Research Article

Differences of Composition Operators on the Space of Bounded Analytic Functions in the Polydisc

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This paper gives some estimates of the essential norm for the difference of composition operators induced by φ and ψ acting on the space, $H^\infty(\mathbb{D}^n)$, of bounded analytic functions on the unit polydisc $\mathbb{D}^n$, where φ and ψ are holomorphic self-maps of $\mathbb{D}^n$. As a consequence, one obtains conditions in terms of the Carathéodory distance on $\mathbb{D}^n$ that characterizes those pairs of holomorphic self-maps of the polydisc for which the difference of two composition operators on $H^\infty(\mathbb{D}^n)$ is compact.

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1. Introduction

Let $\mathbb{D}^n$ be the unit polydisc of $\mathbb{C}^n$ with boundary $\partial\mathbb{D}^n$. If $n = 1$, we will denote the unit disk $\mathbb{D}^1$ simply by $\mathbb{D}$. The class of all holomorphic functions on $\mathbb{D}^n$ will be denoted by $H(\mathbb{D}^n)$, while by $H^\infty(\mathbb{D}^n)$, we denote the space of all bounded analytic functions in the unit polydisc with the norm $\|f\|_\infty = \sup_{z \in \mathbb{D}^n} |f(z)|$.

Let $\varphi(z) = (\varphi_1(z), \dots, \varphi_n(z))$ and $\psi(z) = (\psi_1(z), \dots, \psi_n(z))$ be holomorphic self-maps of $\mathbb{D}^n$. The composition operator, C_φ , is defined by

$$C_\varphi(f)(z) = f(\varphi(z)) \quad (1.1)$$

for any $f \in H(\mathbb{D}^n)$ and $z \in \mathbb{D}^n$.

Let X be a Banach space. Recall that the essential norm of a continuous linear operator $T : X \rightarrow X$ is the distance from T to the compact operators, that is,

$$\|T\|_e = \inf \{ \|T - K\|; K : X \rightarrow X \text{ is compact} \}. \quad (1.2)$$

Notice that $\|T\|_e = 0$ if and only if T is compact, so that estimates on $\|T\|_e$ lead to conditions for T to be compact.

In the past few decades, boundedness, compactness, and essential norms of composition and closely related operators between various spaces of holomorphic functions have been studied by many authors (see, e.g., the following papers mostly in the settings of the unit ball and the unit polydisc [1–23] and the references therein). Recently, several papers focused on studying the mapping properties of the difference of two composition operators, that is, of an operator of the form

$$T = C_\varphi - C_\psi. \quad (1.3)$$

One of the first results of this type, in the setting of the Hardy space $H^2(\mathbb{D})$, belongs to Berkson [24]. There, it was shown that if φ is an analytic self-map of the unit disk $\mathbb{D}$ whose radial limit function φ^* satisfies $|\varphi^*(\zeta)| = 1$ for $\zeta \in E \subset \partial\mathbb{D}$, $\text{meas}(E) > 0$, then for any analytic self-map ψ of the disk, $\psi \neq \varphi$,

$$\|C_\varphi - C_\psi\| \geq \sqrt{\frac{\text{meas}(E)}{2}}, \quad (1.4)$$

where meas denotes the normalized Lebesgue measure on $\partial\mathbb{D}$, which means that C_φ is isolated in the operator norm topology. Some other conditions for isolation in the same setting are obtained in [15].

In [25], MacCluer et al., among other results, characterized the compactness of the difference of two composition operators on $H^\infty(\mathbb{D})$ in terms of the Poincaré distance. In [26], isolated points and essential components of composition operators on $H^\infty(\mathbb{D})$ are studied. In [27, 28], the authors have independently extended the result to $H^\infty(B_n)$ space, where B_n is the unit ball of $\mathbb{C}^n$. In [29], Moorhouse showed that if the pseudohyperbolic distance between the image values φ and ψ converges to zero as $z \rightarrow \zeta$ for every point ζ at which φ and ψ have finite angular derivative, then the difference $C_\varphi - C_\psi$ yields a compact operator. Differences of composition operators on the Bloch and the little Bloch space are studied in [30, 31]. Motivated by these results, we give some upper and lower estimates of the essential norm for the difference of composition operators induced by φ and ψ acting on the space $H^\infty(\mathbb{D}^n)$, where φ and ψ are analytic self-maps of $\mathbb{D}^n$. As a consequence, one obtains conditions in terms of the Carathéodory distance on $\mathbb{D}^n$ that characterize those pairs of holomorphic self-maps of the polydisc for which the difference of two composition operators on $H^\infty(\mathbb{D}^n)$ is compact.

2. Notation and background

The pseudohyperbolic distance on the unit disk is defined by

$$\beta(z, w) = \left| \frac{z - w}{1 - \bar{z}w} \right|, \quad z, w \in \mathbb{D}. \quad (2.1)$$

It is easy to see that $0 \leq \beta(z, w) \leq 1$.

Definition 2.1. The Poincaré distance ρ on $\mathbb{D}$ is

$$\rho(z, w) := \tanh^{-1} \beta(z, w) = \frac{1}{2} \ln \frac{1 + \beta(z, w)}{1 - \beta(z, w)} \quad (2.2)$$

for $z, w \in \mathbb{D}$.

Definition 2.2. The Carathéodory pseudodistance on a domain $G \subset \mathbb{C}^n$ is given by

$$c_G(z, w) := \sup \{ \rho(f(z), f(w)) : f \in H(G, \mathbb{D}) \} \quad (2.3)$$

for $z, w \in G$, where $H(G, \mathbb{D})$ denotes the class of holomorphic mappings from G to $\mathbb{D}$.

If we put

$$c_G^*(z, w) := \sup \{ \beta(f(z), f(w)) : f \in H(G, \mathbb{D}) \}, \quad z, w \in G, \quad (2.4)$$

then by the monotonicity of the function $h(x) = \ln((1+x)/(1-x))$ on $[0, 1)$ and the inequality $h(x) \geq 2x$, $x \in [0, 1)$, we have that

$$c_G = \tanh^{-1}(c_G^*) \geq c_G^*. \quad (2.5)$$

Next, we introduce the following pseudodistance on G :

$$d_G(z, w) := \sup \{ |f(z) - f(w)| : f \in H(G, \mathbb{D}) \}. \quad (2.6)$$

For the case $G = \mathbb{D}$, it is known that (see [32])

$$d_{\mathbb{D}}(z, w) = \frac{2 - 2\sqrt{1 - \beta(z, w)^2}}{\beta(z, w)}. \quad (2.7)$$

Hence, the Poincaré metric on $\mathbb{D}$ is

$$\rho(z, w) = \tanh^{-1} \beta(z, w) = \ln \frac{2 + d_{\mathbb{D}}(z, w)}{2 - d_{\mathbb{D}}(z, w)}. \quad (2.8)$$

It is easy to see that for $z, w \in G$,

$$\begin{aligned} d_G(z, w) &= \sup \{ |g(f(z)) - g(f(w))| : g \in H(\mathbb{D}, \mathbb{D}), f \in H(G, \mathbb{D}) \} \\ &= \sup_{f \in H(G, \mathbb{D})} d_{\mathbb{D}}(f(z), f(w)). \end{aligned} \quad (2.9)$$

Since the map $t \rightarrow \ln((2+t)/(2-t))$ is strictly increasing on $[0, 2)$, it follows that

$$\begin{aligned} \ln \frac{2+d_G}{2-d_G} &= \sup_{f \in H(G, \mathbb{D})} \ln \frac{2+d_{\mathbb{D}}(f(z), f(w))}{2-d_{\mathbb{D}}(f(z), f(w))} \\ &= \sup_{f \in H(G, \mathbb{D})} \rho(f(z), f(w)) \\ &= c_G(z, w), \end{aligned} \quad (2.10)$$

or equivalently for any domain G and any $z, w \in G$,

$$\begin{aligned} d_G(z, w) &= \frac{2 - 2\sqrt{1 - (\tanh c_G(z, w))^2}}{\tanh c_G(z, w)} \\ &= \frac{2 - 2\sqrt{1 - (c_G^*(z, w))^2}}{c_G^*(z, w)}. \end{aligned} \quad (2.11)$$

It is well known that $c_{\mathbb{D}^n}^*(z, w) = \max_{1 \leq j \leq n} \beta(z_j, w_j)$ (see [33, Corollary 2.2.4]). So we have

$$d_{\mathbb{D}^n}(z, w) = \frac{2 - 2\sqrt{1 - (\max_{1 \leq j \leq n} \beta(z_j, w_j))^2}}{\max_{1 \leq j \leq n} \beta(z_j, w_j)}. \quad (2.12)$$

Before formulating and proving the main theorem, we give some notations. For any $\delta \in (0, 1)$, define

$$E_\delta^j := \{z \in \mathbb{D}^n : |\varphi_j(z)| \vee |\psi_j(z)| > 1 - \delta\}, \quad (2.13)$$

and we put $E_\delta = \cup_{j=1}^n E_\delta^j$, where $\vee$ means the maximum of two real numbers.

Lemma 2.3 (see [34]). *Let $\{z_n\}$ be a sequence in $\mathbb{D}$ with $|z_n| \rightarrow 1$ as $n \rightarrow \infty$. Then, there is a subsequence $\{z_{n_j}\}$ of $\{z_n\}$, a positive number M , and a sequence of functions $f_m \in H^\infty(\mathbb{D})$ such that*

- (i) $f_m(z_{n_j}) = \delta_m^j$,
- (ii) $\sum_m |f_m(z)| \leq M < \infty$ for any $z \in \mathbb{D}$,

(the symbol δ_m^j is equal to 1 if $m = j$ and 0, otherwise.)

Lemma 2.4. *Let Ω be a domain in $\mathbb{C}^n$, $f \in H(\Omega)$. If a compact set K and a neighborhood G of K satisfy $K \subset G \subset \subset \Omega$ (i.e., G is relative compact in Ω) and $\eta = \text{dist}(K, \partial G) > 0$, then*

$$\sup_{z \in K} \left| \frac{\partial f}{\partial z_j}(z) \right| \leq \frac{\sqrt{n}}{\eta} \sup_{z \in G} |f(z)|, \quad (2.14)$$

for each $j \in \{1, \dots, n\}$.

Proof. Since $\eta = \text{dist}(K, \partial G) > 0$ for any $a \in K$, the polydisc

$$P_a = \left\{ (z_1, \dots, z_n) \in \mathbb{C}^n : |z_j - a_j| < \frac{\eta}{\sqrt{n}}, j = 1, \dots, n \right\} \quad (2.15)$$

is contained in G . Using Cauchy's inequality, we have

$$\left| \frac{\partial f}{\partial z_j}(a) \right| \leq \frac{\sqrt{n}}{\eta} \sup_{z \in \partial_0 P_a} |f(z)| \leq \frac{\sqrt{n}}{\eta} \sup_{z \in G} |f(z)|, \quad (2.16)$$

as desired (where $\partial_0 P_a$ is the distinguished boundary of P_a). $\square$

Lemma 2.5. *For fixed $0 < \delta < 1$, let $F_\delta = \{z \in \mathbb{D}^n : \max_{1 \leq j \leq n} |z_j| > 1 - \delta\}$. Then,*

$$\lim_{r \rightarrow 1} \sup_{\|f\|_\infty=1} \sup_{z \in F_\delta^c} |f(z) - f(rz)| = 0 \quad (2.17)$$

for any f in the unit ball of $H^\infty(\mathbb{D}^n)$ (where F_δ^c denotes the complement of F_δ relative to $\mathbb{D}^n$).

Proof. We have

$$\begin{aligned} & \sup_{z \in F_\delta^c} |f(z) - f(rz)| \\ &= \sup_{z \in F_\delta^c} \left| \sum_{j=1}^n (f(rz_1, rz_2, \dots, rz_{j-1}, z_j, \dots, z_n) - f(z_1, rz_2, \dots, rz_j, z_{j+1}, \dots, z_n)) \right| \\ &\leq \sup_{z \in F_\delta^c} \sum_{j=1}^n \int_r^1 |z_j| \left| \frac{\partial f}{\partial z_j}(rz_1, \dots, rz_{j-1}, tz_j, z_{j+1}, \dots, z_n) \right| dt \\ &\leq (1-r) \sup_{z \in F_\delta^c} \sum_{j=1}^n \left| \frac{\partial f}{\partial z_j}(z) \right|. \end{aligned} \quad (2.18)$$

Consider $F_{\delta/2}^c$, then $F_\delta^c \subset F_{\delta/2}^c$ and $\text{dist}(F_{\delta/2}^c, \partial \mathbb{D}^n) = \delta/2$.

From Lemma 2.4, we have that for each $j \in \{1, \dots, n\}$

$$\sup_{z \in F_\delta^c} \left| \frac{\partial f}{\partial z_j}(z) \right| \leq \frac{2\sqrt{n}}{\delta} \sup_{z \in F_{\delta/2}^c} |f(z)|. \quad (2.19)$$

From this and (2.18), it follows that

$$\sup_{z \in F_\delta^c} |f(z) - f(rz)| \leq \frac{2(1-r)n\sqrt{n}}{\delta} \|f\|_\infty. \quad (2.20)$$

Taking the supremum in (2.20) over the unit ball in $H^\infty(\mathbb{D}^n)$, then letting $r \rightarrow 1$ in (2.20), the lemma follows. $\square$

3. Main theorem

In this section, we will state our main result and give its proof.

Theorem 3.1. *Let $\varphi, \psi : \mathbb{D}^n \rightarrow \mathbb{D}^n$ and $C_\varphi - C_\psi : H^\infty(\mathbb{D}^n) \rightarrow H^\infty(\mathbb{D}^n)$. Then,*

$$\frac{1}{M}\Psi \leq \|C_\varphi - C_\psi\|_e \leq \frac{4 - 4\sqrt{1 - \Psi^2}}{\Psi}, \quad (3.1)$$

where $\Psi := \max_{1 \leq k \leq n} \lim_{\delta \rightarrow 0} \sup_{z \in E_\delta^k} \max_{1 \leq j \leq n} \beta(\varphi_j(z), \psi_j(z))$ and M is a positive constant.

Proof. First, we consider the upper estimate. For fixed $r \in (0, 1)$, it is easy to check that both $C_{r\varphi}$ and $C_{r\psi}$ are compact operators. Therefore,

$$\|C_\varphi - C_\psi\|_e \leq \|C_\varphi - C_\psi - C_{r\varphi} + C_{r\psi}\|. \quad (3.2)$$

Now, for any $0 < \delta < 1$,

$$\begin{aligned} \|C_\varphi - C_\psi - C_{r\varphi} + C_{r\psi}\| &= \sup_{\|f\|_\infty=1} \|(C_\varphi - C_\psi - C_{r\varphi} + C_{r\psi})f\|_\infty \\ &= \sup_{\|f\|_\infty=1} \sup_{z \in \mathbb{D}^n} |f(\varphi(z)) - f(\psi(z)) - f(r\varphi(z)) + f(r\psi(z))| \\ &\leq \sup_{\|f\|_\infty=1} \sup_{z \in E_\delta} |f(\varphi(z)) - f(\psi(z)) - f(r\varphi(z)) + f(r\psi(z))| \\ &\quad + \sup_{\|f\|_\infty=1} \sup_{z \in E_\delta^c} |f(\varphi(z)) - f(\psi(z)) - f(r\varphi(z)) + f(r\psi(z))| \\ &= I_1 + I_2. \end{aligned} \quad (3.3)$$

From Lemma 2.5, we can choose r sufficiently close to 1 such that I_2 is sufficiently small.

Applying the Schwarz-Pick lemma on the function $\phi(z) = rz$, $r \in (0, 1)$, and by the monotony of the function $f(x) = (2 - 2\sqrt{1 - x^2})/x$, we obtain

$$\begin{aligned} I_1 &\leq \sup_{\|f\|_\infty=1} \sup_{z \in E_\delta} (|f(\varphi(z)) - f(\psi(z))| + |-f(r\varphi(z)) + f(r\psi(z))|) \\ &= \sup_{z \in E_\delta} \sup_{\|f\|_\infty=1} (|f(\varphi(z)) - f(\psi(z))| + |-f(r\varphi(z)) + f(r\psi(z))|) \\ &\leq \sup_{z \in E_\delta} (d_{\mathbb{D}^n}(\varphi(z), \psi(z)) + d_{\mathbb{D}^n}(r\varphi(z), r\psi(z))) \end{aligned} \quad (3.4)$$

$$\begin{aligned} &\leq 2 \sup_{z \in E_\delta} \frac{2 - 2(1 - \max_{1 \leq j \leq n} (\beta(\varphi_j(z), \psi_j(z)))^2)^{1/2}}{\max_{1 \leq j \leq n} \beta(\varphi_j(z), \psi_j(z))} \\ &= \frac{4 - 4(1 - \sup_{z \in E_\delta} \max_{1 \leq j \leq n} (\beta(\varphi_j(z), \psi_j(z)))^2)^{1/2}}{\sup_{z \in E_\delta} \max_{1 \leq j \leq n} \beta(\varphi_j(z), \psi_j(z))} \\ &\leq \frac{4 - 4(1 - \max_{1 \leq k \leq n} \sup_{z \in E_\delta^k} \max_{1 \leq j \leq n} (\beta(\varphi_j(z), \psi_j(z)))^2)^{1/2}}{\max_{1 \leq k \leq n} \sup_{z \in E_\delta^k} \max_{1 \leq j \leq n} \beta(\varphi_j(z), \psi_j(z))}. \end{aligned} \quad (3.5)$$

By direct calculation, it is easy to check that

$$\lim_{\delta \rightarrow 0} \max_{1 \leq k \leq n} \sup_{z \in E_\delta^k} \max_{1 \leq j \leq n} \beta(\varphi_j(z), \psi_j(z)) = \max_{1 \leq k \leq n} \lim_{\delta \rightarrow 0} \sup_{z \in E_\delta^k} \max_{1 \leq j \leq n} \beta(\varphi_j(z), \psi_j(z)). \quad (3.6)$$

From which, and letting $\delta \rightarrow 0$ in (3.5), the upper estimate in (3.1) follows.

Now, we turn to the lower estimate.

Let

$$a_k = \lim_{\delta \rightarrow 0} \sup_{z \in E_\delta^k} \max_{1 \leq j \leq n} \beta(\varphi_j(z), \psi_j(z)), \quad k = 1, \dots, n. \quad (3.7)$$

If we set $\delta_m = 1/m$, then $\delta_m \rightarrow 0$ as $m \rightarrow \infty$, and there exists $z_m \in E_{\delta_m}^j$ and some j such that

$$\lim_{m \rightarrow \infty} \beta(\varphi_j(z_m), \psi_j(z_m)) = a_k. \quad (3.8)$$

Since $z_m \in E_{\delta_m}^j$ and $\delta_m \rightarrow 0$, we have $|\varphi_j(z_m)| \rightarrow 1$ or $|\phi_j(z_m)| \rightarrow 1$. Without loss of generality, we can assume $|\varphi_j(z_m)| \rightarrow 1$. Let $w_m = \varphi_j(z_m)$, by Lemma 2.3, we have that there is a subsequence of w_m (we may denote it again by w_m), a positive number M , and a sequence of functions $f_m \in H^\infty(\mathbb{D})$ such that

$$(i) \quad f_m(w_k) = \delta_m^k,$$

$$(ii) \quad \sum_m |f_m(z)| \leq M < \infty \text{ for any } z \in \mathbb{D}.$$

Now, for any $z \in \mathbb{D}^n$, we define $\tilde{f}_m(z) := f_m(z^j)$, where z^j is the j th component of z , then $\sum_m |\tilde{f}_m(z)| \leq M < \infty$.

Next we claim that $\tilde{f}_m$ converge weakly to 0. Let $\lambda \in H^\infty(\mathbb{D}^n)^*$. For any natural N , there exist some unimodular sequences α_m such that

$$\begin{aligned} \sum_{m=0}^N |\lambda(\tilde{f}_m)| &= \sum_{m=0}^N \alpha_m \lambda(\tilde{f}_m) \\ &= \lambda \left(\sum_{m=0}^N \alpha_m \tilde{f}_m \right) \\ &\leq \|\lambda\| \left\| \sum_{m=0}^N \alpha_m \tilde{f}_m \right\|_\infty \\ &\leq \|\lambda\| M. \end{aligned} \quad (3.9)$$

Thus, $\lambda(\tilde{f}_m) \rightarrow 0$ as $m \rightarrow \infty$, that is, $\tilde{f}_m$ converge weakly to 0.

Set

$$g_m(z) = \frac{\tilde{f}_m(z)}{M} \frac{z^j - \varphi_j(z_m)}{1 - \overline{\varphi_j(z_m)} z^j}, \quad m \in \mathbb{N}. \quad (3.10)$$

Then, $\|g_m\|_\infty \leq 1$ and similarly to $\tilde{f}_m$, it is easy to see that g_m converge weakly to 0. Thus, for any compact operator K , we have $\|Kg_m\|_\infty \rightarrow 0$ as $m \rightarrow \infty$.

Now, we have

$$\begin{aligned}
J &= \|C_\varphi - C_\psi - K\| \\
&\geq \limsup_{m \rightarrow \infty} \|(C_\varphi - C_\psi - K)g_m\|_\infty \\
&\geq \limsup_{m \rightarrow \infty} (\|(C_\varphi - C_\psi)g_m\|_\infty - \|Kg_m\|_\infty) \\
&= \limsup_{m \rightarrow \infty} \sup_{z \in \mathbb{D}^n} |g_m(\varphi(z)) - g_m(\psi(z))| \\
&\geq \frac{1}{M} \limsup_{m \rightarrow \infty} \sup_{z \in \mathbb{D}^n} \left| \frac{\varphi_j(z) - \psi_j(z_m)}{1 - \overline{\psi_j(z_m)}\varphi_j(z)} \tilde{f}_m(\varphi(z)) - \frac{\psi_j(z) - \psi_j(z_m)}{1 - \overline{\psi_j(z_m)}\psi_j(z)} \tilde{f}_m(\psi(z)) \right| \\
&\geq \frac{1}{M} \limsup_{m \rightarrow \infty} \frac{|\varphi_j(z_m) - \psi_j(z_m)|}{|1 - \overline{\psi_j(z_m)}\varphi_j(z_m)|} \\
&= \frac{1}{M} \lim_{m \rightarrow \infty} \beta(\varphi_j(z_m), \psi_j(z_m)) \\
&= \frac{1}{M} a_k.
\end{aligned} \tag{3.11}$$

Then, we have

$$\|C_\varphi - C_\psi\|_e \geq \frac{1}{M} \max_{1 \leq k \leq n} \limsup_{\delta \rightarrow 0} \max_{z \in E_\delta^k} \max_{1 \leq j \leq n} \beta(\varphi_j(z), \psi_j(z)), \tag{3.12}$$

finishing the proof of the theorem. $\square$

Corollary 3.2. *The operator $C_\varphi - C_\psi$ is compact if and only if*

$$\max_{1 \leq k \leq n} \limsup_{\delta \rightarrow 0} \max_{z \in E_\delta^k} \max_{1 \leq j \leq n} \beta(\varphi_j(z), \psi_j(z)) = 0. \tag{3.13}$$

Proof. By using the inequality $(1 - \sqrt{1 - x^2})/x \leq x$ ($0 < x \leq 1$) and the fact that T is compact if and only if $\|T\|_e = 0$, the corollary follows by Theorem 3.1. $\square$

Example 3.3. Let $n = 2$, $\varphi(z) = (z_1, (1/2)z_2)$, and $\psi(z) = (z_1, (1/3)z_2)$. Then, $\beta(\varphi_1(z), \psi_1(z)) = 0$ and $\beta(\varphi_2(z), \psi_2(z)) = (1/6)(|z_2|/(1 - (1/6)|z_2|^2))$. A direct calculation shows that

$$\max_{1 \leq k \leq 2} \limsup_{\delta \rightarrow 0} \max_{z \in E_\delta^k} \max_{1 \leq j \leq 2} \beta(\varphi_j(z), \psi_j(z)) = \frac{1}{5} > 0; \tag{3.14}$$

so by Corollary 3.2, $C_\varphi - C_\psi$ is not compact.

Example 3.4. Let $n = 2$, $p > 1$, $0 < c \leq 1$, $\varphi(z) = ((z_1 + 1)/2, (1/2)z_2)$, and

$$\psi(z) = \left(\frac{z_1 + 1}{2} + c \left(\frac{1 - z_1}{2} \right)^p, \frac{1}{2}z_2 \right), \quad (3.15)$$

where we choose the usual branch of the logarithm of w_1 , $\operatorname{Re} w_1 > 0$, in order to define $((1 - z_1)/2)^p$. By [35], ψ is a self-map of $\mathbb{D}^2$, whenever c is small. Moreover, $\max_{1 \leq j \leq 2} \beta(\varphi_j(z), \psi_j(z)) = \beta(\varphi_1(z), \psi_1(z))$. By Corollary 3.2 and the proof of Example 1 of [25], we have, for these c ,

- (1) if $1 < p \leq 2$, then $C_\varphi - C_\psi$ is noncompact;
- (2) if $2 < p < \infty$, then $C_\varphi - C_\psi$ is compact.

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References

- [1] D. D. Clahane and S. Stević, "Norm equivalence and composition operators between Bloch/Lipschitz spaces of the ball," *Journal of Inequalities and Applications*, vol. 2006, Article ID 61018, 11 pages, 2006.
- [2] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, Fla, USA, 1995.
- [3] P. Gorkin and B. D. MacCluer, "Essential norms of composition operators," *Integral Equations and Operator Theory*, vol. 48, no. 1, pp. 27–40, 2004.
- [4] S. Li and S. Stević, "Composition followed by differentiation between Bloch type spaces," *Journal of Computational Analysis and Applications*, vol. 9, no. 2, pp. 195–205, 2007.
- [5] S. Li and S. Stević, "Weighted composition operators from α -Bloch space to H^∞ on the polydisc," *Numerical Functional Analysis and Optimization*, vol. 28, no. 7-8, pp. 911–925, 2007.
- [6] S. Li and S. Stević, "Weighted composition operators from H^∞ to the Bloch space on the polydisc," *Abstract and Applied Analysis*, vol. 2007, Article ID 48478, 13 pages, 2007.
- [7] S. Li and S. Stević, "Generalized composition operators on Zygmund spaces and Bloch type spaces," *Journal of Mathematical Analysis and Applications*, vol. 338, no. 2, pp. 1282–1295, 2008.
- [8] S. Li and H. Wulan, "Composition operators on Q_K spaces," *Journal of Mathematical Analysis and Applications*, vol. 327, no. 2, pp. 948–958, 2007.
- [9] B. D. MacCluer and R. Zhao, "Essential norms of weighted composition operators between Bloch-type spaces," *The Rocky Mountain Journal of Mathematics*, vol. 33, no. 4, pp. 1437–1458, 2003.
- [10] A. Montes-Rodríguez, "Weighted composition operators on weighted Banach spaces of analytic functions," *Journal of the London Mathematical Society*, vol. 61, no. 3, pp. 872–884, 2000.
- [11] S. Ohno, K. Stroethoff, and R. Zhao, "Weighted composition operators between Bloch-type spaces," *The Rocky Mountain Journal of Mathematics*, vol. 33, no. 1, pp. 191–215, 2003.
- [12] J. H. Shapiro, *Composition Operators and Classical Function Theory*, Universitext: Tracts in Mathematics, Springer, New York, NY, USA, 1993.
- [13] J. H. Shapiro, "The essential norm of a composition operator," *The Annals of Mathematics*, vol. 125, no. 2, pp. 375–404, 1987.
- [14] J. H. Shapiro, "Compact composition operators on spaces of boundary-regular holomorphic functions," *Proceedings of the American Mathematical Society*, vol. 100, no. 1, pp. 49–57, 1987.
- [15] J. H. Shapiro and C. Sundberg, "Isolation amongst the composition operators," *Pacific Journal of Mathematics*, vol. 145, no. 1, pp. 117–152, 1990.
- [16] J. Shi and L. Luo, "Composition operators on the Bloch space of several complex variables," *Acta Mathematica Sinica*, vol. 16, no. 1, pp. 85–98, 2000.

- [17] S. Stević, "Composition operators between H^∞ and α -Bloch spaces on the polydisc," *Zeitschrift für Analysis und ihre Anwendungen*, vol. 25, no. 4, pp. 457–466, 2006.
- [18] S. Stević, "Weighted composition operators between mixed norm spaces and H^∞_α spaces in the unit ball," *Journal of Inequalities and Applications*, vol. 2007, Article ID 28629, 9 pages, 2007.
- [19] S.-I. Ueki and L. Luo, "Compact weighted composition operators and multiplication operators between Hardy spaces," *Abstract and Applied Analysis*, vol. 2008, Article ID 196498, 12 pages, 2008.
- [20] Z.-H. Zhou and R.-Y. Chen, "Weighted composition operators from $F(p, q, s)$ to Bloch type spaces on the unit ball," *International Journal of Mathematics*, vol. 19, no. 8, pp. 899–926, 2008.
- [21] Z. Zhou and Y. Liu, "The essential norms of composition operators between generalized Bloch spaces in the polydisc and their applications," *Journal of Inequalities and Applications*, vol. 2006, Article ID 90742, 22 pages, 2006.
- [22] Z. Zhou and J. Shi, "Composition operators on the Bloch space in polydiscs," *Complex Variables*, vol. 46, no. 1, pp. 73–88, 2001.
- [23] Z. Zhou and J. Shi, "Compactness of composition operators on the Bloch space in classical bounded symmetric domains," *The Michigan Mathematical Journal*, vol. 50, no. 2, pp. 381–405, 2002.
- [24] E. Berkson, "Composition operators isolated in the uniform operator topology," *Proceedings of the American Mathematical Society*, vol. 81, no. 2, pp. 230–232, 1981.
- [25] B. MacCluer, S. Ohno, and R. Zhao, "Topological structure of the space of composition operators on H^∞ ," *Integral Equations and Operator Theory*, vol. 40, no. 4, pp. 481–494, 2001.
- [26] T. Hosokawa, K. Izuchi, and D. Zheng, "Isolated points and essential components of composition operators on H^∞ ," *Proceedings of the American Mathematical Society*, vol. 130, no. 6, pp. 1765–1773, 2002.
- [27] P. Gorkin, R. Mortini, and D. Suárez, "Homotopic composition operators on $H^\infty(B_N)$," in *Function Spaces (Edwardsville, IL, 2002)*, vol. 328 of *Contemporary Mathematics*, pp. 177–188, American Mathematical Society, Providence, RI, USA, 2003.
- [28] C. Toews, "Topological components of the set of composition operators on $H^\infty(B_N)$," *Integral Equations and Operator Theory*, vol. 48, no. 2, pp. 265–280, 2004.
- [29] J. Moorhouse, "Compact differences of composition operators," *Journal of Functional Analysis*, vol. 219, no. 1, pp. 70–92, 2005.
- [30] T. Hosokawa and S. Ohno, "Topological structures of the sets of composition operators on the Bloch spaces," *Journal of Mathematical Analysis and Applications*, vol. 314, no. 2, pp. 736–748, 2006.
- [31] T. Hosokawa and S. Ohno, "Differences of composition operators on the Bloch spaces," *Journal of Operator Theory*, vol. 57, no. 2, pp. 229–242, 2007.
- [32] H. S. Bear, *Lectures on Gleason Parts*, vol. 121 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1970.
- [33] M. Jarnicki and P. Pflug, *Invariant Distances and Metrics in Complex Analysis*, vol. 9 of *de Gruyter Expositions in Mathematics*, Walter de Gruyter, Berlin, Germany, 1993.
- [34] J. B. Garnett, *Bounded Analytic Functions*, vol. 96 of *Pure and Applied Mathematics*, Academic Press, New York, NY, USA, 1981.
- [35] R. Mortini and R. Rupp, "Sums of holomorphic selfmaps of the unit disk," *Annales Universitatis Mariae Curie-Skłodowska. Sectio A*, vol. 61, pp. 107–115, 2007.