Research Article

A Note on “Common Fixed Point of Multistep Noor Iteration with Errors for a Finite Family of Generalized Asymptotically Quasi-Nonexpansive Mappings”

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1. Introduction

Let C be a nonempty subset of a Banach space X . A mapping $T : C \rightarrow C$ is said to be

- (i) *asymptotically nonexpansive* [1] if there exists a sequence $\{k_n\}$ in $[0, \infty)$ such that $k_n \rightarrow 0$ and

$$\|T^n x - T^n y\| \leq (1 + k_n)\|x - y\| \quad (1.1)$$

for all $x, y \in C$ and $n \geq 1$;

- (ii) *asymptotically quasi-nonexpansive* [2] if $F(T) = \{p \in C : Tp = p\} \neq \emptyset$ and there exists a sequence $\{k_n\}$ in $[0, \infty)$ such that $k_n \rightarrow 0$ and

$$\|T^n x - p\| \leq (1 + k_n)\|x - p\| \quad (1.2)$$

for all $x \in C, p \in F(T)$ and $n \geq 1$;

- (iii) *generalized asymptotically nonexpansive* if there exist sequences $\{k_n\}, \{l_n\}$ in $[0, \infty)$ such that $k_n, l_n \rightarrow 0$ and

$$\|T^n x - T^n y\| \leq (1 + k_n)\|x - y\| + l_n \quad (1.3)$$

for all $x, y \in C$ and $n \geq 1$;

- (iv) *generalized asymptotically quasi-nonexpansive* [3] if $F(T) \neq \emptyset$ and there exist sequences $\{k_n\}, \{l_n\}$ in $[0, \infty)$ such that $k_n, l_n \rightarrow 0$ and

$$\|T^n x - p\| \leq (1 + k_n)\|x - p\| + l_n \quad (1.4)$$

for all $x \in C, p \in F(T)$ and $n \geq 1$.

Many researchers have paid their attention on the approximation of a fixed point of a single mapping or a common fixed point of a family of mappings. One effective way is to use a sequence generated by an appropriate iteration. In this paper, we propose a general and short principle for proving some convergence results of certain types of iterative sequences. We also discuss and correct a small gap in the recent paper by Innang and Suantai [4]. In the last section, we give a remark on the generalized asymptotically quasi-nonexpansive mapping in the sense of Lan [5].

Let $\{T_i\}_{i=1}^N$ be a finite family of self-mappings of a closed convex subset C of X . The sequence $\{x_n\}$ is generated from $x_1 \in C$, and

$$\begin{aligned} y_{1n} &= \alpha_{1n} T_1^n x_n + \beta_{1n} x_n + \gamma_{1n} u_{1n}, \\ y_{2n} &= \alpha_{2n} T_2^n y_{1n} + \beta_{2n} x_n + \gamma_{2n} u_{2n}, \\ &\vdots \\ y_{(N-1)n} &= \alpha_{(N-1)n} T_{N-1}^n y_{(N-2)n} + \beta_{(N-1)n} x_n + \gamma_{(N-1)n} u_{(N-1)n}, \\ x_{n+1} &= \alpha_{Nn} T_N^n y_{(N-1)n} + \beta_{Nn} x_n + \gamma_{Nn} u_{Nn}, \end{aligned} \quad (1.5)$$

where $\{u_{1n}\}, \{u_{2n}\}, \dots, \{u_{Nn}\}$ are bounded sequences in C , and $\{\alpha_{in}\}, \{\beta_{in}\}$, and $\{\gamma_{in}\}$ are sequences in $[0, 1]$ such that $\alpha_{in} + \beta_{in} + \gamma_{in} = 1$ for all $i = 1, 2, \dots, N$ and $n \geq 1$.

2. Main Results

2.1. Sequences of Monotone Types (1) and (2)

Definition 2.1. Let $\{x_n\}$ be a sequence in a metric space (X, d) and F a subset of X . We say that $\{x_n\}$ is of

- (i) *monotone type (1) with respect to F* [6] if there exist sequences $\{r_n\}$ and $\{s_n\}$ of nonnegative real numbers such that $\sum_{n=1}^{\infty} r_n < \infty$, $\sum_{n=1}^{\infty} s_n < \infty$ and

$$d(x_{n+1}, p) \leq (1 + r_n)d(x_n, p) + s_n \quad (2.1)$$

for all $n \geq 1$ and $p \in F$;

(ii) *monotone type (2) with respect to F* if for each $p \in F$ there exist sequences $\{r_n\}$ and $\{s_n\}$ of nonnegative real numbers such that $\sum_{n=1}^{\infty} r_n < \infty$, $\sum_{n=1}^{\infty} s_n < \infty$ and

$$d(x_{n+1}, p) \leq (1 + r_n)d(x_n, p) + s_n \quad (2.2)$$

for all $n \geq 1$.

Proposition 2.2. *If $\{x_n\}$ is of monotone type (1) with respect to F , then it is of monotone type (2) with respect to F .*

Lemma 2.3 ([7, Lemma 1]). *Let $\{a_n\}$, $\{b_n\}$, and $\{\alpha_n\}$ be sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1 + \alpha_n)a_n + b_n, \quad n \geq 1. \quad (2.3)$$

If $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n \rightarrow \infty} a_n$ exists.

Theorem 2.4. *Let (X, d) be a complete metric space, $F \subset X$, and $\{x_n\}$ a sequence in X . Then one has the following assertions.*

- (a) *If $\{x_n\}$ is of monotone type (2) with respect to F , then $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for all $p \in F$.*
- (b) *If $\{x_n\}$ is of monotone type (1) with respect to F , then $\lim_{n \rightarrow \infty} d(x_n, F)$ exists.*
- (c) *If $\{x_n\}$ is of monotone type (1) with respect to F and $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$, then $x_n \rightarrow p$ for some $p \in X$ satisfying $d(p, F) = 0$. In particular, if F is closed, then $p \in F$.*

Proof. (a) It is easy to see that the result follows from (2.2) and Lemma 2.3.

- (b) Note that $\{r_n\}$ and $\{s_n\}$ are independent of $p \in F$. Taking infimum over all $p \in F$ in (2.1) gives

$$d(x_{n+1}, F) \leq (1 + r_n)d(x_n, F) + s_n \quad \forall n \geq 1. \quad (2.4)$$

Again, by Lemma 2.3, we get that $\lim_{n \rightarrow \infty} d(x_n, F)$ exists.

(c) It follows from (b) and $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$ that

$$\lim_{n \rightarrow \infty} d(x_n, F) = 0. \quad (2.5)$$

To show that $\{x_n\}$ is a Cauchy sequence, let $\varepsilon > 0$. Since $\lim_{n \rightarrow \infty} d(x_n, F) = 0$, we may assume without loss of generality that there is a sequence $\{p_n\}$ in F such that $d(x_n, p_n) \leq \varepsilon/4$ for all $n \geq 1$. As $\{x_n\}$ is bounded, we put $M = \sup\{d(x_m, p_n) : m, n \geq 1\}$. From (2.1), we have

$$d(x_{n+1}, p_k) \leq d(x_n, p_k) + t_n \quad \forall n, k \geq 1, \quad (2.6)$$

where $t_n \equiv r_n M + s_n$. Consequently,

$$d(x_{n+k}, p_n) \leq d(x_n, p_n) + \sum_{j=n}^{n+k-1} t_j \leq \frac{\varepsilon}{4} + \sum_{j=n}^{\infty} t_j \quad \forall n, k \geq 1. \quad (2.7)$$

Notice that $\sum_{n=1}^{\infty} t_n < \infty$. So there exists $N \geq 1$ such that $\sum_{n=N}^{\infty} t_n < \varepsilon/2$. Then for all $n \geq N, k \geq 1$, we have

$$d(x_{n+k}, x_n) \leq d(x_{n+k}, p_n) + d(x_n, p_n) < \varepsilon. \quad (2.8)$$

Hence, $\{x_n\}$ is a Cauchy sequence in X . By the completeness of X , we assume that $x_n \rightarrow p$ for some $p \in X$. Since

$$|d(x_n, F) - d(p, F)| \leq d(x_n, p) \rightarrow 0, \quad (2.9)$$

we obtain $d(p, F) = 0$. This completes the proof. $\square$

2.2. A Correction of Recent Results of Innang and Suantai

The following observation is an auxiliary result.

Proposition 2.5. *Let C be a nonempty subset of a Banach space X , and let $T_1, T_2, \dots, T_N : C \rightarrow C$ be N generalized asymptotically quasi-nonexpansive mappings with $F := \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Then there exist sequences $\{k_n\}, \{l_n\}$ in $[0, \infty)$ such that $k_n, l_n \rightarrow 0$ and*

$$\|T_i^n x - p\| \leq (1 + k_n)\|x - p\| + l_n, \quad (2.10)$$

for all $x \in C, p \in F, n \geq 1$, and $i = 1, 2, \dots, N$.

From now on, we assume that N generalized asymptotically quasi-nonexpansive mappings $T_1, T_2, \dots, T_N : C \rightarrow C$ are equipped with the sequences $\{k_n\}, \{l_n\}$ in $[0, \infty)$ as mentioned in the preceding proposition.

Theorem 2.6. Let C be a nonempty closed convex subset of a Banach space X , and $\{T_1, T_2, \dots, T_N\}$ a finite family of generalized asymptotically quasi-nonexpansive self-mappings of C with the sequence $\{(k_n, l_n)\}$ such that $\sum_{n=1}^{\infty} k_n < \infty$ and $\sum_{n=1}^{\infty} l_n < \infty$. Assume that $F := \bigcap_{i=1}^N F(T_i) \neq \emptyset$ is closed, and $\{x_n\}$ is the sequence in C defined by (1.5) such that $\sum_{n=1}^{\infty} \gamma_{in} < \infty$ for each $i = 1, 2, \dots, N$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point of the family of mappings if and only if $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$.

Remark 2.7. There is a small gap in [4, Theorem 3.2]. More precisely, the sequence $\{x_n\}$ generated by (1.5) is shown in [4, Theorem 3.2] to be of monotone type (2) with respect to F , that is, $\|x_{n+1} - p\| \leq (1 + k_n)^N \|x_n - p\| + e_{kn}$ where each e_{kn} is a nonnegative real number depending on p . Then the expression $d(x_{n+1}, F) \leq (1 + k_n)^N d(x_n, F) + e_{kn}$ cannot warrant.

Remark 2.8. The same gap also appears in [8, Lemma 2.3] and [6, Theorem 3.2].

Proof of Theorem 2.6. Necessity is obvious. Conversely, we show first that $\{x_n\}$ is of monotone type (2) with respect to F . Let $p \in F$. We have that

$$\begin{aligned} \|y_{1n} - p\| &= \|\alpha_{1n} T_1^n x_n + \beta_{1n} x_n + \gamma_{1n} u_{1n} - p\| \\ &\leq \alpha_{1n} \|T_1^n x_n - p\| + \beta_{1n} \|x_n - p\| + \gamma_{1n} \|u_{1n} - p\| \\ &\leq (\alpha_{1n} + \beta_{1n})(1 + k_n) \|x_n - p\| + \alpha_{1n} l_n + \gamma_{1n} \|u_{1n} - p\| \end{aligned} \quad (2.11)$$

$$\leq (1 + k_n) \|x_n - p\| + \tilde{l}_{1n}, \quad (2.12)$$

where $\tilde{l}_{1n} \equiv \alpha_{1n} l_n + \gamma_{1n} \|u_{1n} - p\|$. Notice that $\sum_{n=1}^{\infty} l_n < \infty$ and $\{u_{1n}\}$ is bounded. Then $\sum_{n=1}^{\infty} \tilde{l}_{1n} < \infty$. It follows from (2.12) that

$$\begin{aligned} \|y_{2n} - p\| &\leq \alpha_{2n} \|T_2^n y_{1n} - p\| + \beta_{2n} \|x_n - p\| + \gamma_{2n} \|u_{2n} - p\| \\ &\leq \alpha_{2n} (1 + k_n) \|y_{1n} - p\| + \alpha_{2n} l_n + \beta_{2n} \|x_n - p\| + \gamma_{2n} \|u_{2n} - p\| \\ &\leq (\alpha_{2n} + \beta_{2n})(1 + k_n)^2 \|x_n - p\| + \alpha_{2n} ((1 + k_n) \tilde{l}_{1n} + l_n) + \gamma_{2n} \|u_{2n} - p\| \\ &\leq (1 + k_n)^2 \|x_n - p\| + \tilde{l}_{2n}, \end{aligned} \quad (2.13)$$

where $\tilde{l}_{2n} \equiv \alpha_{2n} ((1 + k_n) \tilde{l}_{1n} + l_n) + \gamma_{2n} \|u_{2n} - p\|$. Notice that $\sum_{n=1}^{\infty} k_n < \infty$, $\sum_{n=1}^{\infty} l_n < \infty$, $\sum_{n=1}^{\infty} \tilde{l}_{1n} < \infty$ and $\{u_{2n}\}$ is bounded. Then $\sum_{n=1}^{\infty} \tilde{l}_{2n} < \infty$. By continuing this process, there is a sequence $\{\tilde{l}_{kn}\}$ of nonnegative real numbers such that $\sum_{n=1}^{\infty} \tilde{l}_{kn} < \infty$ and

$$\|x_{n+1} - p\| \leq (1 + k_n)^N \|x_n - p\| + \tilde{l}_{kn}. \quad (2.14)$$

Then $\{x_n\}$ is of monotone type (2) with respect to F . By Theorem 2.4(a), we get that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists and $\{x_n\}$ is bounded. Next, we show that $\{x_n\}$ is of monotone type (1) with respect to F . It follows from (2.11) that

$$\begin{aligned} \|y_{1n} - p\| &\leq (\alpha_{1n} + \beta_{1n})(1 + k_n)\|x_n - p\| + \alpha_{1n}l_n + \gamma_{1n}\|u_{1n} - p\| \\ &\leq (\alpha_{1n} + \beta_{1n})(1 + k_n)\|x_n - p\| + \alpha_{1n}l_n + \gamma_{1n}(\|x_n - p\| + \|x_n - u_{1n}\|) \\ &\leq (1 + k_n)\|x_n - p\| + \tilde{l}_{1n}, \end{aligned} \quad (2.15)$$

where $\tilde{l}_{1n} \equiv \alpha_{1n}l_n + \gamma_{1n}\|x_n - u_{1n}\|$. Notice that $\{u_{1n}\}, \{x_n\}$ are bounded and $\sum_{n=1}^{\infty} l_n < \infty$. Then $\sum_{n=1}^{\infty} \tilde{l}_{1n} < \infty$ and $\{\tilde{l}_{1n}\}$ is independent of p . Again, by continuing this process, we obtain a sequence $\{\tilde{l}_{kn}\}$ of nonnegative real numbers such that it is independent of p , $\sum_{n=1}^{\infty} \tilde{l}_{kn} < \infty$ and

$$\|x_{n+1} - p\| \leq (1 + k_n)^N \|x_n - p\| + \tilde{l}_{kn} \quad (2.16)$$

for all $n \geq 1$ and $p \in F$. Then $\{x_n\}$ is of monotone type (1) with respect to F . Hence the result follows from (2.16) and Theorem 2.4(c). This completes the proof. $\square$

Remark 2.9. Theorem 2.4 is a correction of [4, Theorem 3.2]. In fact, the closedness of F is not assumed there (this defect is now corrected *after* the submission of this article). Moreover, it is shown in the following example that the fixed point set of a generalized asymptotically nonexpansive mapping is not necessarily closed even in a Hilbert space.

Example 2.10 (A generalized asymptotically nonexpansive mapping whose fixed point set is not closed). Let $T : [-1/2, 1/2] \rightarrow [-1/2, 1/2]$ be a mapping defined by

$$Tx = \begin{cases} x, & \text{if } x \in \left[-\frac{1}{2}, 0\right), \\ \frac{1}{4}, & \text{if } x = 0, \\ x^2, & \text{if } x \in \left(0, \frac{1}{2}\right]. \end{cases} \quad (2.17)$$

Then T is generalized asymptotically nonexpansive.

Proof. Notice that $F(T) = [-1/2, 0)$ is not closed. We prove that

$$|T^n x - T^n y| \leq |x - y| + \frac{1}{2^{2^n}} \quad (2.18)$$

for all $x, y \in [-1/2, 1/2]$ and $n \geq 1$. The inequality above holds trivially if $x = y = 0$ or $x, y \in [-1/2, 0)$. Then it suffices to consider the following cases.

Case 1 ($x, y \in (0, 1/2]$). Then

$$|T^n x - T^n y| = |x^{2^n} - y^{2^n}| \leq \frac{1}{2^{2^n}}. \quad (2.19)$$

Case 2 ($x \in [-1/2, 0)$ and $y = 0$). Then

$$|T^n x - T^n y| = \left| x - \frac{1}{2^{2^n}} \right| \leq |x - y| + \frac{1}{2^{2^n}}. \quad (2.20)$$

Case 3 ($x \in [-1/2, 0)$ and $y \in (0, 1/2]$). Then

$$|T^n x - T^n y| = |x - y^{2^n}| \leq |x - y|. \quad (2.21)$$

Case 4 ($x = 0$ and $y \in (0, 1/2]$). Then

$$|T^n x - T^n y| = \left| \frac{1}{2^{2^n}} - y^{2^n} \right| \leq |x - y| + \frac{1}{2^{2^n}}. \quad (2.22)$$

Hence, (2.18) holds. This completes the proof. $\square$

Remark 2.11. For T which is defined in Example 2.10 and $x_1 \in (0, 1/2]$, we define

$$x_{n+1} = \alpha_n T^n x_n + (1 - \alpha_n) x_n, \quad (2.23)$$

where $0 < \alpha_n \leq 1$ and $n \geq 1$. It is not hard to show that $x_n \rightarrow 0 \notin F(T)$ and $d(x_n, F(T)) \rightarrow 0$. Hence [4, Theorems 3.2 and 3.6] do not hold even for a single mapping if the closedness of the fixed point set is not assumed.

We present a sufficient condition guaranteeing the closedness of the fixed point set of a generalized asymptotically quasi-nonexpansive mapping.

Theorem 2.12. *Let C be a nonempty subset of a Banach space X and $T : C \rightarrow C$ a generalized asymptotically quasi-nonexpansive mapping. If $G(T) := \{(x, Tx) : x \in C\}$ is closed, then $F(T)$ is closed.*

Proof. Let $\{p_n\}$ be a sequence in $F(T)$ such that $p_n \rightarrow p$. Since T is a generalized asymptotically quasi-nonexpansive mapping with the sequence $\{(k_n, l_n)\}$, we have

$$\begin{aligned} \|T^n p - p\| &\leq \|T^n p - p_n\| + \|p_n - p\| \\ &\leq (1 + k_n)\|p - p_n\| + l_n + \|p_n - p\| \rightarrow 0. \end{aligned} \quad (2.24)$$

Then $T^n p \rightarrow p$, and so $T(T^n p) = T^{n+1} p \rightarrow p$. Hence, by the closedness of $G(T)$, $TP = p$. This completes the proof. $\square$

Remark 2.13. It is also worth mentioning that the $(L - \gamma)$ uniform Lipschitz condition of mappings in [4, Theorems 4.2 and 4.3] implies the closedness of their graphs.

The following result shows that the closedness of $G(T)$ can be dropped if T is asymptotically quasi-nonexpansive.

Theorem 2.14. *Let C be a nonempty subset of a Banach space X , and $T : C \rightarrow C$ an asymptotically quasi-nonexpansive mapping. Then $F(T)$ is closed.*

Proof. Suppose that T is an asymptotically quasi-nonexpansive mapping with the sequence $\{k_n\}$. Let $\{p_n\}$ be a sequence in $F(T)$ such that $p_n \rightarrow p$. We have

$$\begin{aligned} \|Tp - p\| &\leq \|Tp - p_n\| + \|p_n - p\| \\ &\leq (1 + k_1)\|p - p_n\| + \|p_n - p\| \rightarrow 0. \end{aligned} \quad (2.25)$$

Then $Tp = p$. This completes the proof. $\square$

Remark 2.15. Not every generalized asymptotically quasi-nonexpansive mapping is asymptotically quasi-nonexpansive. In fact, the mapping T in Example 2.10 is not asymptotically quasi-nonexpansive since $F(T)$ is not closed.

3. Remark on Lan's Generalized Asymptotically Quasi-Nonexpansive Mappings

The following mapping introduced by Lan [5] also bears the name generalized asymptotically quasi-nonexpansive mappings. We recall his definition here.

Definition 3.1 (see [5, Definition 2.1(4)]). Let C be a subset of a Banach space X . A mapping $T : C \rightarrow C$ is called *generalized asymptotically quasi-nonexpansive in the sense of Lan* if there exists two sequences $\{r_n\} \subset [0, \infty)$ and $\{s_n\} \subset [0, 1)$ such that $r_n, s_n \rightarrow 0$ and

$$\|T^n x - p\| \leq (1 + r_n)\|x - p\| + s_n\|x - T^n x\| \quad (3.1)$$

for all $x \in C, p \in F(T)$, and $n \geq 1$.

Lan [5] and many authors (e.g., [8–11]) have investigated convergence theorems for such mappings without awareness that Lan's mappings are not new ones.

Proposition 3.2. *If $T : C \rightarrow C$ is generalized asymptotically quasi-nonexpansive in the sense of Lan, then it is asymptotically quasi-nonexpansive.*

Proof. By Lan's definition, there exist two sequences $\{r_n\} \subset [0, \infty)$ and $\{s_n\} \subset [0, 1)$ such that $r_n, s_n \rightarrow 0$ and

$$\|T^n x - p\| \leq (1 + r_n)\|x - p\| + s_n\|x - T^n x\| \quad (3.2)$$

for all $x \in C$, $p \in F(T)$, and $n \in \mathbb{N}$. Consequently,

$$\|T^n x - p\| \leq (1 + r_n)\|x - p\| + s_n(\|x - p\| + \|T^n x - p\|). \quad (3.3)$$

This implies

$$\|T^n x - p\| \leq \frac{1 + r_n + s_n}{1 - s_n} \|x - p\| = \left(1 + \frac{r_n + 2s_n}{1 - s_n}\right) \|x - p\|. \quad (3.4)$$

It is also clear that $(r_n + 2s_n)/(1 - s_n) \rightarrow 0$ and this completes the proof. $\square$

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