Research Article

Global Behavior of the Max-Type Difference Equation $x_{n+1} = \max\{1/x_n, A_n/x_{n-1}\}$

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We study global behavior of the following max-type difference equation $x_{n+1} = \max\{1/x_n, A_n/x_{n-1}\}$, $n = 0, 1, \ldots$, where $\{A_n\}_{n=0}^{\infty}$ is a sequence of positive real numbers with $0 \leq \inf A_n \leq \sup A_n < 1$. The special case when $\{A_n\}_{n=0}^{\infty}$ is a periodic sequence with period $k$ and $A_n \in (0, 1)$ for every $n \geq 0$ has been completely investigated by Y. Chen. Here we extend his results to the general case.

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1. Introduction

In the recent years, there has been a lot of interest in studying the global behavior of, the so-called, max-type difference equations; see, for example, [1–17] (see also references therein). In [1, 3–5, 7, 8], the second order max-type difference equation

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{A_n}{x_{n-1}}\right\}, \quad n = 0, 1, \ldots$$

(1.1)

has been studied for positive coefficients $A_n$, which are periodic with period $k$. The case $k = 1$ was studied in [1], the case $k = 2$ was studied in [3], the case $k = 3$ was studied in [4, 8], and the more difficult case $k = 4$ was studied in [7]. Chen [5] found that every positive solution of (1.1) is eventually periodic with period 2 when $\{A_n\}_{n=0}^{\infty}$ is a periodic sequence of positive real numbers with period $k \geq 2$ and $A_n \in (0, 1)$ for all $n \geq 0$. These results were also included in the recent monograph [9] along with other related references. In this paper, we study global behavior of (1.1) when $\{A_n\}_{n=0}^{\infty}$ is a sequence of positive real numbers with $0 \leq \inf A_n \leq \sup A_n < 1$. 
2. Main Results

The main results of this paper are established through the following lemmas.

Lemma 2.1. Let \( \{x_n\}_{n=1}^{\infty} \) be a positive solution of (1.1), then

1. \( x_{n+1}x_n \geq 1 \) for all \( n \geq 0 \);
2. if \( x_{k+1}x_k > 1 \) for some \( k \geq 1 \), then \( x_{k+2}x_{k+1} = 1 \).

Proof. (1) is obvious since \( x_{n+1} \geq 1/x_n \) for all \( n \geq 0 \).

(2) If \( x_{k+1}x_k > 1 \) for some \( k \geq 1 \), then \( x_{k+1}x_k - 1 = A_k \). Suppose for the sake of contradiction that \( x_{k+2}x_k = A_k \) and

\[
A_{k+1} = x_{k+1}x_{k-1}x_{k+2}x_k \geq 1.
\]

This is a contradiction since \( A_{k+1} < 1 \) and \( A_k < 1 \). The proof is complete.

Lemma 2.2. Let \( \{x_n\}_{n=1}^{\infty} \) be a positive solution of (1.1) and \( P_n = \max\{x_n, x_{n-1}\} \) for all \( n \geq 1 \). Then

1. \( x_{n+1} \leq P_n \) and \( P_n \) is nonincreasing;
2. \( x_n \) is bounded, and moreover \( 1/P_1 \leq x_n \leq P_1 \) for any \( n \geq 1 \).

Proof. By Lemma 2.1(1) and the assumption \( A_n < 1 \), we obtain that for any \( n \geq 1 \),

\[
x_{n+1} = \max\left\{ \frac{x_{n-1}}{x_n x_{n-1}}, \frac{A_n x_n}{x_n x_{n-1}} \right\} \leq \max\{x_{n-1}, x_n\} = P_n.
\]

Hence

\[
P_{n+1} = \max\{x_{n+1}, x_n\} \leq P_n,
\]

which implies that for all \( n \geq 1 \),

\[
x_n \leq P_1.
\]

Furthermore, it follows that for all \( n \geq 1 \),

\[
x_{n+1} = \max\left\{ \frac{1}{x_n}, \frac{A_n}{x_n x_{n-1}} \right\} \geq \frac{1}{x_n} \geq \frac{1}{P_1},
\]

The proof is complete.

Remark 2.3. Note that from the proof of Lemma 2.2 we have that \( P_1 \geq 1 \).

Remark 2.4. Various sequences which satisfy inequality in Lemma 2.2(1), that is, \( x_{n+1} \leq P_n \) have been studied, for example, in [18–24].
Lemma 2.5. Let \( \{x_n\}_{n=1}^{\infty} \) be a positive solution of (1.1) and \( \lim_{n \to \infty} P_n = S \). Then \( S = \limsup_{n \to \infty} x_n \).

Proof. Since \( P_n \) is a subsequence of \( x_n \), it follows that
\[
S \leq \limsup_{n \to \infty} x_n. \tag{2.6}
\]

On the other hand, by \( x_{n+1} \leq P_n \) for all \( n \geq 1 \), we obtain
\[
\limsup_{n \to \infty} x_n \leq \limsup_{n \to \infty} P_n = S. \tag{2.7}
\]

The proof is complete.

Remark 2.6. Let \( \{x_n\}_{n=1}^{\infty} \) be a positive solution of (1.1). By Lemma 2.2, we see that if \( S = \limsup_{n \to \infty} x_n \) and \( x_N < S \) for some \( N > 0 \), then \( x_{N-1}, x_{N+1} \in [S, +\infty) \). For example, if it were \( x_{N-1} < S \), then it would be \( P_N < S \), which would imply \( \limsup_{n \to \infty} x_n < S \).

Lemma 2.7. Suppose that \( \{x_n\}_{n=1}^{\infty} \) is a positive solution of (1.1) and \( S = \limsup_{n \to \infty} x_n \). Write
\[
\omega(x_n) = \left\{ x : \text{there exist } -1 \leq k_1 < k_2 < \cdots < k_n < \cdots \text{ such that } \lim_{n \to \infty} x_{k_n} = x \right\}. \tag{2.8}
\]

Then \( \omega(x_n) = \{S, 1/S\} \).

Proof. If \( \omega(x_n) \) contains only one point, we may assume by taking a subsequence that \( A_{n_k} \to \mu(<1) \). By taking the limit in the following relationship:
\[
x_{n_k+1} = \max\left\{ \frac{1}{x_{n_k}}, \frac{A_{n_k}}{x_{n_k-1}} \right\}, \tag{2.9}
\]
as \( k \to \infty \), we obtain
\[
S = \max\left\{ \frac{1}{S}, \frac{\mu}{S} \right\} = \frac{1}{S}, \tag{2.10}
\]
which implies that \( S = 1 \).

If \( \omega(x_n) \) contains at least two points, let \( L \in \omega(x_n) - \{S\} \), then there exists a subsequence \( x_{n_k} \) of \( x_n \) such that
\[
x_{n_k} \to L < S. \tag{2.11}
\]

By Remark 2.6, we see that there exists \( N > 0 \) such that for every \( n_k > N \),
\[
x_{n_k} < S, \quad x_{n_k+1}, x_{n_k-1} \in [S, +\infty), \tag{2.12}
\]
from which it follows that

\[ x_{n_1 + 1} \to S, \quad x_{n-1} \to S. \]  

(2.13)

By taking a subsequence we may assume that \( A_{n_k} \to \mu (\mu < 1). \) By taking the limit in the following relationship:

\[ x_{n_1 + 1} = \max \left\{ \frac{1}{x_{n_k}}, \frac{A_{n_k}}{x_{n_k - 1}} \right\}, \]  

(2.14)

as \( k \to \infty, \) we obtain

\[ S = \max \left\{ \frac{1}{L'}, \frac{\mu}{S} \right\} = \frac{1}{L'}, \]  

(2.15)

which implies

\[ L = \frac{1}{S}. \]  

(2.16)

The proof is complete. \( \square \)

**Theorem 2.8.** Let \( \{x_n\}_{n=1}^{\infty} \) be a positive solution of (1.1) and \( S = \lim \sup_{n \to \infty} x_n. \) Then one of the following two statements is true.

1. If there exist infinitely many \( n \) such that \( x_n \geq S \) and \( x_{n+1} \geq S, \) then \( \{x_n\}_{n=1}^{\infty} \) is eventually equal to 1.

2. If there exists \( N \) such that \( x_{N+2k} < S \) and \( x_{N+2k-1} \geq S \) for all \( k \geq 0, \) then \( x_{N+2k} \to 1/S \) and \( x_{N+2k-1} \to S. \)

**Proof.** (1) We assume that there exists an infinite sequence \( n_1 < n_2 < n_3 < \cdots < n_k < \cdots \) such that

\[ x_{n_k} \geq S, \quad x_{n_k+1} \geq S. \]  

(2.17)

By taking a subsequence we may assume from Lemma 2.7 that

\[ A_{n_k} \to \mu < 1, \quad x_{n_k - 1} \to l \in \left\{ S, \frac{1}{S} \right\}. \]  

(2.18)

By taking the limit in the following relationship:

\[ x_{n_1 + 1} = \max \left\{ 1, \frac{A_{n_k} x_{n_k}}{x_{n_k - 1}} \right\}, \]  

(2.19)
as \( k \to \infty \), we get

\[
S^2 = \max \left\{ 1, \frac{S\mu}{I} \right\}. \tag{2.20}\]

Since \( S\mu/I \in \{ \mu, \mu S^2 \} \) and \( \mu < 1 \), it follows that \( S^2 = 1 \) and \( \omega(x_n) = \{ 1 \} \).

In the following, we show that \( \{ x_n \}_{n=1}^\infty \) is eventually equal to 1. It only needs to prove that there exists \( N \geq 0 \) such that for all \( n \geq N \),

\[
\frac{1}{x_n} > \frac{A_n}{x_{n-1}}. \tag{2.21}\]

Indeed, if there exist infinitely many \( n_k \) such that

\[
x_{n_k+1} = \frac{A_{n_k}}{x_{n_k-1}}, \tag{2.22}\]

by taking a subsequence we may assume that \( A_{n_k} \to \mu < 1 \), then it follows that

\[
1 = \frac{\mu}{1}, \quad \mu = 1, \tag{2.23}\]

which is a contradiction. Therefore there exists \( N \) such that for all \( n \geq N \),

\[
x_{n+1} = \frac{1}{x_n}. \tag{2.24}\]

Thus

\[
x_n = x_N, \quad \text{for } n = N + 2k, \tag{2.25}\]

\[
x_n = x_{N+1}, \quad \text{for } n = N + 2k + 1.\]

Since \( x_n \to 1 \), we have \( x_{N+1} = x_N = 1 \).

(2) If \( S = 1 \), then the result follows from Lemma 2.7. In the following, we assume \( S \neq 1 \). Suppose for the sake of contradiction that there exists a subsequence \( x_{N+2k} \) of \( x_{N+2k} \) such that

\[
x_{N+2k} \to S. \tag{2.26}\]

By taking a subsequence we may assume that

\[
A_{N+2k} \to \mu. \tag{2.27}\]
By taking the limit in the following relationship:

\[ x_{N+2k_i+1} = \max \left\{ \frac{1}{x_{N+2k_i}} \cdot \frac{A_{N+2k_i}}{x_{N+2k_i-1}} \right\}, \quad (2.28) \]

as \( k_i \to \infty \), we get

\[ S = \max \left\{ \frac{1}{S}, \frac{\mu}{S} \right\}, \quad (2.29) \]

which implies

\[ S = 1. \quad (2.30) \]

This is a contradiction. The proof is complete. \( \square \)

**Corollary 2.9.** Let \( \{A_n\}_{n=0}^{\infty} \) be a periodic sequence of positive real numbers, then every positive solution of (1.1) is eventually periodic with period 2.

**Proof.** Let \( \{x_n\}_{n=-1}^{\infty} \) be a positive solution of (1.1) and \( S = \lim \sup_{n \to \infty} x_n \). By Remark 2.6 and Theorem 2.8, we may assume without loss of generality that \( x_{2k} < S, x_{2k-1} \geq S \geq 1 \) for all \( k \geq 0 \). Suppose for the sake of contradiction that there exists a sequence \( m_1 < m_2 < \cdots < m_k < \cdots \) such that

1. \( x_{m_{k+1}} x_{m_k-1} = A_{m_k}, \) and \( x_{m_{k+1}} x_{m_k} > 1; \)
2. \( x_{n+1} x_n = 1, \) for \( n \neq m_k. \)

Then \( m_k \) is odd for every \( k \geq 1 \). Let \( m_k = 2n_k + 1 \), then it follows from Lemma 2.1 that

\[ x_{2n_k+2} x_{2n_k} = A_{2n_k+1} < 1 = x_{2n_k+1} x_{2n_k} < x_{2n_k+1} x_{2n_k+2}. \quad (2.31) \]

From this and by (2) it follows that

\[ \frac{A_{2n_k+1}}{x_{2n_k+2}} = x_{2n_k} < x_{2n_k+2} = x_{2n_k+4} = \cdots = x_{2n_{k+1}} < x_{2n_{k+1}+2} = \frac{A_{2n_{k+1}+1}}{x_{2n_{k+1}}}. \quad (2.32) \]

Therefore for every \( k \geq 1, \)

\[ A_{2n_k+1} < x_{2n_k+2} = x_{2n_{k+1}} < A_{2n_{k+1}+1}, \quad (2.33) \]

which is a contradiction since \( \{A_n\}_{n=0}^{\infty} \) is a periodic sequence. The proof is complete. \( \square \)

**Remark 2.10.** Corollary 2.9 is the main result of [5].
3. Example

In this section, we give an example for \( \{ A_n \}_{n=0}^{\infty} \) to be no periodic sequence.

Example 3.1. Consider

\[
x_{n+1} = \max \left\{ \frac{1}{x_n} A_n \right\}, \quad n = 0, 1, \ldots,
\]

(3.1)

where \( A_{2n} = A_{2n+1} = (2 - 1/2^n)(2 - 1/2^{n+1})/16 \) for any \( n \geq 0 \). Then solution \( \{ x_n \}_{n=1}^{\infty} \) of (3.1) with the initial values \( x_{-1} = 1/4 \) and \( x_0 = 4 \) satisfies the following.

1. \( x_{2p-1} x_{2p} = 1 \), for any \( p \geq 0 \).
2. \( x_{2p-1} < x_{2p+1} = \frac{A_{2p}}{x_{2p-1}} < \frac{1}{2} < x_{2p+2} < x_{2p} \), for any \( p \geq 0 \).

Proof. By simple computation, we have

\[
A_{2p} = \frac{(2 - 1/2^p)(2 - 1/2^{p+1})}{16} = \left\{ \begin{array}{ll}
\frac{x_{-1}^2}{}, & \text{if } p = 0, \\
\left( \frac{A_0}{x_{-1}} \right)^2, & \text{if } p = 1, \\
\left( \frac{A_{2p-2} A_{2p-6} \cdots A_2}{A_{2p-4} A_{2p-8} \cdots A_0} x_{-1} \right)^2, & \text{if } p \geq 2 \text{ is even}, \\
\left( \frac{A_{2p-2} A_{2p-6} \cdots A_4 A_0}{A_{2p-4} A_{2p-8} \cdots A_2 x_{-1}} \right)^2, & \text{if } p \geq 2 \text{ is odd}.
\end{array} \right.
\]

(3.2)

It follows from (3.1) and (3.2) that

\[
x_1 x_{-1} = \max \left\{ \frac{x_{-1}}{x_0}, A_0 \right\} = \max \left\{ x_{-1}^2, A_0 \right\} = A_0,
\]

\[
x_2 x_1 = \max \left\{ 1, \frac{x_1 A_1}{x_0} \right\} = \max \left\{ 1, \frac{A_0 A_1}{x_1 x_0} \right\} = 1,
\]

\[
x_3 x_1 = \max \left\{ \frac{x_1}{x_2}, A_2 \right\} = \max \left\{ \frac{x_1^2}{x_2 x_1}, A_2 \right\} = \max \left\{ \left( \frac{A_0}{x_{-1}} \right)^2, A_2 \right\} = A_2,
\]

\[
x_4 x_3 = \max \left\{ 1, \frac{x_3 A_3}{x_2} \right\} = \max \left\{ 1, \frac{A_2 A_3}{x_2 x_1} \right\} = 1,
\]

\[
x_5 x_3 = \max \left\{ \frac{x_3}{x_4}, A_4 \right\} = \max \left\{ \frac{x_3^2}{x_4 x_3}, A_4 \right\} = \max \left\{ \left( \frac{x_3 x_1}{x_1 x_{-1}} \right)^2, A_4 \right\}
\]

\[
= \max\left\{ \left( \frac{A_2}{A_0} x_{-1} \right)^2, A_4 \right\} = A_4,
\]
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By induction, we have from (3.1) and (3.2) that for any \( p \geq 1, \)

\[
x_{4p+1} = \max \left\{ \frac{x_{4p+1}}{x_{4p}}, A_{4p} \right\} = \max \left\{ \frac{x_{4p+1}^2}{x_{4p}x_{4p+1}}, A_{4p} \right\} = \max \left\{ x_{4p+1}^2, A_{4p} \right\}
\]

\[
x_{4p+2} = \max \left\{ \frac{x_{4p+2}}{x_{4p+1}}, A_{4p+1} \right\} = \max \left\{ \frac{x_{4p+2}^2}{x_{4p+1}x_{4p+2}}, A_{4p+1} \right\} = \max \left\{ x_{4p+2}^2, A_{4p+1} \right\}
\]

\[
x_{4p+3} = \max \left\{ \frac{x_{4p+3}}{x_{4p+2}}, A_{4p+2} \right\} = \max \left\{ \frac{x_{4p+3}^2}{x_{4p+2}x_{4p+3}}, A_{4p+2} \right\} = \max \left\{ x_{4p+3}^2, A_{4p+2} \right\}
\]

\[
x_{4p+4} = \max \left\{ \frac{x_{4p+4}}{x_{4p+3}}, A_{4p+3} \right\} = \max \left\{ \frac{x_{4p+4}^2}{x_{4p+3}x_{4p+4}}, A_{4p+3} \right\} = \max \left\{ x_{4p+4}^2, A_{4p+3} \right\}
\]

from which the result follows. The proof is complete.
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References


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