}(\mathbb{Z}_p) \). For details see [1–10].

Classical Euler numbers are defined by the generating function

\[ \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \]

and these numbers are interpolated by the Euler zeta function which is defined as

\[ \zeta_E(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}, \quad s \in \mathbb{C}. \]

After Carlitz [11] gave \( q \)-extensions of the classical Bernoulli numbers and polynomials, the \( q \)-extensions of Bernoulli and Euler numbers and polynomials have been studied by several authors (cf. [1–16, 18–26, 34–39]).
By using $p$-adic $q$-integral, the $q$-Euler numbers $E_{n,q}$ are defined as

$$E_{n,q} = \int_{\mathbb{Z}_p} [t]^n d\mu_q(t), \quad \text{for } n \in \mathbb{N}. \quad (1.9)$$

The $q$-Euler numbers $E_{n,q}$ are defined by means of the generating function

$$F_q(t) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n e^{[n]_q t} \quad (1.10)$$

(cf. [8, 26]). Kim [22] gave a new construction of the $q$-Euler numbers $E_{n,q}$ which can be uniquely determined by

$$E_{0,q} = \frac{[2]_q}{2},$$

$$(qE + 1)^n + E_{n,q} = \begin{cases} [2]_q & \text{if } n = 0, \\ 0 & \text{if } n \neq 0, \end{cases} \quad (1.11)$$

with the usual convention of replacing $E^n$ by $E_{n,q}$.

The twisted $q$-Euler numbers and $q$-Euler polynomials are very important in several fields of mathematics and physics, and so they have been studied by many authors. Simsek [37, 38] constructed generating functions of $q$-generalized Euler numbers and polynomials and twisted $q$-generalized Euler numbers and polynomials. Recently, Y. H. Kim et al. [27] gave the twisted $q$-Euler zeta function associated with twisted $q$-Euler numbers and obtained $q$-Euler’s identity. They also have a $q$-extension of the Euler zeta function for negative integers and the $q$-analog of twisted Euler zeta function. Kim [24] defined twisted $q$-Euler numbers and polynomials of higher order and studied multiple twisted $q$-Euler zeta functions.

The Apostol-Bernoulli and the Apostol-Euler polynomials and numbers have been studied by several authors (cf. [15, 17, 32, 33, 40, 41]). Recently, $q$-extensions of the Apostol-Bernoulli and the Apostol-Euler polynomials and numbers have been studied by many authors with great interest. In [15], Cenkci and Can introduced and investigated $q$-extensions of the Bernoulli polynomials. Choi et al. [16] have studied some $q$-extensions of the Apostol-Bernoulli and the Apostol-Euler polynomials of order $n$ and multiple Hurwitz zeta function.

In this paper, we define Apostol’s type $q$-Euler numbers and $q$-Euler polynomials. Then, we have the generating functions of Apostol’s type $q$-Euler numbers and $q$-Euler polynomials and the distribution relation for Apostol’s type $q$-Euler polynomials. In Section 2, we define Apostol’s type $q$-Euler numbers $E_{n,q,\xi}$ and $q$-Euler polynomials $E_{n,q,\xi}(x)$. Then, we obtain the generating functions of $E_{n,q,\xi}$ and $E_{n,q,\xi}(x)$, respectively. We also have the distribution relation for Apostol’s type $q$-Euler polynomials. In Section 3, we obtain $q$-zeta function associated with Apostol’s type $q$-Euler numbers and Hurwitz’s type $q$-zeta function associated with Apostol’s type $q$-Euler polynomials for negative integers.
2. On the $q$-extensions of the Apostol-Euler numbers and polynomials

In this section, we will assume $q \in \mathbb{C}$ with $|q - 1| < 1$. For $n \in \mathbb{Z}$, let $C_{p^n} = \{ \xi \mid \xi^{p^n} = 1 \}$ be the cyclic group of order $p^n$, and let $T_p$ be the space of locally constant functions, that is,

$$T_p = \lim_{n \to \infty} C_{p^n} = \bigcup_{n \geq 0} C_{p^n}. \tag{2.1}$$

Let $\xi \in T_p$. We define Apostol's type $q$-Euler numbers by

$$E_{n,q,\xi} = \int_{Z_p} q^{-x} \xi^x [x]_q^n d\mu_q(x). \tag{2.2}$$

Then, we have

$$E_{n,q,\xi} = \frac{[2]_q}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1 + q^l \xi} \tag{2.3}$$

where $\binom{n}{l}$ are the binomial coefficients.

Apostol's type $q$-Euler polynomials are defined as

$$E_{n,q,\xi}(x) = \int_{Z_p} q^{-y} \xi^y [x+y]_q^n d\mu_q(y). \tag{2.4}$$

Since

$$[x+y]_q^n = ([x]_q + q^x [y]_q^n) = \sum_{l=0}^{n} \binom{n}{l} [x]^n_q q^l x \tag{2.5}$$

we have from (2.4) that

$$E_{n,q,\xi}(x) = \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^l x E_{l,q,\xi}. \tag{2.6}$$

By (2.2) and (2.6), we have

$$E_{n,q,\xi}(x) = \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^l x E_{l,q,\xi}. \tag{2.7}$$

Since

$$[x+y]_q^n = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^l x y^l = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^l x q^l y^l, \tag{2.8}$$
we have

\[
\int_{\mathbb{Z}_p} q^{-y} \xi^{x+y} d\mu_{-q}(y) = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^l x \int_{\mathbb{Z}_p} q^{(l-1)y} \xi^y d\mu_{-q}(y). \tag{2.9}
\]

Therefore, we also have

\[
E_{n,q,\xi}(x) = [2]_q \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^l \xi}. \tag{2.10}
\]

Note that (2.7) and (2.10) are two representations for \(E_{n,q,\xi}(x)\). Hence, we have the following result.

**Theorem 2.1.** For \(n \in \mathbb{Z}_+\) and \(\xi \in T_p\), one has

\[
E_{n,q,\xi} = [2]_q \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^l \xi},
\]

\[
E_{n,q,\xi}(x) = [2]_q \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^l x \frac{1}{1+q^l \xi}, \tag{2.11}
\]

\[
= \sum_{l=0}^{n} \binom{n}{l} [x]^{n-l} q^l x E_{l,q,\xi}.
\]

Now, we will find the generating function of \(E_{n,q,\xi}\) and \(E_{n,q,\xi}(x)\), respectively. Let \(F(t)\) be the generating function of \(E_{n,q,\xi}\). Then, we have

\[
F(t) = \sum_{n=0}^{\infty} E_{n,q,\xi} \frac{t^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \left[2\right]_q \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^l \xi} \frac{t^n}{n!}
\]

\[
= \left[2\right]_q \sum_{n=0}^{\infty} \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \left(\sum_{m=0}^{\infty} q^{m} \xi^{m} (-1)^m \right) \frac{t^n}{n!}
\]

\[
= \left[2\right]_q \sum_{m=0}^{\infty} (-1)^m \xi^m \sum_{n=0}^{\infty} \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^m \frac{t^n}{n!}
\]

\[
= \left[2\right]_q \sum_{m=0}^{\infty} (-1)^m \xi^m \sum_{n=0}^{\infty} \frac{1}{(1-q)^n} (1-q^m)^n \frac{t^n}{n!}
\]
\[ F(t) = [2]_q \sum_{m=0}^{\infty} (-1)^m \xi^m \sum_{n=0}^{\infty} \frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty} (-1)^m \xi^m e^{[m]_q t}. \]  
(2.12)

Therefore, the generating function \( F(t) \) of \( E_{n,q,\xi} \) equals

\[ F(t) = \sum_{n=0}^{\infty} E_{n,q,\xi} \frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty} (-1)^m \xi^m e^{[m]_q t}. \]  
(2.13)

Note that

\[ \int_{\mathbb{Z}_p} q^{-x} \xi^x e^{[x]_q t} d\mu_{-q}(x) = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} q^{-x} \xi^x [x]_q^n d\mu_{-q}(x) \frac{t^n}{n!} \]
\[ = \sum_{n=0}^{\infty} E_{n,q,\xi} \frac{t^n}{n!} = F(t). \]  
(2.14)

For the generating function of \( E_{n,q,\xi}(x) \), we have

\[ \int_{\mathbb{Z}_p} q^{-y} \xi^y e^{[x+y]_q t} d\mu_{-q}(y) = [2]_q \sum_{m=0}^{\infty} (-1)^m \xi^m e^{[m+x]_q t}. \]  
(2.15)

Hence, we obtain the following theorem.

**Theorem 2.2.** For \( \xi \in T_p \), one has

\[ \int_{\mathbb{Z}_p} q^{-x} \xi^x e^{[x]_q t} d\mu_{-q}(x) = [2]_q \sum_{m=0}^{\infty} (-1)^m \xi^m e^{[m]_q t}, \]  
(2.16)

\[ \int_{\mathbb{Z}_p} q^{-y} \xi^y e^{[x+y]_q t} d\mu_{-q}(y) = [2]_q \sum_{m=0}^{\infty} (-1)^m \xi^m e^{[m+x]_q t}. \]  
(2.17)

Since (2.16) equals to the generating functions (2.17) equals to the generating functions \( \sum_{n=0}^{\infty} E_{n,q,\xi}(x) (t^n/n!) \), we have the following result.

**Corollary 2.3.** For \( n \in \mathbb{Z}_+ \) and \( \xi \in T_p \), one has

\[ E_{n,q,\xi} = [2]_q \sum_{m=0}^{\infty} (-1)^m \xi^m [m]_q^n, \]  
(2.18)

\[ E_{n,q,\xi}(x) = [2]_q \sum_{m=0}^{\infty} (-1)^m \xi^m [m + x]_q^n. \]
Now, we will find the distribution relation for $E_{n,q,ξ}(x)$. By (2.4), we have

$$E_{n,q,ξ}(x) = \int X q^{-y} s^y [x + y]_q^n dμ_ξ(y)$$

$$= \lim_{N \to \infty} \frac{1}{[dp^N]_{-q}} \sum_{y=0}^{dp^N-1} s^y (-1)^y [x + y]_q^n$$

$$= \lim_{N \to \infty} \frac{1}{[dp^N]_{-q}} \sum_{a=0}^{d-1} \sum_{y=0}^{p^N-1} s^y (-1)^y [x + a + dy]_q^n.$$  \hspace{1cm} (2.19)

Note that for odd numbers $d$ and $p$,

$$[dp^N]_{-q} = [d]_{-q}[p^N]_{-q^d},$$

$$[x + a + dy]_q = [d]_q \left[ \frac{x + a}{d} + y \right]_{q^d}. \hspace{1cm} (2.20)$$

By (2.19), we have

$$E_{n,q,ξ}(x) = \frac{1}{[d]_{-q}} \sum_{a=0}^{d-1} s^a (-1)^a \lim_{N \to \infty} \frac{1}{[p^N]_{-q^d}} \sum_{y=0}^{p^N-1} s^y (-1)^y [d]_q^n \left[ \frac{x + a}{d} + y \right]_{q^d}^n$$

$$= \frac{[d]_q^{n-1}}{[d]_{-q}} \sum_{a=0}^{d-1} s^a (-1)^a \int_{\mathbb{Z}_q} (s^d)^y (q^d)^{-y} \left[ \frac{x + a}{d} + y \right]_{q^d}^n dμ_{-q^d}(y). \hspace{1cm} (2.21)$$

Therefore, we obtain the distribution relation for $E_{n,q,ξ}(x)$ as follows.

**Theorem 2.4.** For $n \in \mathbb{Z}_+$, $ξ \in T_p$, and $d \in \mathbb{Z}_+$ with $d \equiv 1 \pmod{2}$, one has

$$E_{n,q,ξ}(x) = \frac{[d]_q^{n-1}}{[d]_{-q}} \sum_{a=0}^{d-1} s^a (-1)^a E_{n,q^d,ξ} \left( \frac{x + a}{d} \right). \hspace{1cm} (2.22)$$

**3. Further remark on the basic $q$-zeta functions associated with Apostol’s type $q$-Euler numbers and polynomials**

In this section, we assume that $q \in \mathbb{C}$ with $|q| < 1$. Let $ξ \in T_p$. For $s \in \mathbb{C}$, $q$-zeta function associated with Apostol’s type $q$-Euler numbers is defined as

$$ζ_{q,ξ}(s) = [2]_q \sum_{n=1}^{\infty} \frac{ζ_n(-1)^n}{[n]_q^s}.$$  \hspace{1cm} (3.1)
Hence, we obtain which is analytic in whole complex $s$-plane. Substituting $s = -k$ with $k \in \mathbb{Z}_+$ into $\zeta_{q,k}(s)$ and using Corollary 2.3, then we arrive at

$$\zeta_{q,k}(-k) = [2]_q \sum_{n=1}^{\infty} \frac{t^n(-1)^n}{[n]_q^k} = E_{k,q,k}.$$  \quad (3.2)

Now, we also consider Hurwitz’s type $q$-zeta function associated with the Apostol’s type $q$-Euler polynomials as follows:

$$\zeta_{q,k}(s,x) = [2]_q \sum_{n=0}^{\infty} \frac{t^n(-1)^n}{[n+x]_q^k}.$$  \quad (3.3)

Substituting $s = -k$ with $k \in \mathbb{Z}_+$ into $\zeta_{q,k}(s,x)$ and using Corollary 2.3, then we arrive at

$$\zeta_{q,k}(-k,x) = [2]_q \sum_{n=0}^{\infty} \frac{t^n(-1)^n}{[n+x]_q^k} = E_{k,q,k}(x).$$  \quad (3.4)

Hence, we obtain $q$-zeta function associated with Apostol’s type $q$-Euler numbers and Hurwitz’s type $q$-zeta function associated with Apostol’s type $q$-Euler polynomials for negative integers.

References


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