Research Article

Spectral Singularities of Sturm-Liouville Problems with Eigenvalue-Dependent Boundary Conditions

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Let $L$ denote the operator generated in $L^2(R_+)$ by Sturm-Liouville equation $-y'' + q(x)y = \lambda^2 y$, $x \in R_+ = [0, \infty)$, $y'(0)/y(0) = \alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2$, where $q$ is a complex-valued function and $\alpha_i \in \mathbb{C}$, $i = 0, 1, 2$ with $\alpha_2 \neq 0$. In this article, we investigate the eigenvalues and the spectral singularities of $L$ and obtain analogs of Naimark and Pavlov conditions for $L$.

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1. Introduction

Let $L_0$ denote Sturm-Liouville operator generated in $L^2(R_+)$ by the differential expression

$$l_0(y) := -y'' + q(x)y, \quad x \in R_+, \quad (1.1)$$

and the boundary condition $y(0) = 0$, where $q : R_+ \to \mathbb{C}$. Since $q$ is a complex-valued function, the operator $L_0$ is a non-selfadjoint. The spectral analysis of $L_0$ has been investigated by Naimark [1]. He proved that some of the poles of the kernel of resolvent of $L_0$ are not the eigenvalues of the operator. He also showed that those poles (which are called spectral singularities by Schwartz [2]) are on the continuous spectrum. Moreover, he has shown the spectral singularities play an important role in the spectral analysis of $L_0$, and if

$$\int_0^\infty e^{\varepsilon x} |q(x)| \, dx < \infty, \quad \varepsilon > 0, \quad (N)$$

then the eigenvalues and the spectral singularities are of a finite number and each of them is of a finite multiplicity.
One very important step in the spectral analysis of $L_0$ was taken by Pavlov [3]. He studied the dependence of the structure of the eigenvalues and the spectral singularities of $L_0$ on the behavior of potential function at infinity. He also proved that if

$$\sup_{x \in \mathbb{R}} \left| e^{\varepsilon \sqrt{x}} |q(x)| \right| < \infty, \quad \varepsilon > 0,$$  \hspace{1cm} \text{(P)}

then the eigenvalues and the spectral singularities are of a finite number and each of them is of a finite multiplicity.

Conditions (N) and (P) are called Naimark and Pavlov conditions for $L_0$, respectively. Lyance showed that the spectral singularities play an important role in the spectral analysis of $L_0$ [4, 5]. He also investigated the effect of the spectral singularities in the spectral expansion.

The spectral singularities of non-selfadjoint operator generated in $L_2(\mathbb{R}_+)$ by (1.1) and the boundary condition

$$\int_0^\infty K(x)y(x)dx + ay'(0) - by(0) = 0 \hspace{1cm} \text{(1.2)}$$

was investigated in detail by Krall [6, 7].

Some problems of spectral theory of differential operator and some other types of operators with spectral singularities were studied by some authors [8–14]. Note that in all papers the boundary conditions are not depending on the spectral parameter.

In a recent series of papers, Binding et al. and Browne [15–18] have studied the spectral theory of regular Sturm-Liouville operators with boundary conditions depending on the spectral parameter.

Let $L$ denote the operator generated in $L_2(\mathbb{R}_+)$ by

$$-y'' + q(x)y = \lambda^2 y, \quad x \in \mathbb{R}_+, \hspace{1cm} \text{(1.3)}$$

$$\frac{y'(0)}{y(0)} = a_0 + a_1 \lambda + a_2 \lambda^2, \hspace{1cm} \text{(1.4)}$$

where $q$ is a complex-valued function, $a_i \in \mathbb{C}, i = 0, 1, 2$, with $a_2 \neq 0$. In this paper, we investigate the eigenvalues and the spectral singularities of $L$. In particular, we show that the analogs of Naimark and Pavlov conditions for $L$ are

$$q \in AC(\mathbb{R}_+), \quad \lim_{x \to \infty} q(x) = 0, \quad \int_0^\infty e^{\varepsilon x}|q'(x)|dx < \infty, \quad \varepsilon > 0,$$  \hspace{1cm} \text{(1.5)}

$$q \in AC(\mathbb{R}_+), \quad \lim_{x \to \infty} q(x) = 0, \quad \sup_{x \in \mathbb{R}_+} \left| e^{\varepsilon \sqrt{x}} |q(x)| \right| < \infty, \quad \varepsilon > 0,$$

respectively, where $AC(\mathbb{R}_+)$ denotes the class of complex-valued absolutely continuous functions on $\mathbb{R}_+$. 
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2. Jost Functions of (1.3)-(1.4)

Under the condition

\[ \int_0^\infty x|q(x)|dx < \infty, \]  

(2.1)

(1.3) has a solution \( e(x, \lambda) \) satisfying

\[ \lim_{x \to \infty} e(x, \lambda)e^{-i\lambda x} = 1, \quad \lambda \in \mathbb{C}_+, \]  

(2.2)

where \( \mathbb{C}_+ = \{ \lambda : \lambda \in \mathbb{C}, \text{Im} \lambda \geq 0 \} \). The solution \( e(x, \lambda) \) is called Jost solution of (1.3). Note that Jost solution has a representation [19]

\[ e(x, \lambda) = e^{i\lambda x} + \int_x^{\infty} K(x, t)e^{i\lambda t}dt, \quad \lambda \in \mathbb{C}_+, \]  

(2.3)

where \( K(x, t) \) is the solution of the integral equation

\[ K(x, t) = \frac{1}{2} \int_0^{\infty} q(s)ds + \frac{1}{2} \int_x^{(x+t)/2} q(s)K(s, u)du ds + \frac{1}{2} \int_0^{\infty} \int_s^{s+x} q(s)K(s, u)du ds, \]  

(2.4)

and \( K(x, t) \) are continuously differentiable with respect to their arguments. We also have

\[ |K(x, t)| \leq cw\left(\frac{x + t}{2}\right), \]  

\[ |K_x(x, t)|, |K_t(x, t)| \leq \frac{1}{4} |q\left(\frac{x + t}{2}\right)| + cw\left(\frac{x + t}{2}\right), \]  

(2.5)

where \( w(x) = \int_x^{\infty} |q(s)|ds \) and \( c > 0 \) is a constant.

Let

\[ E^+(\lambda) := e'(0, \lambda) - (a_0 + a_1\lambda + a_2\lambda^2)e(0, \lambda), \quad \lambda \in \mathbb{C}_+, \]  

\[ E^-(\lambda) := e'(0, -\lambda) - (a_0 + a_1\lambda + a_2\lambda^2)e(0, -\lambda), \quad \lambda \in \mathbb{C}_-, \]  

(2.6)

where \( \mathbb{C}_- = \{ \lambda : \lambda \in \mathbb{C}, \text{Im} \lambda \leq 0 \} \). Therefore, \( E^+ \) and \( E^- \) are analytic in \( \mathbb{C}_+ = \{ \lambda : \lambda \in \mathbb{C}, \text{Im} \lambda > 0 \} \) and \( \mathbb{C}_- = \{ \lambda : \lambda \in \mathbb{C}, \text{Im} \lambda < 0 \} \), respectively, and continuous up to real axis. The functions \( E^+ \) and \( E^- \) are called Jost functions of \( L \).
Let us denote the eigenvalues and the spectral singularities of $L$ by $\sigma_d(L)$ and $\sigma_{ss}(L)$, respectively. It is evident that

\[
\sigma_d(L) = \{ \lambda : \lambda \in \mathbb{C}_+, E^+(\lambda) = 0 \} \cup \{ \lambda : \lambda \in \mathbb{C}_-, E^-(\lambda) = 0 \}, \tag{2.7}
\]

\[
\sigma_{ss}(L) = \{ \lambda : \lambda \in \mathbb{R}^+, E^+(\lambda) = 0 \} \cup \{ \lambda : \lambda \in \mathbb{R}^-, E^-(\lambda) = 0 \}, \tag{2.8}
\]

where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

**Definition 2.1.** The multiplicity of a zero $E^+(\text{or } E^-)$ in $\overline{\mathbb{C}}_+$ (or $\overline{\mathbb{C}}_-$) is defined as the multiplicity of the corresponding eigenvalue and spectral singularity of $L$.

In order to investigate the quantitative properties of the eigenvalues and the spectral singularities of $L$, we need to discuss the quantitative properties of the zeros of $E^+$ and $E^-$ in $\overline{\mathbb{C}}_+$ and $\overline{\mathbb{C}}_-$, respectively.

Define

\[
M^+_1 = \{ \lambda : \lambda \in \mathbb{C}_+, E^+(\lambda) = 0 \}, \quad M^-_1 = \{ \lambda : \lambda \in \mathbb{R}^+, E^+(\lambda) = 0 \}, \quad M^+_2 = \{ \lambda : \lambda \in \mathbb{R}^+, E^+(\lambda) = 0 \}, \quad M^-_2 = \{ \lambda : \lambda \in \mathbb{R}^-, E^-(\lambda) = 0 \}, \tag{2.9}
\]

then by (2.7), we have

\[
\sigma_d(L) = M^+_1 \cup M^-_1, \quad \sigma_{ss}(L) = M^+_2 \cup M^-_2. \tag{2.10}
\]

Now, let us assume that

\[
q \in AC(\mathbb{R}_+), \quad \lim_{x \to -\infty} q(x) = 0, \quad \int_0^\infty x^3 |q'(x)| dx < \infty. \tag{2.11}
\]

**Theorem 2.2.** Under condition (2.11), the functions $E^+$ and $E^-$ have the representations

\[
E^+(\lambda) = -\alpha_2 \lambda^2 + \beta^+ \lambda + \delta^+ + \int_0^\infty f^+(t)e^{i\lambda t} dt, \quad \lambda \in \overline{\mathbb{C}}_+, \tag{2.12}
\]

\[
E^-(\lambda) = -\alpha_2 \lambda^2 + \beta^- \lambda + \delta^- + \int_0^\infty f^-(t)e^{-i\lambda t} dt, \quad \lambda \in \overline{\mathbb{C}}_-, \tag{2.13}
\]

where $\beta^\pm, \delta^\pm \in \mathbb{C}$, and $f^\pm \in L_1(\mathbb{R}_+)$. 

**Proof.** Using (2.3), (2.4), and (2.6), we get (2.12), where

\[
\beta^+ = i - \alpha_1 - i\alpha_2 K(0,0),
\]

\[
\delta^+ = -K(0,0) - \alpha_0 - i\alpha_1 K(0,0) + \alpha_2 K_i(0,0), \tag{2.14}
\]

\[
f^+(t) = K_x(0,t) - \alpha_0 K(0,t) - i\alpha_1 K_i(0,t) + \alpha_2 K_{ii}(0,t).
\]
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From (2.4), we see that

\[
|K_n(0, t)| \leq c \left[ t \left| q \left( \frac{t}{2} \right) \right| + \left| q' \left( \frac{t}{2} \right) \right| + tw \left( \frac{t}{2} \right) + f(t) \frac{t}{2} \right]
\]  

(2.15)

holds, where \( w(t) = \int_1^T w(s)ds \) and \( c > 0 \) is a constant. It follows from (2.5), (2.14), and (2.15) that \( f^+ \in L_1(\mathbb{R}_+) \). In a similar way, we obtain (2.13).

\[ \square \]

**Theorem 2.3.** Under condition (2.11), we have the following.

(i) The set of \( \sigma_d(L) \) is bounded and has at most a countable number of elements, and its limit points can lie only in a bounded subinterval of the real axis.

(ii) The set of \( \sigma_{ss}(L) \) is bounded and its linear Lebesgue measure is zero.

**Proof.** From (2.14) and (2.15), we see that

\[
E^+(\lambda) = -\alpha_2 \lambda^2 + \beta^+ \lambda + \delta^+ + o(1), \quad \lambda \in \mathbb{C}_+, \ |\lambda| \to \infty,
\]

\[
E^-(\lambda) = -\alpha_2 \lambda^2 + \beta^- \lambda + \delta^- + o(1), \quad \lambda \in \mathbb{C}_-, \ |\lambda| \to \infty.
\]  

(2.16)

Using (2.10), (2.16), and the uniqueness theorem of analytic functions [20], we get (i) and (ii). \[ \square \]

3. **Naǐmark and Pavlov Conditions for L**

We will denote the set of all limit points of \( M^+_1 \) and \( M^-_1 \) by \( M^+_3 \) and \( M^-_3 \), respectively, and the set of all zeros of \( E^+ \) and \( E^- \) with infinity multiplicity in \( \mathbb{C}_+ \) and \( \mathbb{C}_- \), by \( M^+_4 \) and \( M^-_4 \), respectively. It is obvious that

\[
M^+_3 \subset M^+_2, \quad M^+_4 \subset M^+_3, \quad M^-_3 \subset M^-_4,
\]  

(3.1)

and the linear Lebesgue measures of \( M^+_3 \) and \( M^-_4 \) are zero.

**Theorem 3.1.** If

\[
q \in AC(\mathbb{R}_+), \quad \lim_{x \to \infty} q(x) = 0, \quad \int_0^\infty e^{-\varepsilon x} |q'(x)| dx < \infty, \quad \varepsilon > 0,
\]  

(3.2)

then the operator \( L \) has a finite number of eigenvalues and spectral singularities, and each of them is of a finite multiplicity.

**Proof.** From (2.5), (2.14), (2.15), and (3.2), we find that

\[
|f^+(t)| \leq ce^{-(\varepsilon/2)t},
\]  

(3.3)

where \( c > 0 \) is a constant. By (2.12) and (3.3), we observe that the function \( E^+ \) has an analytic continuation to the half-plane \( \text{Im} \lambda > -\varepsilon/4 \). So we get that \( M^+_4 = \emptyset \). It follows from (3.1)
that \( M^+_3 = \emptyset \). Therefore the sets \( M^+_1 \) and \( M^+_2 \) have a finite number of elements with a finite multiplicity. We obtain similar results for the sets \( M^-_1 \) and \( M^-_2 \). By (2.10) we have the proof of the theorem.

Now let us assume that

\[
q \in AC(\mathbb{R}_+), \quad \lim_{x \to \infty} q(x) = 0, \quad \sup_{x \in \mathbb{R}_+} \left| e^{\sqrt{x}} |q'(x)| \right| < \infty, \quad \epsilon > 0.
\] (3.4)

Hence, we have the following lemma.

**Lemma 3.2.** It holds that \( M^+_4 = M^-_4 = \emptyset \).

**Proof.** From (2.12) and (3.4), we find that the function \( E^+ \) is analytic in \( \mathbb{C}_+ \), and all of its derivatives are continuous in \( \mathbb{C}_+ \). For a sufficiently large \( T > 0 \), we have

\[
\left| \frac{d^k}{d\lambda^k} E^+(\lambda) \right| \leq A_k, \quad \lambda \in \mathbb{C}_+, \quad |\lambda| \leq T, \quad k = 0, 1, 2, \ldots, \tag{3.5}
\]

where

\[
A_k = 2^k c \int_0^\infty t^k e^{-(\epsilon/2)\sqrt{t}} dt, \quad k = 0, 1, 2, \ldots, \tag{3.6}
\]

and \( c > 0 \) is a constant. Since the function \( E^+ \) is not equal to zero identically, then by Pavlov’s theorem, \( M^+_4 \) satisfies

\[
\int_0^h \ln A(s) d\mu(M^+_4, s) > -\infty, \tag{3.7}
\]

where \( A(s) = \inf_k (A_k s^k/k!) \), \( \mu(M^+_4, s) \) is the linear Lebesgue measure of \( s \)-neighborhood of \( M^+_4 \)[3]. Now, we obtain the following estimates for \( A_k \):

\[
A_k \leq B b^k k^k k!, \tag{3.8}
\]

where \( B \) and \( b \) are constants depending on \( c \) and \( \epsilon \). From (3.8), we get that

\[
A(s) \leq B \inf_k \left( b^k s^k k^k \right) \leq B \exp \left( -b^{-1} e^{-1} s^{-1} \right). \tag{3.9}
\]

Now, (3.7) yields that

\[
\int_0^h \frac{1}{s} d\mu(M^+_4, s) < \infty. \tag{3.10}
\]
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However, (3.10) holds for an arbitrary $s$, if and only if $\mu(M^+_4, s) = 0$ or $M^+_4 = \emptyset$. In a similar way we can prove that $M^+_4 = \emptyset$. □

**Theorem 3.3.** Under condition (3.4), the operator $L$ has a finite number of eigenvalues and spectral singularities, and each of them is of a finite multiplicity.

**Proof.** To be able to prove the theorem, we have to show that the functions $E^+$ and $E^-$ have a finite number of zeros with finite multiplicities in $\overline{C}_+$ and $\overline{C}_-$, respectively. We give the proof for $E^+$.

From Lemma 3.2 and (3.1), we find that $M^+_3 = \emptyset$. So the bounded sets $M^+_1$ and $M^+_2$ have no limit points, that is, the function $E^+$ has only a finite number of zeros in $\overline{C}_+$. Since $M^+_4 = \emptyset$, these zeros are of finite multiplicity. □

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**References**


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