Research Article

Multivariate \(p\)-Adic Fermionic \(q\)-Integral on \(\mathbb{Z}_p\) and Related Multiple Zeta-Type Functions

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In 2008, Jang et al. constructed generating functions of the multiple twisted Carlitz’s type \(q\)-Bernoulli polynomials and obtained the distribution relation for them. They also raised the following problem: “are there analytic multiple twisted Carlitz’s type \(q\)-zeta functions which interpolate multiple twisted Carlitz’s type \(q\)-Euler (Bernoulli) polynomials?” The aim of this paper is to give a partial answer to this problem. Furthermore we derive some interesting identities related to twisted \(q\)-extension of Euler polynomials and multiple twisted Carlitz’s type \(q\)-Euler polynomials.

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1. Introduction, definitions, and notations

Let \(p\) be an odd prime. \(\mathbb{Z}_p, \mathbb{Q}_p,\) and \(\mathbb{C}_p\) will always denote, respectively, the ring of \(p\)-adic integers, the field of \(p\)-adic numbers, and the completion of the algebraic closure of \(\mathbb{Q}_p\). Let \(\nu_p : \mathbb{C}_p \to \mathbb{Q} \cup \{\infty\}\) (\(\mathbb{Q}\) is the field of rational numbers) denote the \(p\)-adic valuation of \(\mathbb{C}_p\) normalized so that \(\nu_p(p) = 1\). The absolute value on \(\mathbb{C}_p\) will be denoted as \(|\cdot|_p\), and \(|x|_p = p^{-\nu_p(x)}\) for \(x \in \mathbb{C}_p\). We let \(\mathbb{Z}_p^\times = \{x \in \mathbb{Z}_p\mid 1/x \in \mathbb{Z}_p\}\). A \(p\)-adic integer in \(\mathbb{Z}_p^\times\) is sometimes called a \(p\)-adic unit. For each integer \(N \geq 0\), \(\mathbb{C}_{p^N}\) will denote the multiplicative group of the primitive \(p^N\)th roots of unity in \(\mathbb{C}_p^\times\). Set

\[
T_p = \{\omega \in \mathbb{C}_p \mid \omega^{p^N} = 1 \text{ for some } N \geq 0\} = \bigcup_{N \geq 0} \mathbb{C}_{p^N}.
\]

The dual of \(\mathbb{Z}_p\), in the sense of \(p\)-adic Pontryagin duality, is \(T_p = \mathbb{C}_{p^{\infty}}\), the direct limit (under inclusion) of cyclic groups \(\mathbb{C}_{p^N}\) of order \(p^N(N \geq 0)\), with the discrete topology.
When one talks of \( q \)-extension, \( q \) is variously considered as an indeterminate, a complex number \( q \in \mathbb{C} \), or a \( p \)-adic number \( q \in \mathbb{C}_p \). If \( q \in \mathbb{C}_p \), then we normally assume \( |1 - q|_p < p^{-1/(p-1)} \), so that \( q^x = \exp(x \log q) \) for \( |x|_p \leq 1 \). If \( q \in \mathbb{C} \), then we assume that \( |q| < 1 \).

Let

\[
\mathbb{Z}_p = \lim_{N \to \infty} \left( \frac{\mathbb{Z}}{p^N \mathbb{Z}} \right), \quad \mathbb{Z}_p^* = \bigcup_{a \in \mathbb{C}_p} a + p\mathbb{Z}_p,
\]

\[
a + p^N \mathbb{Z}_p = \{ x \in \mathbb{Z}_p \mid x \equiv a \pmod{p^N} \},
\]

where \( a \in \mathbb{Z} \) lies in \( 0 \leq a < p^N \).

We use the following notation:

\[
[x]_q = \frac{1 - q^x}{1 - q}.
\]

Hence

\[
\lim_{q \to 1} [x]_q = x
\]

for any \( x \) with \( |x|_p \leq 1 \) in the present \( p \)-adic case. The distribution \( \mu_q(a + p^N \mathbb{Z}_p) \) is given as

\[
\mu_q(a + p^N \mathbb{Z}_p) = \frac{q^a}{[p^N]_q}
\]

(cf. [1–9]). For the ordinary \( p \)-adic distribution \( \mu_0 \) defined by

\[
\mu_0(a + p^N \mathbb{Z}_p) = \frac{1}{p^N},
\]

we see

\[
\lim_{q \to 1} \mu_q = \mu_0.
\]

We say that \( f \) is a uniformly differentiable function at a point \( a \in \mathbb{Z}_p \), we write \( f \in \text{UD}(\mathbb{Z}_p, \mathbb{C}_p) \) if the difference quotient

\[
F_f(x, y) = \frac{f(x) - f(y)}{x - y}
\]

has a limit \( f'(a) \) as \((x, y) \to (a, a)\). Also we use the following notation:

\[
[x]_{-q} = \frac{1 - (-q)^x}{1 + q}.
\]

(cf.[1–5]).
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In [1–3], Kim gave a detailed proof of fermionic $p$-adic $q$-measures on $\mathbb{Z}_p$. He treated some interesting formulae-related $q$-extension of Euler numbers and polynomials; and he defined fermionic $p$-adic $q$-measures on $\mathbb{Z}_p$ as follows:

$$\mu_q(a + p^N\mathbb{Z}_p) = \frac{(-q)^a}{[p^N]_{-q}}.$$ \hspace{1cm} (1.10)

By using the fermionic $p$-adic $q$-measures, he defined the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$ as follows:

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x$$ \hspace{1cm} (1.11)

for $f \in \text{UD}(\mathbb{Z}_p, C_p)$ (cf. [1–3]). Observe that

$$I_1(f) = \lim_{q \to 1} I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x.$$ \hspace{1cm} (1.12)

From (1.12), we obtain

$$I_1(f_1) + I_1(f) = 2f(0),$$ \hspace{1cm} (1.13)

where $f_1(x) = f(x + 1)$. By substituting $f(x) = e^{tx}$ into (1.13), classical Euler numbers are defined by means of the following generating function:

$$\int_{\mathbb{Z}_p} e^{tx} d\mu_1(x) = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}.$$ \hspace{1cm} (1.14)

These numbers are interpolated by the Euler zeta function which is defined as follows:

$$\zeta_E(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}, \quad s \in \mathbb{C},$$ \hspace{1cm} (1.15)

(cf. [1–9]). From (1.12), we also obtain

$$qI_q(f_1) + I_q(f) = [2]_q f(0),$$ \hspace{1cm} (1.16)

where $f_1(x) = f(x + 1)$. By substituting $f(x) = e^{tx}$ into (1.13), $q$-Euler numbers are defined by means of the following generating function:

$$\int_{\mathbb{Z}_p} e^{tx} d\mu_q(x) = \frac{[2]_q}{qe^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}.$$ \hspace{1cm} (1.17)
These numbers are interpolated by the Euler $q$-zeta function which is defined as follows:

$$\zeta_{q,E}(s) = [2]_q \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{n^s}, \quad s \in \mathbb{C},$$

\hspace{1cm} (1.18)

(cf. [4]).

In [6], Ozden and Simsek defined generating function of $q$-Euler numbers by

$$\frac{2}{q + 1} \int_{\mathbb{Z}_p} e^{tx} d\mu_q(x) = \frac{2}{qe^x + 1},$$

which are different from (1.17). But we observe that all these generating functions were obtained by the same fermionic $p$-adic $q$-measures on $\mathbb{Z}_p$ and the fermionic $p$-adic $q$-integrals on $\mathbb{Z}_p$.

In this paper, we define a multiple twisted Carlitz’s type $q$-zeta functions, which interpolated multiple twisted Carlitz’s type $q$-Euler polynomials at negative integers. This result gave us a partial answer of the problem proposed by Jang et al. [10], which is given by: “Are there analytic multiple twisted Carlitz’s type $q$-zeta functions which interpolate multiple twisted Carlitz’s type $q$-Euler (Bernoulli) polynomials?”

2. Preliminaries

In [10], Jang and Ryoo defined $q$-extension of Euler numbers and polynomials of higher order and studied multivariate $q$-Euler zeta functions. They also derived sums of products of $q$-Euler numbers and polynomials by using fermionic $p$-adic $q$-integral.

In [5, 7], Ozden et al. defined multivariate Barnes-type Hurwitz $q$-Euler zeta functions and $l$-functions. They also gave relation between multivariate Barnes-type Hurwitz $q$-Euler zeta functions and multivariate $q$-Euler $l$-functions.

In this section, we consider twisted $q$-extension of Euler numbers and polynomials of higher order and study multivariate twisted Barnes-type Hurwitz $q$-Euler zeta functions and $l$-functions.

Let $\text{UD}(\mathbb{Z}_p^h, \mathbb{C}_p)$ denote the space of all uniformly (or strictly) differentiable $\mathbb{C}_p$-valued functions on $\mathbb{Z}_p^h = \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$. For $f \in \text{UD}(\mathbb{Z}_p^h, \mathbb{C}_p)$, the $p$-adic $q$-integral on $\mathbb{Z}_p^h$ is defined by

$$I_{-q}^{(h)}(f) = \int_{\mathbb{Z}_p^h} \cdots \int_{\mathbb{Z}_p^h} f(x_1, \ldots, x_h) d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_h)$$

\hspace{1cm} (2.1)

$$= \lim_{N \to \infty} \frac{1}{[pN]^h} \sum_{x_1=0}^{pN-1} \cdots \sum_{x_h=0}^{pN-1} f(x_1, \ldots, x_h)(-q)^{x_1 + \cdots + x_h}$$

(cf. [3]). If $q \to 1$, then

$$I_{-1}^{(h)}(f) = \lim_{q \to -1} I_{-q}^{(h)}(f) = \lim_{N \to \infty} \sum_{x_1=0}^{pN-1} \cdots \sum_{x_h=0}^{pN-1} f(x_1, \ldots, x_h)(-1)^{x_1 + \cdots + x_h}.$$
For a fixed positive integer $d$ with $(d, p) = 1$, we set
\[ X_p = \lim_{N \to \infty} \left( \frac{Z}{dp^N \mathbb{Z}} \right). \]  
(2.3)

For $f \in \text{UD}(\mathbb{Z}_p^h, \mathbb{C}_p)$,
\[ I_{-1}^{(h)}(f) = \int_{X_p} \cdots \int_{X_p} f(x_1, \ldots, x_h) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_h), \]  
(2.4)
(cf. [2]).

We set $f(x_1, \ldots, x_h) = \omega^{x_1+\cdots+x_h} e^{(x+x_1+\cdots+x_h)h}$ in (2.2) and (2.4). Then we have
\[ \int_{\mathbb{Z}_p^h} \cdots \int_{\mathbb{Z}_p^h} \omega^{x_1+\cdots+x_h} e^{(x+x_1+\cdots+x_h)h}d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_h) = \left( \frac{2}{\omega q^0 + 1} \right) \cdots \left( \frac{2}{\omega q^h + 1} \right) = \sum_{n=0}^{\infty} E_{n, \omega}^{(h)}(x) \frac{t^n}{n!}, \]  
where $E_{n, \omega}^{(h)}(x)$ are the twisted Euler polynomials of order $h$. From (2.5), we note that
\[ \int_{\mathbb{Z}_p^h} \cdots \int_{\mathbb{Z}_p^h} \omega^{x_1+\cdots+x_h} (x + x_1 + \cdots + x_h)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_h) = E_{n, \omega}^{(h)}(x). \]  
(2.6)

We give an application of the multivariate $q$-deformed $p$-adic integral on $\mathbb{Z}_p^h$ in the fermionic sense related to [3]. Let
\[ \int_{\mathbb{Z}_p^h} = \int_{\mathbb{Z}_p^h} \cdots \int_{\mathbb{Z}_p^h}. \]  
(2.7)

By substituting
\[ f(x_1, \ldots, x_h) = \omega^{x_1+\cdots+x_h} e^{(x+x_1+\cdots+x_h)h} \]  
(2.8)
into (2.1), we define twisted $q$-extension of Euler numbers of higher order by means of the following generating function:
\[ \int_{\mathbb{Z}_p^h} \omega^{x_1+\cdots+x_h} e^{(x+x_1+\cdots+x_h)h} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_h) = \left( \frac{[2]_q^{\omega q^0 + 1}}{\omega q^0 + 1} \right) \cdots \left( \frac{[2]_q^{\omega q^h + 1}}{\omega q^h + 1} \right) = \sum_{n=0}^{\infty} E_{n, \omega}^{(h)}(x) \frac{t^n}{n!}, \]  
(2.9)
Then we have
\[
\int \omega x^{m+n} \frac{d \mu_{-q}(x_1) \cdots d \mu_{-q}(x_h)}{\mu_{-q}^{h}(x)} = E^{(h)}_{n,q,\omega}. \tag{2.10}
\]

From (2.9), we obtain
\[
\int \omega x^{m+n} e^{(x+x_1+\cdots+x_h)t} \frac{d \mu_{-q}(x_1) \cdots d \mu_{-q}(x_h)}{\mu_{-q}^{h}(x)} = \sum_{n=0}^{\infty} E^{(h)}_{n,q,\omega}(x) \frac{t^n}{n!},
\]
\[
\tag{2.11}
\]

where \(E^{(h)}_{n,q,\omega}(x)\) is called twisted \(q\)-extension of Euler polynomials of higher order (cf. [11]). We note that if \(\omega = 1\), then \(E^{(h)}_{n,q,\omega}(x) = E^{(h)}_{n,q}(x)\) and \(E^{(h)}_{n,q,\omega} = E^{(h)}_{n,q}\) (cf. [6]). We also note that
\[
E^{(h)}_{n,q,\omega}(x) = \sum_{k=0}^{n} \binom{n}{k} E^{(h)}_{k,q,\omega} x^{n-k}.
\tag{2.12}
\]

The twisted \(q\)-extension of Euler polynomials of higher order, \(E^{(h)}_{n,q,\omega}(x)\), is defined by means of the following generating function:
\[
G^{(h)}_{q,\omega}(x,t) = \frac{[2]^h_q e^{xt}}{(\omega q e^t + 1)} \left( \frac{[2]^h_q e^{xt}}{\omega q e^t + 1} \right)^h = \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\infty} \cdots \sum_{l_h=0}^{\infty} \frac{(-\omega)^{l_1} q^{l_1} e^{lt_1} \cdots (-\omega)^{l_h} q^{l_h} e^{lt_h}}{(\omega q e^t + 1)^h} = \sum_{n=0}^{\infty} E^{(h)}_{n,q,\omega}(x) \frac{t^n}{n!},
\tag{2.13}
\]
where \(|t + \log(\omega q)| < \pi\). From these generating functions of twisted \(q\)-extension of Euler polynomials of higher order, we construct twisted multiple \(q\)-Euler zeta functions as follows.

For \(s \in \mathbb{C}\) and \(x \in \mathbb{R}\) with \(0 < x \leq 1\), we define
\[
\zeta^{(h)}_{q,\omega}(s,x) = [2]^h_q \sum_{l_1,\ldots,l_h=0}^{\infty} (-\omega)^{l_1+\cdots+l_h} q^{l_1+\cdots+l_h} (l_1 + \cdots + l_h + x)^s.
\tag{2.14}
\]

By the \(m\)th differentiation on both sides of (2.13) at \(t = 0\), we obtain the following
\[
E^{(h)}_{m,q,\omega}(x) = \left( \frac{d}{dt} \right)^m G^{(h)}_{q,\omega}(x,t) \bigg|_{t=0} = [2]^h_q \sum_{l_1,\ldots,l_h=0}^{\infty} (-\omega)^{l_1+\cdots+l_h} q^{l_1+\cdots+l_h} (x + l_1 + \cdots + l_h)^m \tag{2.15}
\]
for \(m = 0, 1, \ldots\).
From (2.14) and (2.15), we arrive at the following

\[ b_{q,a;E}^{(h)}(-m, x) = E_{m,q,a;E}^{(h)}(x), \quad m = 0, 1, \ldots \]  

(2.16)

We set

\[
\int_{X_p^h} = \int_{X_p} \cdots \int_{X_p} \text{h-times}
\]

(2.17)

Let \( \chi \) be Dirichlet's character with odd conductor \( d \). We define twisted \( q \)-extension of generalized Euler polynomials of higher order by means of the following generating function (cf. [11]):

\[
\int_{X_p^h} \chi(x_1 + \cdots + x_h) \omega^{x_1 + \cdots + x_h} e^{(x_1 + \cdots + x_h)t} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_h) = \sum_{n=0}^{\infty} E_{n,q,a;\chi}^{(h)}(x) \frac{t^n}{n!}
\]

(2.18)

Note that

\[
\sum_{n=0}^{\infty} E_{n,q,a;\chi}^{(h)}(x) \frac{t^n}{n!} = e^{xt} \int_{X_p} \cdots \int_{X_p} \chi(x_1 + \cdots + x_h) \omega^{x_1 + \cdots + x_h} e^{(x_1 + \cdots + x_h)t} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_h)
\]

\[
= e^{xt} [d]^{-h}_{-q} \lim_{N \to \infty} \frac{1}{[p^N]_{-q}} \sum_{a_1 = 0}^{d-1} \sum_{a_2 = 0}^{d-1} \cdots \sum_{a_h = 0}^{d-1} \sum_{x_1 = 0}^{p^{N-1}} \cdots \sum_{x_h = 0}^{p^{N-1}} \chi(a_1 + dx_1 + \cdots + a_h + dx_h)
\]

\[
\times \omega^{a_1 + dx_1 + \cdots + a_h + dx_h} e^{(a_1 + dx_1 + \cdots + a_h + dx_h)t} (-q)^{a_1 + dx_1 + \cdots + a_h + dx_h}
\]

\[
= e^{xt} \frac{1}{[d]^{-h}_{-q}} \sum_{a_1 = 0}^{d-1} \cdots \sum_{a_h = 0}^{d-1} \chi(a_1 + \cdots + a_h) \omega^{a_1 + \cdots + a_h} (-q)^{a_1 + \cdots + a_h} e^{(a_1 + \cdots + a_h)t}
\]

\[
\times \lim_{N \to \infty} \frac{1 + q^d}{1 + q^{dp^N} e^{dp^N}} \cdots \lim_{N \to \infty} \frac{1 + q^d}{1 + q^{dp^N} e^{dp^N}} \frac{1 + q^d}{1 + q^{dp^N} e^{dp^N}}
\]

\[
= e^{xt} \frac{1}{[d]^{-h}_{-q}} \sum_{a_1 = 0}^{d-1} \cdots \sum_{a_h = 0}^{d-1} \chi(a_1 + \cdots + a_h) \omega^{a_1 + \cdots + a_h} (-q)^{a_1 + \cdots + a_h} e^{(a_1 + \cdots + a_h)t}
\]

\[
\times \frac{1 + q^d}{1 + \omega^d q^{dp^N} e^{dp^N}} \cdots \frac{1 + q^d}{1 + \omega^d q^{dp^N} e^{dp^N}}
\]

\[
\text{h-times}
\]

since

\[
\lim_{N \to \infty} q^{dp^N} = 1 \quad \text{for} \quad |1 - q|_p < 1.
\]

(2.20)
This allows us to rewrite (2.18) as

\[
\sum_{n=0}^{\infty} E_{m,q,a,\chi}^{(b)}(x) \frac{t^n}{n!} = e^{xt} \frac{1}{[d]^h_{-q}} \sum_{a_1,\ldots,a_h=0}^{d-1} \chi(a_1 + \cdots + a_h) \omega^{a_1 \cdots a_h} (-q)^{a_1 \cdots a_h} e^{(a_1 \cdots a_h)t} \\
\times \frac{1 + q^d}{1 + \omega d^q e^{dt}} \cdots \frac{1 + q^d}{1 + \omega d^q e^{dt}} \left[ h \text{-times} \right]
\]

\[
= [2]^h_q e^{xt} \sum_{a_1,\ldots,a_h=0}^{d-1} \chi(a_1 + \cdots + a_h) \omega^{a_1 \cdots a_h} (-q)^{a_1 \cdots a_h} e^{(a_1 \cdots a_h)t} \\
\times \sum_{x_1=0}^{\infty} \left( -\omega d^q e^{dt} \right)^{x_1} \cdots \sum_{x_h=0}^{\infty} \left( -\omega d^q e^{dt} \right)^{x_h} \left[ h \text{-times} \right]
\]

\[
= [2]^h_q e^{xt} \sum_{x_1,\ldots,x_h=0}^{\infty} \sum_{a_1,\ldots,a_h=0}^{d-1} \chi(a_1 \cdot d x_1 + \cdots + a_h \cdot d x_h) \\
\times \omega^{a_1 \cdot d x_1 + \cdots + a_h \cdot d x_h} (-q)^{a_1 \cdot d x_1 + \cdots + a_h \cdot d x_h} e^{(a_1 \cdot d x_1 + \cdots + a_h \cdot d x_h)t} \\
= [2]^h_q \sum_{l_1,\ldots,l_h=0}^{\infty} (-1)^{l_1 + \cdots + l_h} \chi(l_1 + \cdots + l_h) \omega^{l_1 + \cdots + l_h} q^{l_1 + \cdots + l_h} e^{(l_1 + \cdots + l_h)t}.
\]

By applying the \(m\)th derivative operator \((d/dt)^m|_{t=0}\) in the above equation, we have

\[
E_{m,q,a,\chi}^{(b)}(x) = [2]^h_q \sum_{l_1,\ldots,l_h=0}^{\infty} \chi(l_1 + \cdots + l_h) \prod_{i=1}^{h} (-1)^i \omega^i q^i (x + l_1 + \cdots + l_h)^m 
\]

(2.22)

for \(m = 0, 1, \ldots\).

From these generating functions of twisted \(q\)-extension of generalized Euler polynomials of higher order, we construct twisted multiple \(q\)-Euler \(l\)-functions as follows. For \(s \in \mathbb{C}\) and \(x \in \mathbb{R}\) with \(0 < x \leq 1\), we define

\[
l_{q,a,\chi}^{(b)}(s,x,\chi) = [2]^h_q \sum_{l_1,\ldots,l_h=0}^{\infty} \chi(l_1 + \cdots + l_h) \prod_{i=1}^{h} (-1)^i \omega^i q^i \frac{(l_1 + \cdots + l_h + x)^s}{(l_1 + \cdots + l_h + x)^s}. 
\]

(2.23)

From (2.22) and (2.23), we arrive at the following

\[
l_{q,a,\chi}^{(b)}(-m,x,\chi) = E_{m,q,a,\chi}^{(b)}(x), \quad m = 0, 1, \ldots.
\]

(2.24)
Let $s \in \mathbb{C}$ and $a_i, F \in \mathbb{Z}$ with $F$ is an odd integer and $0 < a_i < F$, where $i = 1, \ldots, h$. Then twisted partial multiple $q$-Euler $\zeta$-functions are as follows:

$$H^{(h)}_{q,a,E}(s, a_1, \ldots, a_h, x \mid F) = [2]_q^h \sum_{l_i=0}^{\infty} \frac{(-1)^{l_1+\cdots+l_h} \omega^{l_1+\cdots+l_h} q^{l_1+\cdots+l_h}}{(l_1+\cdots+l_h+x)^s}. \quad (2.25)$$

For $i = 1, \ldots, h$, substituting $l_i = a_i + n_i F$ with $F$ is odd into (2.25), we have

$$H^{(h)}_{q,a,E}(s, a_1, \ldots, a_h, x \mid F) = \frac{[2]_q^h}{[2]_q^h} \sum_{m=0}^{\infty} \frac{(-1)^{m+n_i F} \omega^{m+n_i F} q^{m+n_i F}}{(m+n_i F + a_i + n_i F + x)^s}. \quad (2.26)$$

Then we obtain

$$H^{(h)}_{q,a,E}(s, a_1, \ldots, a_h, x \mid F) = \frac{[2]_q^h}{[2]_q^h} \left( \frac{-\omega q}{F} \right)^{a_1+\cdots+a_h} \zeta^{(h)}_{q,q,F,E} \left( s, \frac{a_1 + \cdots + a_h + x}{F} \right). \quad (2.27)$$

By using (2.12) and (2.27) and by substituting $s = -m$, $m = 0, 1, \ldots$, we get

$$H^{(h)}_{q,a,E}(-m, a_1, \ldots, a_h, x \mid F) = \frac{[2]_q^h}{[2]_q^h} \left( \frac{-\omega q}{F} \right)^{a_1+\cdots+a_h} (a_1 + \cdots + a_h + x)^m \times \sum_{k=0}^{m} F \left( a_1 + \cdots + a_h + x \right)^k E^{(h)}_{k,q,q,F,E}. \quad (2.28)$$

Therefore, we modify twisted partial multiple $q$-Euler zeta functions as follows:

$$H^{(h)}_{q,a,E}(s, a_1, \ldots, a_h, x \mid F) = \frac{[2]_q^h}{[2]_q^h} \left( \frac{-\omega q}{F} \right)^{a_1+\cdots+a_h} (a_1 + \cdots + a_h + x)^s \times \sum_{k=0}^{\infty} \frac{(-s)^k}{k} \left( \frac{F}{a_1 + \cdots + a_h + x} \right)^k E^{(h)}_{k,q,q,F,E}. \quad (2.29)$$

Let $\chi$ be a Dirichlet character with conductors $d$ and $d \mid F$. From (2.23) and (2.27), we have

$$l^{(h)}_{q,a,E}(s, x, \chi) = \frac{[2]_q^h}{[2]_q^h} \sum_{a_i=0}^{F-1} (-\omega q)^{a_1+\cdots+a_h} \chi(a_1 + \cdots + a_h) \zeta^{(h)}_{q,q,F,E} \left( s, \frac{a_1 + \cdots + a_h + x}{F} \right) \times \chi(a_1 + \cdots + a_h) H^{(h)}_{q,a,E}(s, x, a_1, \ldots, a_h, x \mid F). \quad (2.30)$$
3. The multiple twisted Carlitz’s type $q$-Euler polynomials and $q$-zeta functions

Let us consider the multiple twisted Carlitz’s type $q$-Euler polynomials as follows:

$$E_{n,q,w}(x) = \int_{\mathbb{Z}_q} \cdots \int_{\mathbb{Z}_q} [x_1 + \cdots + x_h + x]_q^{n} d\mu_q(x_1) \cdots d\mu_q(x_h)$$

(3.1)

(cf. [1, 3]). These can be written as

$$E_{n,q,w}(x) = [2]_q^n \sum_{i=0}^{n} \binom{n}{i} q^{ix} (-1)^i \frac{1}{1 + \omega q^{zi}} \cdots \frac{1}{1 + \omega q^{zi+h-1}}.$$  

(3.2)

We may now mention the following formulae which are easy to prove:

$$\omega q^z E_{n,q,w}(x + 1) + E_{n,q,w}(x) = [2]_q E_{n,q,w+1}(x).$$

(3.3)

From (3.2), we can derive generating function for the multiple twisted Carlitz’s type $q$-Euler polynomials as follows:

$$\sum_{n=0}^{\infty} E_{n,q,w}(x) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{[2]_q^n}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{i} q^{ix} (-1)^i \frac{1}{1 + \omega q^{zi}} \cdots \frac{1}{1 + \omega q^{zi+h-1}} \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{[2]_q^n}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{i} q^{ix} (-1)^i \sum_{i_1=0}^{\infty} (-\omega q^{zi_1}) \cdots \sum_{i_h=0}^{\infty} (-\omega q^{zi_h}) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{[2]_q^n}{(1-q)^n} \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_h=0}^{\infty} (-1)^{i_1+\cdots+i_h} \omega^{i_1+\cdots+i_h} q^{i_1 z + i_2 (z-1) + \cdots + i_h (z-h+1)} \frac{t^n}{n!}$$

$$= [2]_q \sum_{n=0}^{\infty} \sum_{i_1, i_2, \ldots, i_h=0} (-1)^{i_1+\cdots+i_h} \omega^{i_1+\cdots+i_h} q^{i_1 z + i_2 (z-1) + \cdots + i_h (z-h+1)} \frac{t^n}{n!}$$

$$= [2]_q \sum_{n=0}^{\infty} \sum_{i_1, i_2, \ldots, i_h=0} (-\omega)^{i_1+\cdots+i_h} q^{i_1 z + i_2 (z-1) + \cdots + i_h (z-h+1)} e^{[x+i_1+\cdots+i_h]t}$$

(3.4)

Also, an obvious generating function for the multiple twisted Carlitz’s type $q$-Euler polynomials is obtained, from (3.2), by

$$[2]_q^h e^{t/(1-q)} \sum_{j=0}^{\infty} (-1)^j \binom{j}{1-q} x^{j} \frac{1}{1 + \omega q^{z+j}} \cdots \frac{1}{1 + \omega q^{z+j+h-1}} = E_{n,q,w}(x).$$

(3.5)
From now on, we assume that $q \in \mathbb{C}$ with $|q| < 1$. From (3.2) and (3.4), we note that

\[ G_{q,h,\omega}^{(z,h)}(x,t) = [2]^h_q \sum_{l_1,\ldots,l_h=0}^{\infty} (-\omega)^{l_1+\cdots+l_h} q^{l_1z+l_2(z-1)+\cdots+l_h(z-h+1)} e^{[x+l_1+\cdots+l_h]t} \left[ x+l_1+\cdots+l_h \right]_q^n. \]  

(3.6)

Thus we can define the multiple twisted Carlitz’s type $q$-zeta functions as follows:

\[ \zeta_{q,h,\omega}^{(z,h)}(s,x) = [2]^h_q \sum_{l_1,\ldots,l_h=0}^{\infty} \frac{(-\omega)^{l_1+\cdots+l_h} q^{l_1z+l_2(z-1)+\cdots+l_h(z-h+1)}}{[x+l_1+\cdots+l_h]_q^n}. \]  

(3.8)

In [12, Proposition 3], Yamasaki showed that the series $\zeta_{q,h,\omega}^{(z,h)}(s,x)$ converges absolutely for $\text{Re}(z) > h-1$, and it can be analytically continued to the whole complex plane $\mathbb{C}$. Note that if $h=1$, then

\[ \zeta_{q,h,\omega}^{(z,h)}(s,x) \rightarrow \zeta_{q,1,\omega}^{(z)}(s,x) = [2]^1_q \sum_{l=0}^{\infty} \frac{(-\omega)^{l} q^{lz}}{[x+l]_q^s}. \]  

(3.9)

In [13], Wakayama and Yamasaki studied $q$-analogue of the Hurwitz zeta function

\[ \zeta_{q}(s,x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s} \]  

(3.10)

defined by the $q$-series with two complex variable $s, z \in \mathbb{C}$:

\[ \zeta_{q}^{(x)}(s,x) = \sum_{n=0}^{\infty} \frac{q^{(n+x)z}}{[x+n]_q^s}, \text{ Re}(z) > 0, \]  

(3.11)

and special values at nonpositive integers of the $q$-analogue of the Hurwitz zeta function.

Therefore, by the $m$th differentiation on both sides of (3.6) at $t = 0$, we obtain the following:

\[ E_{m,q,h,\omega}^{(z,h)}(x) \left( \frac{d}{dt} \right)^m G_{q,h,\omega}^{(z,h)}(x,t) \bigg|_{t=0} = [2]^h_q \sum_{l_1,\ldots,l_h=0}^{\infty} (-\omega)^{l_1+\cdots+l_h} q^{l_1z+l_2(z-1)+\cdots+l_h(z-h+1)} [x+l_1+\cdots+l_h]_q^m. \]  

(3.12)

for $m = 0,1,\ldots$. 
From (3.7), (3.8), and (3.12), we have (3.13) which shows that the multiple twisted Carlitz’s type \( q \)-zeta functions interpolate the multiple twisted Carlitz’s type \( q \)-Euler numbers and polynomials. For \( m = 0, 1, \ldots \), we have

\[
f_{q,\omega}^{(z,h)}(-m,x) = E_{m,q,\omega}^{(z,h)}(x),
\]

where \( x \in \mathbb{R} \) and \( 0 < x \leq 1 \).

Thus, we derive the analytic multiple twisted Carlitz’s type \( q \)-zeta functions which interpolate multiple twisted Carlitz’s type \( q \)-Euler polynomials. This gives a part of the answer to the question proposed in [10].

4. Remarks

For nonnegative integers \( m \) and \( n \), we define the \( q \)-binomial coefficient \( \left[ \frac{m}{n} \right]_q \) by

\[
\left[ \frac{m}{n} \right]_q = \frac{(q; q)_m}{(q; q)_n(q; q)_{m-n}},
\]

where \( (a; q)_m = \prod_{k=0}^{m-1} (1 - aq^k) \) for \( m \geq 1 \) and \( (a; q)_0 = 1 \). For \( h \in \mathbb{N} \), it holds that

\[
\sum_{l_1, \ldots, l_h \geq 0 \atop l_1 + \cdots + l_h = l} q^{-(l_1+2l_2+\cdots+hl_h)} = q^{-lh} \left[ \frac{l+h-1}{h-1} \right]_q
\]

(cf. [12, Lemma 2.3]). From (3.8), it is easy to see that

5. \[
f_{q,\omega}^{(z,h)}(s, x) = [2]^h \sum_{l=0}^{\infty} \sum_{l_1, \ldots, l_h \geq 0 \atop l_1 + \cdots + l_h = l} \frac{(-\omega)^{l_1+\cdots+l_h}}{[l+x]_q} q^{(z+1)(l_1+\cdots+l_h)-(l_1+2l_2+\cdots+hl_h)}
\]

\[
= [2]^h \sum_{l=0}^{\infty} \frac{(-\omega)^l q^{(z+1)l}}{[l+x]_q} \sum_{l_1, \ldots, l_h \geq 0 \atop l_1 + \cdots + l_h = l} q^{-(l_1+2l_2+\cdots+hl_h)}
\]

\[
= [2]^h \sum_{l=0}^{\infty} \left[ \frac{l+h-1}{h-1} \right]_q (-\omega)^l q^{(z+h+1)l} [l+x]_q^l.
\]

We set \([m]_q! = [m]_q[m-1]_q \cdots [1]_q\) for \( m \in \mathbb{N} \). The following identity has been studied in [12]:

\[
\left[ \frac{l+h-1}{h-1} \right]_q = \frac{1}{[h-1]_q} \prod_{j=1}^{h-1} ([l+x]_q - q^{l+i}[x-j]_q) = \sum_{k=0}^{h-1} q^{l(h-1-k)} P_{q,h}^{k} (x)[l+x]_q^k,
\]

where \( P_{q,h}^{k} (x), 0 \leq k \leq h-1, \) is a function of \( x \) defined by

\[
P_{q,h}^{k} (x) = \frac{(-1)^{h-1-k}}{[h-1]_q} \sum_{1 \leq m_1, \ldots, m_{h-1,k} \leq h-1} q^{m_1+\cdots+m_{h-1,k}} [x-m_1]_q \cdots [x-m_{h-1,k}]_q
\]
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for \(0 \leq k \leq h - 2\) and \(P_{q,h}^{h-1}(x) = 1/[h - 1]_q!\). By using (3.9), (4.3), and (4.5), we have

\[
\zeta_{q,h}^{(z,k)}(s,x) = [2]_q \sum_{k=0}^{h-1} p_{q,h}^k(x) \sum_{l=0}^{\infty} \frac{(-\alpha)^l}{[l + x]_q^{s-k}} = [2]_q \sum_{k=0}^{h-1} p_{q,h}^k(x) \zeta_{q,h}^{(z-k)}(s-k,x),
\]

and so

\[
\zeta_{q,h}^{(z,h)}(-m,x) = [2]_q \sum_{k=0}^{h-1} p_{q,h}^k(x) \zeta_{q,h}^{(z-k)}(m-k,x).
\]

The values of \(\zeta_{q,h}^{(z,h)}(-m,x)\) at \(h = 2, 3\) are given explicitly as follows:

\[
\begin{align*}
\zeta_{b_{q,h}}^{(z,2)}(-m,x) &= (1 + q) \left(\zeta_{b_{q,h}}^{(z-1)}(-m-1,x) - q[x-1]_q \zeta_{b_{q,h}}^{(z)}(-m,x)\right), \\
\zeta_{b_{q,h}}^{(z,3)}(-m,x) &= (1 + q) \left(\zeta_{b_{q,h}}^{(z-2)}(-m-2,x) \\
&\quad - (q[x-1]_q + q^2[x-2]_q) \zeta_{b_{q,h}}^{(z-1)}(-m-1,x) \\
&\quad + q^3[x-1]_q[x-2]_q \zeta_{b_{q,h}}^{(z)}(-m,x)\right).
\end{align*}
\]

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References


