Research Article

On the Symmetries of the q -Bernoulli Polynomials

Taekyun Kim

Division of General Education-Mathematics, Kwangju University, Seoul 139-701, South Korea

Correspondence should be addressed to Taekyun Kim, tkkim@kw.ac.kr

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Kupershmidt and Tuentner have introduced reflection symmetries for the q -Bernoulli numbers and the Bernoulli polynomials in (2005), (2001), respectively. However, they have not dealt with congruence properties for these numbers entirely. Kupershmidt gave a quantization of the reflection symmetry for the classical Bernoulli polynomials. Tuentner derived a symmetry of power sum polynomials and the classical Bernoulli numbers. In this paper, we study the new symmetries of the q -Bernoulli numbers and polynomials, which are different from Kupershmidt's and Tuentner's results. By using our symmetries for the q -Bernoulli polynomials, we can obtain some interesting relationships between q -Bernoulli numbers and polynomials.

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1. Introduction

Let p be a fixed prime. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ will, respectively, denote the ring of p -adic rational integer, the field of p -adic rational numbers, the complex number field, and the completion of algebraic closure of $\mathbb{Q}_p$. The p -adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p < p^{-1}$. Let q be variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p -adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}_p$, we assume that $|1 - q|_p < 1$. We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p$, and we denote this property by $f \in \text{UD}(\mathbb{Z}_p)$ if the difference quotients,

$$F_f : \mathbb{Z}_p \times \mathbb{Z}_p \longrightarrow \mathbb{Z}_p \quad \text{by } F_f(x, y) = \frac{f(x) - f(y)}{x - y}, \quad (1.1)$$

have a limit $l = f'(a)$ as $(x, y) \rightarrow (a, a)$. The p -adic invariant integral on $\mathbb{Z}_p$ is defined as

$$I(f) = \int_{\mathbb{Z}_p} f(x) dx = \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x) \quad (1.2)$$

[1–22]. From this integral, we derive several further interesting properties of symmetry for the q -Bernoulli numbers and polynomials in this paper. Kupershmidt [14] and Tuentler [20] have introduced reflection symmetries for the q -Bernoulli numbers and the Bernoulli polynomials. However, they have not dealt with congruence properties for these numbers entirely. Kupershmidt gave a quantization of the reflection symmetry for the classical Bernoulli polynomials. Tuentler derived a symmetry of power sum polynomials and the classical Bernoulli numbers. In this paper, we study the new symmetries of the q -Bernoulli numbers and polynomials, which are different from Kupershmidt's and Tuentler's results. By using our symmetries for the q -Bernoulli polynomials, we can obtain some interesting relationships between q -Bernoulli numbers and polynomials.

2. On the symmetries of the q -Bernoulli polynomials

For $f \in \text{UD}(\mathbb{Z}_p)$, the p -adic invariant integral on $\mathbb{Z}_p$ is defined as

$$I(f) = \int_{\mathbb{Z}_p} f(x) dx = \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x). \quad (2.1)$$

Let $f_1(x)$ be a translation with $f_1(x) = f(x+1)$. Then, we have

$$I(f_1) = I(f) + f'(0). \quad (2.2)$$

From (2.2), we can also derive

$$I(f_n) = I(f) + \sum_{i=0}^{n-1} f'(i), \quad f'(i) = \frac{df(i)}{dx}. \quad (2.3)$$

Let $f(x) = q^x e^{tx}$, then we have

$$\int_{\mathbb{Z}_p} q^x e^{tx} dx = \frac{t + \log q}{qe^t - 1}. \quad (2.4)$$

It is known that the q -Bernoulli polynomials are defined as

$$\frac{t + \log q}{qe^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{n!} \quad (2.5)$$

[17, 19]. Now we define an integral representation for the q -extension of Bernoulli numbers as follows:

$$\int_{\mathbb{Z}_p} q^x e^{tx} dx = \frac{\log q + t}{qe^t - 1} = \sum_{n=0}^{\infty} B_{n,q} \frac{t^n}{n!}. \quad (2.6)$$

From (2.3), (2.4), and (2.6), we can derive

$$\int_{\mathbb{Z}_p} q^y (x+y)^n dy = B_{n,q}(x), \quad \int_{\mathbb{Z}_p} q^x x^n dx = B_{n,q}. \quad (2.7)$$

By (2.3), we easily see that

$$\begin{aligned} \frac{1}{\log q + t} \left(\int_{\mathbb{Z}_p} q^{n+x} e^{(n+x)t} dx - \int_{\mathbb{Z}_p} q^x e^{xt} dx \right) &= \frac{q^n e^{nt} - 1}{t + \log q} \int_{\mathbb{Z}_p} q^x e^{xt} dx = \frac{q^n e^{nt} - 1}{q e^t - 1} \\ &= \sum_{i=0}^{n-1} q^i e^{it} = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{n-1} i^k q^i \right) \frac{t^k}{k!}. \end{aligned} \quad (2.8)$$

In (2.2), it is not difficult to show that

$$\frac{1}{\log q + t} \left(\int_{\mathbb{Z}_p} q^{n+x} e^{(n+x)t} dx - \int_{\mathbb{Z}_p} q^x e^{xt} dx \right) = \frac{n \int_{\mathbb{Z}_p} e^{xt} q^x dx}{\int_{\mathbb{Z}_p} e^{nxt} q^{nx} dx}. \quad (2.9)$$

For each integer $k \geq 0$, let

$$S_{k,q}(n) = 0^k + 1^k q + 2^k q^2 + \cdots + q^n n^k. \quad (2.10)$$

From (2.8) and (2.9), we derive

$$\frac{1}{\log q + t} \left(\int_{\mathbb{Z}_p} q^{n+x} e^{(n+x)t} dx - \int_{\mathbb{Z}_p} q^x e^{xt} dx \right) = \frac{n \int_{\mathbb{Z}_p} e^{xt} q^x dx}{\int_{\mathbb{Z}_p} e^{nxt} q^{nx} dx} = \sum_{k=0}^{\infty} S_{k,q}(n-1) \frac{t^k}{k!}. \quad (2.11)$$

From (2.11), we note that

$$B_{k,q}(n) - B_{k,q} = k S_{k-1,q}(n-1) + \log q S_{k,q}(n-1), \quad \text{where } k, n \in \mathbb{N}. \quad (2.12)$$

Let $w_1, w_2 \in \mathbb{N}$, then we have

$$\frac{\iint_{\mathbb{Z}_p} e^{(w_1 x_1 + w_2 x_2)t} q^{w_1 x_1 + w_2 x_2} dx_1 dx_2}{\int_{\mathbb{Z}_p} e^{w_1 w_2 x t} q^{w_1 w_2 x} dx} = (t + \log q) \frac{q^{w_1 w_2} e^{w_1 w_2 t} - 1}{(q^{w_1} e^{w_1 t} - 1)(q^{w_2} e^{w_2 t} - 1)}. \quad (2.13)$$

By (2.11), we see that

$$\frac{w_1 \int_{\mathbb{Z}_p} e^{xt} q^x dx}{\int_{\mathbb{Z}_p} q^{w_1 x} e^{w_1 x t} dx} = \sum_{l=0}^{\infty} \left(\sum_{k=0}^{w_1-1} k^l q^k \right) \frac{t^l}{l!} = \sum_{l=0}^{\infty} S_{l,q}(w_1-1) \frac{t^l}{l!}. \quad (2.14)$$

Let

$$T(w_1, w_2; x, t) = \frac{\iint_{\mathbb{Z}_p} q^{w_1 x_1 + w_2 x_2} e^{(w_1 x_1 + w_2 x_2 + w_1 w_2 x)t} dx_1 dx_2}{\int_{\mathbb{Z}_p} e^{w_1 w_2 x_3 t} q^{w_1 w_2 x_3} dx_3}, \quad (2.15)$$

then we have

$$T(w_1, w_2; x, t) = \frac{(t + \log q) e^{w_1 w_2 x t} (q^{w_1 w_2} e^{w_1 w_2 t} - 1)}{(q^{w_1} e^{w_1 t} - 1)(q^{w_2} e^{w_2 t} - 1)}. \quad (2.16)$$

From (2.15) we derive

$$T(w_1, w_2; x, t) = \left(\frac{1}{w_1} \int_{\mathbb{Z}_p} e^{w_1(x_1 + w_2 x)t} q^{w_1 x_1} dx_1 \right) \left(\frac{w_1 \int_{\mathbb{Z}_p} e^{w_2 x_2 t} q^{w_2 x_2} dx_2}{\int_{\mathbb{Z}_p} e^{w_1 w_2 x t} q^{w_1 w_2 x} dx} \right). \quad (2.17)$$

By (2.5), (2.14), and (2.17), we see that

$$\begin{aligned} T(w_1, w_2; x, t) &= \frac{1}{w_1} \left(\sum_{i=0}^{\infty} B_{i, q^{w_1}}(w_2 x) \frac{w_1^i t^i}{i!} \right) \left(\sum_{l=0}^{\infty} S_{l, q^{w_2}}(w_1 - 1) \frac{w_1^l t^l}{l!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^n \binom{n}{i} B_{i, q^{w_1}}(w_2 x) S_{n-i, q^{w_2}}(w_1 - 1) w_1^{i-1} w_2^{n-i} \right) \frac{t^n}{n!}. \end{aligned} \quad (2.18)$$

By the symmetry of p -adic invariant integral on $\mathbb{Z}_p$, we also see that

$$\begin{aligned} T(w_1, w_2; x, t) &= \left(\frac{1}{w_2} \int_{\mathbb{Z}_p} e^{w_2(x_2 + w_1 x)t} q^{w_2 x_2} dx_2 \right) \left(\frac{w_2 \int_{\mathbb{Z}_p} e^{w_1 x_1 t} q^{w_1 x_1} dx_1}{\int_{\mathbb{Z}_p} e^{w_1 w_2 x t} q^{w_1 w_2 x} dx} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^n \binom{n}{i} B_{i, q^{w_2}}(w_1 x) S_{n-i, q^{w_1}}(w_2 - 1) w_2^{i-1} w_1^{n-i} \right) \frac{t^n}{n!}. \end{aligned} \quad (2.19)$$

By comparing the coefficients $t^n/n!$ on the both sides of (2.18) and (2.19), we obtain the following theorem.

Theorem 2.1. For all $w_1, w_2 \in \mathbb{N}$, we have

$$\sum_{i=0}^n \binom{n}{i} B_{i, q^{w_1}}(w_2 x) S_{n-i, q^{w_2}}(w_1 - 1) w_1^{i-1} w_2^{n-i} = \sum_{i=0}^n \binom{n}{i} B_{i, q^{w_2}}(w_1 x) S_{n-i, q^{w_1}}(w_2 - 1) w_2^{i-1} w_1^{n-i}, \quad (2.20)$$

where $\binom{n}{i}$ is the binomial coefficient.

If we take $w_2 = 1$ in Theorem 2.1, then we have

$$B_{n,q}(w_1 x) = \sum_{i=0}^n \binom{n}{i} B_{i,q^{w_1}}(x) S_{n-i,q}(w_1 - 1) w_1^{i-1}. \quad (2.21)$$

Therefore, we obtain the following corollary.

Corollary 2.2. *For $n \geq 0$, we have*

$$B_{n,q}(w_1 x) = \sum_{i=0}^n \binom{n}{i} B_{i,q^{w_1}}(x) S_{n-i,q}(w_1 - 1) w_1^{i-1}. \quad (2.22)$$

By (2.17), (2.18), and (2.19), we also see that

$$\begin{aligned} T(w_1, w_2; x, t) &= \left(\frac{e^{w_1 w_2 x t}}{w_1} \int_{\mathbb{Z}_p} e^{w_1 x_1 t} q^{w_1 x_1} dx_1 \right) \left(\frac{w_1 \int_{\mathbb{Z}_p} e^{w_2 x_2 t} q^{w_2 x_2} dx_2}{\int_{\mathbb{Z}_p} e^{w_1 w_2 x t} q^{w_1 w_2 x} dx} \right) \\ &= \left(\frac{e^{w_1 w_2 x t}}{w_1} \int_{\mathbb{Z}_p} e^{w_1 x_1 t} q^{w_1 x_1} dx_1 \right) \left(\sum_{i=0}^{w_1-1} q^{w_2 i} e^{w_2 i t} \right) \\ &= \frac{1}{w_1} \sum_{i=0}^{w_1-1} q^{w_2 i} \int_{\mathbb{Z}_p} e^{(x_1 + w_2 x + (w_2/w_1)t) w_1} q^{x w_1} dx_1 \\ &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{w_1-1} B_{n,q^{w_1}} \left(w_2 x + \frac{w_2}{w_1} i \right) w_1^{n-1} q^{w_2 i} \right) \frac{t^n}{n!}. \end{aligned} \quad (2.23)$$

From the symmetry of $T(w_1, w_2; x, t)$, we can also derive

$$T(w_1, w_2; x, t) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{w_2-1} B_{n,q^{w_2}} \left(w_1 x + \frac{w_1}{w_2} i \right) w_2^{n-1} q^{w_1 i} \right) \frac{t^n}{n!}. \quad (2.24)$$

By comparing the coefficients $t^n/n!$ on the both sides of (2.23) and (2.24), we obtain the following theorem.

Theorem 2.3. *For $n \in \mathbb{Z}_+$, $w_1, w_2 \in \mathbb{N}$, we have*

$$\sum_{i=0}^{w_1-1} B_{n,q^{w_1}} \left(w_2 x + \frac{w_2}{w_1} i \right) w_1^{n-1} q^{w_2 i} = \sum_{i=0}^{w_2-1} B_{n,q^{w_2}} \left(w_1 x + \frac{w_1}{w_2} i \right) w_2^{n-1} q^{w_1 i}. \quad (2.25)$$

Remark 2.4. Setting $w_2 = 1$ in Theorem 2.3, we get the multiplication theorem for the q -Bernoulli polynomials as follows:

$$B_{n,q}(w_1 x) = w_1^{n-1} \sum_{i=0}^{w_1-1} B_{n,q^{w_1}} \left(x + \frac{i}{w_1} \right) q^i. \quad (2.26)$$

I cannot obtain the extended formulae of Theorems 2.1 and 2.3 related to the Carlitz's q -Bernoulli numbers and polynomials. So, we suggest the following two questions.

Question 1. Find the extended formulae of Theorems 2.1 and 2.3, which are related to the Carlitz's q -Bernoulli numbers and polynomials.

Question 2. Find the twisted formulae of Theorems 2.1 and 2.3, which are related to the twisted Carlitz's q -Bernoulli polynomials.

Remark 2.5. In [12], q -Volkenborn integral is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x. \quad (2.27)$$

Thus, we note that Carlitz's q -Bernoulli numbers can be written by

$$\beta_{n,q} = \int_{\mathbb{Z}_p} [x]_q^n d\mu_q(x), \quad \text{Witt's type formula.} \quad (2.28)$$

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