Research Article

On Two-Parameter Regularized Semigroups and the Cauchy Problem

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Suppose that $X$ is a Banach space and $C$ is an injective operator in $B(X)$, the space of all bounded linear operators on $X$. In this note, a two-parameter $C$-semigroup (regularized semigroup) of operators is introduced, and some of its properties are discussed. As an application we show that the existence and uniqueness of solution of the 2-abstract Cauchy problem $\frac{\partial}{\partial t_1}(\partial/\partial t_2)u(t_1, t_2) = H_iu(t_1, t_2), i = 1, 2, t_i > 0, u(0, 0) = x, x \in C(D(H_1) \cap D(H_2))$ is closely related to the two-parameter $C$-semigroups of operators.

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1. Introduction and Preliminaries

Suppose that $X$ is a Banach space and $A$ is a linear operator in $X$ with domain $D(A)$ and range $R(A)$. For a given $x \in D(A)$, the abstract Cauchy problem for $A$ with the initial value $x$ consists of finding a solution $u(t)$ to the initial value problem

$$ACP(A; x) \begin{cases} \frac{du(t)}{dt} = Au(t), & t \in \mathbb{R}_+, \\ u(0) = x, \end{cases}$$

(1.1)

where by a solution we mean a function $u : \mathbb{R}_+ \to X$, which is continuous for $t \geq 0$, continuously differentiable for $t > 0$, $u(t) \in D(A)$ for $t \in \mathbb{R}_+$, and $ACP(A; x)$ is satisfied.

If $C \in B(X)$, the space of all bounded linear operators on $X$, is injective, then a one-parameter $C$-semigroup (regularized semigroup) of operators is a family $\{T(t)\}_{t \in \mathbb{R}_+} \subset B(X)$
for which \( T(0) = C, T(s + t)C = T(s)T(t) \), and for each \( x \in X \), the mapping \( t \mapsto T(t)x \) is continuous. An operator \( A : D(A) \to X \) with

\[
D(A) = \left\{ x \in X : \lim_{t \to 0} \frac{T(t)x - Cx}{t} \text{ exists in the range of } C \right\},
\]

and where, for \( x \in D(A) \), \( Ax := C^{-1}\lim_{t \to 0}((T(t)x - Cx)/t) \) is called the infinitesimal generator of \( T(t) \).

Regularized semigroups and their connection with the ACP\((A; x)\) have been studied in [1–6] and some other papers. Also the concept of local C-semigroups and their relation with the \( ACP(A; x) \) have been considered in [7–10].

In Section 2, we introduce the concept of two-parameter regularized semigroups of operators and their generator. Some basic properties of two-parameter regularized semigroups and their relation with the generators are studied in this section.

In Section 3, two-parameter abstract Cauchy problems are considered. It is proved that the existence and uniqueness of its solutions is closely related with two-parameter regularized semigroups of operators.

### 2. Two-Parameter Regularized Semigroups

In this section we introduce two-parameter regularized semigroup and its generator on Banach spaces. Then some properties of two-parameter regularized semigroups are studied.

**Definition 2.1.** Suppose that \( X \) is a Banach space and \( C \in B(X) \) is an injective operator. A family \( \{W(s, t)\}_{s,t \in \mathbb{R}_+} \subset B(X) \) is called a two-parameter regularized semigroup (or two parameter C-semigroup) if

1. \( W(0, 0) = C \),
2. \( W(s + s', t + t')C = W(s, t)W(s', t') \), for all \( s, s', t, t' \in \mathbb{R}_+ \),
3. \( \lim_{(s, t) \to (s', t')} W(s', t')x = W(s, t)x, \) for all \( x \in X \).

It is called exponentially bounded if \( \|W(s, t)\| \leq Me^{\omega(s + t)} \), for some \( M, \omega > 0 \).

Suppose that \( \{W(s, t)\}_{s,t \in \mathbb{R}_+} \) is a two-parameter C-semigroup. Put \( u(s) := W(s, 0) \) and \( v(t) := W(0, t) \), then it is easy to see that these families are two commuting one-parameter C-semigroups such that \( W(s, t)C = u(s)v(t) \). Also \( u(s) \) and \( v(t) \) commute with \( C \). If \( H_1 \) and \( H_2 \) are their generators, respectively, then we will think of \( (H_1, H_2) \) as the generator of \( W(s, t) \).

From the one-parameter case (see [8]), one can prove that \( R(C) \subseteq D(H_1) \cap D(H_2) \), and \( C^{-1}H_iC = H_i, i = 1, 2 \).

Also if \( \{U(s)\}_{s \in \mathbb{R}_+} \) and \( \{V(t)\}_{t \in \mathbb{R}_+} \) are two commuting one-parameter C-semigroups, then one can see that \( W(s, t) := U(s)V(t) \) is a two-parameter \( C^2 \)-semigroup of operators.

The following is an example of a two-parameter C-semigroup which is not exponentially bounded.

**Example 2.2.** Let \( X = L^2(\mathbb{C}) \), and \( [W(s, t)f](z) := e^{-|z|^2+(s+t)z}f(z), (Cf)(z) := e^{-|z|^2}f(z) \), then \( W(s, t) \) is a two-parameter C-semigroup which is not exponentially bounded.

In the following theorem we can see some elementary properties of a two-parameter C-semigroup.
Theorem 2.3. Suppose that \( W(s,t) \) is a two-parameter \( C \)-semigroup with the infinitesimal generator \((H_1, H_2)\). Then, one has the following.

(i) For each \( x \in X \) and for every \( s, t \geq 0 \), \( \int_0^s \int_0^t W(\mu, \nu) x \, d\mu \, d\nu \), is in \( D(H_1) \cap D(H_2) \). Also

\[
\lim_{(h,k) \to (0,0)} \frac{1}{hk} \int_t^{t+h} \int_s^{s+k} W(\mu, \nu) x \, d\mu \, d\nu = W(s,t)x. \tag{2.1}
\]

(ii) For each \( x \in X \), and for every \( s, t \in \mathbb{R}_+ \), \( \int_0^s W(\mu, t)x \, d\mu \in D(H_1) \) and \( \int_0^t W(s, \nu)x \, d\nu \in D(H_2) \); furthermore

\[
H_1 \int_0^s W(\mu, t)x \, d\mu = W(s,t)x - W(0,t)x,
\]

\[
H_2 \int_0^t W(s, \nu)x \, d\nu = W(s,t)x - W(s,0)x. \tag{2.2}
\]

(iii) \( \overline{R(C)} \subseteq \overline{D(H_1) \cap D(H_2)} \) and \( H_1 \) and \( H_2 \) are closed.

(iv) For any \( x \in D(H_1) \cap D(H_2) \), and each \( s, t > 0 \), \( u(s)x \) and \( v(t)x \) are in \( D(H_1) \cap D(H_2) \). Also for this \( x \), and \( i = 1,2 \),

\[
\frac{\partial}{\partial t_i} W(t_1, t_2)x = H_i W(t_1, t_2)x = W(t_1, t_2)H_i x. \tag{2.3}
\]

(v) For any \( a, b > 0 \), \( T(t) := W(ta, tb) \) is a one-parameter \( C \)-semigroup whose generator is an extension of \( aH_1 + bH_2 \).

Proof. To prove (i), suppose \( x \in X \). First we note that for any \( \nu \geq 0 \),

\[
\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} W(\mu, \nu)x \, d\mu = W(0,\nu) \lim_{h \to 0} \frac{1}{h} \int_t^{t+h} W(\mu, 0)x \, d\mu
\]

\[
= W(0,\nu)W(t,0)x = W(t,\nu)x. \tag{2.4}
\]

Thus

\[
\frac{1}{h} \left( W(h,0) \int_0^t W(\mu, \nu)x \, d\mu \, d\nu - C \int_0^t \int_0^t W(\mu, \nu)x \, d\mu \, d\nu \right)
\]

\[
= \frac{1}{h} C \left( \int_0^s \int_0^t W(\mu, \nu)x \, d\mu \, d\nu - \int_0^t \int_0^s W(\mu, \nu)x \, d\mu \, d\nu \right) \tag{2.5}
\]

\[
= \int_0^s \left( \frac{1}{h} \int_t^{t+h} W(\mu, \nu)x \, d\mu \right) d\nu.
\]
which tends to $C_0^1(W(t, v) - W(0, v))x d\nu$ as $h \to 0$. This implies that $\int_0^1 W(x) x d\mu d\nu$ is in $D(H_1)$ and

$$H_1 \int_0^s \int_0^t W(\mu, v) x d\mu d\nu = \int_0^s (W(t, v) - W(0, v)) x d\nu.$$  

(2.6)

A similar argument implies that it is in $D(H_2)$ and

$$H_2 \int_0^s \int_0^t W(\mu, v) x d\mu d\nu = \int_0^s (W(\mu, s) - W(\mu, 0)) x d\nu.$$  

(2.7)

For the second part, from the continuity of $C$ we have

$$C \lim_{(h,k) \to (0,0)} \frac{1}{hk} \int_t^{t+h} \int_s^{s+k} W(\mu, v) x d\mu d\nu$$

$$= \lim_{(h,k) \to (0,0)} \frac{1}{hk} \int_t^{t+h} \int_s^{s+k} W(\mu, v) Cx d\mu d\nu$$

$$= \lim_{(h,k) \to (0,0)} \frac{1}{hk} \int_t^{t+h} W(0, v) \frac{1}{k} \int_s^{s+k} W(\mu, 0) x d\mu d\nu$$

$$= \lim_{h \to 0} \int_t^{t+h} W(0, v) \lim_{k \to 0} \frac{1}{k} \int_s^{s+k} W(\mu, 0) x d\mu d\nu$$

$$= W(0, t) W(s, 0)x$$

$$= Cx.$$

Now the fact that $C$ is injective completes the proof of this part.

The proof of (ii) has a process similar to the first part of (i).

To prove (iii), we first note that $H_1$ and $H_2$ are closed as a trivial consequence of the one-parameter case (see [2]). For any $x \in X$ we saw that

$$\frac{1}{h} \int_0^h \int_0^s W(\mu, v) x d\mu d\nu \in D(H_1) \cap D(H_2),$$  

(2.9)

which tends to $W(0, 0)x = Cx \in R(C)$ as $h \to 0$. This implies that $\overline{R(C)} \subseteq \overline{D(H_1) \cap D(H_2)}$.

To prove (iv), we let $x \in D(H_1) \cap D(H_2)$. If $u(s) = W(s, 0)$ and $v(t) = W(s, t)$, there is $y \in X$ such that

$$\lim_{s \to 0} \frac{u(s)x - Cx}{s} = Cy.$$  

(2.10)
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Hence

\[
\lim_{s \to 0} \frac{u(s)v(t)x - Cv(t)x}{s} = v(t)Cy = Cv(t)y,
\]

which is in the \(R(C)\), and this implies that \(v(t)x\) is in \(D(H_1)\), similarly it is in \(D(H_2)\).

Now from [2, Theorem 2.4(b)], for \(x \in D(H_1) \cap D(H_2)\), from the fact that \(v(t)x\) is in \(D(H_1)\),

\[
\frac{\partial}{\partial s} W(s,t)Cx = \frac{d}{ds} (u(s)(v(t)x)) = H_1u(s)(v(t)x) = H_1W(s,t)Cx = CH_1W(s,t)x.
\]

On the other hand from the part (ii) and closedness of \(H_1\),

\[
\int_0^s H_1W(\mu,t)x d\mu = H_1\int_0^s W(\mu,t)x d\mu = W(s,t)x - W(0,t)x,
\]

which implies that \((\partial/\partial s)W(s,t)x\) exists. Hence from the continuity of \(C\)

\[
C \frac{\partial}{\partial s} W(s,t)x = \frac{\partial}{\partial s} W(s,t)Cx = CH_1W(s,t)x.
\]

But \(C\) is injective so

\[
\frac{\partial}{\partial s} W(s,t)x = H_1W(s,t)x = W(s,t)H_1x.
\]

The second one is similar.

To prove (v), first we note that \(T(t)\) is a one-parameter \(C\)-semigroup. Now if \(x \in D(aH_1 + bH_2) = D(H_1) \cap D(H_2)\),

\[
C \lim_{t \to 0} \frac{T(t)x - Cx}{t} = \lim_{t \to 0} \frac{W(ta,0)W(0,tb)x - W(ta,0)Cx + W(ta,0)Cx - C^2x}{t} = b \lim_{t \to 0} W(ta,0) \frac{W(0,tb)x - Cx}{bt} + a \lim_{t \to 0} \frac{W(at,0)Cx - C^2x}{t} = bC^2H_2x + aH_1C^2x.
\]
Now the fact that $C$ is injective implies that
\[ C^{-1} \lim_{t \to 0^-} \frac{T(t)x - CX}{t} = aH_1x + bH_2x. \] (2.17)

For an exponentially bounded one-parameter $C$-semigroup $T(t)$ with the generator $A$, from [1] the existence of $L_\lambda(A)x = \int_0^\infty e^{-\lambda t}T(t)x \, dt$ is guaranteed for sufficiently large $\lambda \in \mathbb{R}$. Now we have the following lemma for one-parameter $C$-semigroups of operators which is similar to the Yosida-approximation theorem for strongly continuous semigroups. This will be applied in our study of two-parameter regularized semigroups.

**Lemma 2.4.** Let $\{T(t)\}_{t \in \mathbb{R}^+}$ be a one-parameter $C$-semigroup satisfying the condition $\|T(t)\| \leq Me^{\omega t}$, for some $\omega > 0$ and $M > 0$, with the generator $A$. If for $\lambda > \omega$, $A_1 := \lambda AL_\lambda(A)$, then one has the following.

(i) For any $x \in X$, $\|L_\lambda(A)x\| \leq (M/(\lambda - \omega))\|x\|$, $A_1 = \lambda^2 L_\lambda(A) - \lambda C$, and so $A_1$ is bounded. Also $S(t) := Ce^{tA_1}$ is a one-parameter $C$-semigroup which is exponentially bounded.

(ii) For any $x \in D(A)$, $\lim_{s \to -\infty} L_\lambda(A)x = CX$ and for all $x \in D(A)$, $\lim_{s \to -\infty} A_1x = CAx$. Also if $R(C)$ is dense in $X$, then the first equality holds on $X$.

(iii) For any $x \in D(A)$, $T(t)x = \lim_{s \to -\infty} Ce^{sA_1}x$.

**Proof.** The first inequality of (i) is trivial. From [2, Lemma 2.8], we know that for any $x \in X$, $(\lambda - A)L_\lambda(A)x = CX$; thus,
\[ -\lambda(\lambda - A)L_\lambda(A)x = -\lambda CX. \] (2.18)

This implies our desired equality.

For the second part, first we show that $CA_1 = A_1C$. For this we note that
\[
CL_\lambda(A) = C \int_0^\infty e^{-\lambda t}T(t)x \, dx \\
= \int_0^\infty Ce^{-\lambda t}T(t)x \, dx \\
= \int_0^\infty e^{-\lambda t}T(t)Cx \, dx \\
= L_\lambda(A)Cx.
\] (2.19)

This and the first part imply that $CA_1 = A_1C$. Now we prove the $C$-semigroup properties of $S(t)$. Trivially $S(0) = C$. Also from the last equality,
\[
S(s + t)C = Ce^{(s + t)A_1}C = Ce^{sA_1}Ce^{tA_1} = S(s)S(t).
\] (2.20)

The fact that $A_1$, $\lambda > \omega$, is a bounded operator trivially implies that $S(\cdot)$ is exponentially bounded. Now the continuity of the mapping $t \mapsto S(t)x$ at zero implies the strongly continuity of $S(t)$.
To prove (ii), for \( x \in D(A) \), from (i) and the fact that \( A \) is closed, we have

\[
\|AL_1(A)x - Cx\| = \|AL_1(A)x\| \\
= \|L_1(A)Ax\| \\
\leq \|L_1(A)\|\|Ax\| \\
\leq \frac{M}{(\lambda - \omega)}\|Ax\| \to 0 \text{ as } \lambda \to \infty.
\]

(2.21)

The continuity of \( C \) and \( L_1(A) \) implies that for any \( x \in \overline{D(A)} \), \( \lim_{\lambda \to \infty} \lambda L_1(A)x = Cx \).

Now for \( x \in D(A) \),

\[
\lim_{\lambda \to \infty} A_1x = \lim_{\lambda \to \infty} \lambda L_1(A)Ax = CAx = ACx.
\]

(2.22)

For the last part of (ii), if \( C \) has a dense range, then by [8, Lemma 1.1.3], \( R(C) \subseteq \overline{D(A)} \), and so \( X = \overline{R(C)} \subseteq \overline{D(A)} \subseteq X \), which means that \( \overline{D(A)} = X \).

To prove (iii), for any \( x \in D(A) \), we have

\[
\left\| Ce^{tA_1}x - Ce^{tA_\mu}x \right\| = \left\| \int_0^1 \frac{d}{ds} \left( Ce^{sA_1}e^{t(1-s)A_\mu)x} \right) ds \right\| \\
\leq \int_0^1 t\left\| Ce^{sA_1}e^{t(1-s)A_\mu}(A_1x - A_\mu x) \right\| ds \\
\leq t\|C\|\|A_1x - A_\mu x\| \\
\leq t\|C\|(\|A_1x - ACx\| + \|ACx - A_\mu x\|).
\]

This and the previous part prove the existence of \( \lim_{\lambda \to \infty} Ce^{tA_1}x \).

\( \square \)

Using this theorem we may find the following approximation theorem for two-parameter regularized semigroups.

**Corollary 2.5.** Suppose that \((H, K)\) is the infinitesimal generator of an exponentially bounded two-parameter \(C\)-semigroup \(W(s, t)\), then for each \( x \in D(H) \cap D(K) \),

\[
W(s, t)x = C \lim_{\lambda \to \infty} e^{sH_\lambda + tK_\lambda}x.
\]

(2.24)

For exponentially bounded \(C\)-semigroup \(W(s, t)\) satisfying \( \|W(s, t)\| \leq Me^{(s+t)\omega} \), with the infinitesimal generator \((H, K)\), define \( L_1(H)x := \int_0^\infty e^{-\lambda_1 s}W(s, 0)x ds \) and \( L_1(K)x := \int_0^\infty e^{-\lambda_2 s}W(0, t)x dt \), where \( \text{Re}(\lambda_i) > \omega \). From the previous Lemma \( L_1(H) \) and \( L_1(K) \) are bounded operators.
Theorem 2.6. (i) Let \((H, K)\) be the generator of an exponentially bounded two-parameter \(C\)-semigroup, then for large enough \(\lambda_1, \lambda_2\)

\[
L_{\lambda_1}(H)L_{\lambda_2}(K) = L_{\lambda_2}(K)L_{\lambda_1}(H).
\]  

(ii) Let \((H, K)\) be the generator of an exponentially bounded two-parameter \(C\)-semigroup, then

\[
D(H) \cap D(HK) \subseteq D(KH), \text{ and for } x \in D(H) \cap D(HK),
\]

\[
HKx = KHx.
\]

(iii) Suppose that \(H\) and \(K\) are the generators of two exponentially bounded one-parameter \(C\)-semigroups \(\{u(s)\}_{s \in \mathbb{R}_+}\) and \(\{v(t)\}_{t \in \mathbb{R}_+}\), respectively. If their resolvents commute and \(R(C)\) is dense in \(X\), then \(W(s, t) := u(s)v(t)\) is a two-parameter \(C^2\)-semigroup.

**Proof.** The proof of (i) follows trivially from the properties of two-parameter \(C\)-semigroups.

To prove (ii), we let \(x \in D(H) \cap D(HK)\); from the strongly continuity of \(W(s, t)\) and the fact that \(K\) is closed, we have

\[
C^2HKx = C \lim_{s \to 0} \frac{W(s, 0)Kx - CKx}{s}
= \lim_{s \to 0} \frac{1}{s} \left( W(s, 0) \left( \lim_{t \to 0} \frac{W(0, t)x - Cx}{t} \right) - \lim_{t \to 0} \frac{W(0, t)x - Cx}{t} \right)
= \lim_{s \to 0} \lim_{t \to 0} \frac{1}{st} (W(s, 0)W(0, t)x - W(s, 0)Cx - W(0, t)x + Cx)
= \lim_{s \to 0} \lim_{t \to 0} \frac{1}{st} (W(0, t)W(s, 0)x - W(s, 0)Cx - W(0, t)x + Cx)
= \lim_{s \to 0} \lim_{t \to 0} \frac{1}{st} \left( W(0, t) \left( \frac{W(0, s)Cx - Cx}{s} \right) - \frac{W(s, 0)x - Cx}{s} \right)
= C \lim_{s \to 0} \left( \frac{W(s, 0)x - Cx}{s} \right)
= C^2KHx.
\]

However, \(C\) is injective, and this completes the proof of (i).

To prove (iii), from our hypothesis, for sufficiently large \(\lambda, \lambda'\), we know that

\[
L_{\lambda}(H)L_{\lambda'}(K) = L_{\lambda'}(K)L_{\lambda}(H).
\]

By Lemma 2.4, \(H_\lambda = \lambda^2L_{\lambda}(H) - \lambda C\) and \(K_{\lambda'} = \lambda'^2L_{\lambda'}(H) - \lambda' C\), thus \(H_\lambda K_{\lambda'} = K_{\lambda'}H_\lambda\). From (iii) of Lemma 2.4, for each \(x \in D(H) \cap D(K),\)

\[
u(t) = \lim_{\lambda' \to \infty} C e^{tK_{\lambda'}}x.
\]
So

\[ u(s)v(t)x = C \lim_{\lambda \to \infty} e^{sH_1} v(t)x \]

\[ = C^2 \lim_{\lambda \to \infty} e^{sH_1} \left( \lim_{\lambda' \to \infty} e^{\lambda' K_1} x \right). \] 

\[ \left( e^{sH_1} \text{ is continuous} \right) = C^2 \lim_{\lambda \to \infty} \lim_{\lambda' \to \infty} e^{sH_1} e^{\lambda' K_1} x \]

\[ = C^2 \lim_{\lambda \to \infty} e^{\lambda' K_1} e^{sH_1} x \]

\[ = C \lim_{\lambda \to \infty} v(t)e^{sH_1} x \]

\[ = v(t)u(s)x. \]

Now the continuity of \( u(s) \) and \( v(t) \) and the fact that \( \overline{D(H) \cap D(K)} = \overline{R(C)} = X \) imply that for each \( x \in X, u(s)v(t)x = v(t)u(s)x. \) Thus

\[ W(s,t)W(s',t') = u(s)v(t)u(s')v(t') \]

\[ = u(s)u(s')v(t)v(t') \]

\[ = Cu(s+s')Cv(t+t') \]

\[ = W(s+s',t+t')C^2. \]

On the other hand \( W(0,0) = C^2 \), which completes the proof. \( \square \)

If \( H \) and \( K \) are two closed operators on \( X \), then \( X_1 := D(H) \cap D(K) \) with \( \|x\|_1 = \|x\| + \|Hx\| + \|Kx\| \), \( x \in X_1 \), is a Banach space.

**Proposition 2.7.** Suppose that \( C \in B(X) \) is injective and \( \{W(s,t)\} \) is a two-parameter \( C \)-semigroup with the generator \((H,K)\). Then \( W_1(s,t) := W(s,t)|_{X_1} \) defines a two-parameter \( C_1 \)-semigroup, with the generator \((H_1,K_1)\), where \( C_1 = C|_{X_1} \), and \( H_1, K_1 \) are the part of \( H \) and \( K \) on \( X_1 \), respectively.

**Proof.** The \( C_1 \)-semigroup properties of \( W_1(s,t) \) are obvious. Let \((A,B)\) be the generator of \( W_1(s,t)\); we show that \( A = H_1 \) and \( B = H_2 \). First we note that

\[ D(H_1) = \{ x \in X_1 : Hx \in X_1 \} \]

\[ = \left\{ x \in D(H) \cap D(K) : x \in D(H^2) \cap D(KH) \right\} \]

\[ = D(K) \cap D(H^2) \cap D(KH). \]
Let $x \in D(H_1)$. So we have
\[
\frac{W_1(s,0)x - C_1x}{t} = \frac{W(s,0)x - Cx}{t} \to CHx = C_1H_1x,
\]
\[
H \frac{W_1(s,0)x - C_1x}{t} = \frac{W(s,0)x - CHx}{t} \to CH^2x = HC_1H_1x,
\]
\[
K \frac{W_1(s,0)x - C_1x}{t} = \frac{W(s,0)x - CKx}{t} \to CHKx = KCHx = KC_1H_1x.
\]
These show that $(W_1(s,0)x - C_1x)/t \to C_1H_1x$ in $\| \cdot \|_1$, that is, $x \in D(A)$ and $Ax = H_1x$.
Hence $H_1 \subseteq A$. Conversely, if $x \in D(A) \subseteq X_i$, then
\[
\| \cdot \|_1 - \lim_{t \to 0} \frac{W(s,0)x - Cx}{t} = \| \cdot \|_1 - \lim_{t \to 0} \frac{W_1(s,0)x - C_1x}{t}
\]
\[
= C_1Ax
\]
\[
= CAx,
\]
so $Hx = Ax \in X_i$. Hence $x \in D(K) \cap D(H^2) \cap D(KH) = D(H_1)$ and $H_1x = Hx = Ax$.

A similar argument shows that $K_1 = B$, which completes the proof. 

3. Two-Parameter Abstract Cauchy Problems

Suppose that $H_i : D(H_i) \subseteq X \to X$, $i = 1, 2$, is linear operator. Consider the following two-parameter Cauchy problem:
\[
2-ACP(H_1, H_2; x) \left\{ \begin{array}{l}
\frac{\partial}{\partial t_i}u(t_1, t_2) = H_iu(t_1, t_2), \quad t_i > 0, \quad i = 1, 2, \\
u(0, 0) = x,
\end{array} \right. \quad x \in C(D(H_1) \cap D(H_2)).
\]

We mean by a solution a continuous Banach-valued function $u(\cdot, \cdot) : [0, \infty) \times [0, \infty) \to X$ which has continuous partial derivative and satisfies $2-ACP(H_1, H_2; x)$.

In this section first we prove that if $(H_1, H_2)$ is the infinitesimal generator of a two-parameter $C$-semigroup of operators, then $2-ACP(H_1, H_2; x)$ has a unique solution for any $x \in C(D(H_1) \cap D(H_2))$. Next it is proved that under some condition on $C$, existence and uniqueness of solutions of $2-ACP(H_1, H_2; Cx)$, for every $x \in D(H_1) \cap D(H_2)$, imply that this unique solution is induced by a two-parameter regularized semigroup.

**Theorem 3.1.** Suppose that an extension of $(H_1, H_2)$ is the generator of a two-parameter $C$-semigroup $W(s, t)$, then $2-ACP(H_1, H_2; x)$ has the unique solution $u(s, t; x) := W(s, t)C^{-1}x$, for all $x \in C(D(H_1) \cap D(H_2))$. 
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Proof. The fact that \( u(s,t;x) := W(s,t)C^{-1}x \) is a solution of \( 2\text{-ACP}(H_1, H_2; x) \) is obvious from Theorem 2.3. It is enough to show that \( 2\text{-ACP}(H_1, H_2; x) \) has the unique solution \( u(s,t) = 0 \), for the initial value \( x = 0 \). From one-parameter case (see [2]), we know that the systems

\[
\begin{align*}
\frac{du(t)}{dt} &= H_1u(t), \quad t \in \mathbb{R}_+, \quad u(0) = 0, \\
\frac{dv(t)}{dt} &= H_2v(t), \quad t \in \mathbb{R}_+, \quad v(0) = 0
\end{align*}
\]  

have the unique solution zero. Now if \( u(s,t;0) \) is a solution of \( 2\text{-ACP}(H_1, H_2; 0) \), then

\[
u_1(s) := W(s,0)C^{-1}u(0,t;0), \quad u_2(s) := u(s,t;0)
\]

are two solutions of (3.2), for the initial value \( u(0,t;0) \), since

\[
\begin{align*}
\frac{d}{ds}u_1(s) &= \frac{d}{ds}W(s,0)C^{-1}u(0,t;0) \\
&= H_1W(s,0)C^{-1}u(0,t;0) \\
&= H_1u_1(s),
\end{align*}
\]

\[
\begin{align*}
\frac{d}{ds}u_2(s) &= \frac{\partial}{\partial s}u(s,t;0) \\
&= H_1u(s,t;0) \\
&= H_1u_2(s).
\end{align*}
\]

The uniqueness of solution in one-parameter case implies that \( u_1(s) = u_2(s) \). So

\[
W(s,0)C^{-1}u(0,t;0) = u(s,t;0).
\]  

Also \( v_1(t) := W(0,t)C^{-1}u(s,0;0) \) and \( v_2(t) := u(s,t;0) \) are two solutions of (3.3) for the initial value \( u(s,0;0) \). From the uniqueness of solution in (3.3), \( W(0,t)C^{-1}u(s,0;0) = u(s,t;0) \), for all \( s, t \geq 0 \). Thus

\[
u(s,t;0) = W(s,0)C^{-1}u(0,t;0) = W(s,0)C^{-1}W(0,t)u(0,0;0) = 0.
\]

The uniqueness of solution \( 2\text{-ACP}(H, K; Cx) \), for all \( x \in D(H) \cap D(K) \), also leads us to a two-parameter C-semigroup. This will be shown in the following theorem.
In this theorem $X_1$ and $C_1$ have their meaning in Proposition 2.7.

**Theorem 3.2.** Suppose that $C \in B(X)$ is injective and $H, K$ are two closed operators satisfying

$$Cx \in X_1, \quad KCx = CKx, \quad HCx = CHx, \quad \forall x \in X_1.$$  \hspace{1cm} (3.8)

If, for each $x \in X_1$, the Cauchy problem $2$-ACP$(H, K; Cx)$ has a unique solution $u(\cdot, ; Cx)$, then there exists a two-parameter $C_1$-semigroup $W_1(\cdot, \cdot)$ on $X_1$ such that $u(\cdot, ; Cx) = W_1(\cdot, \cdot)x$. Moreover, the infinitesimal generator of $W_1(\cdot, \cdot)$ is a restriction of $(H_1, K_1)$, where $H_1$ and $K_1$ are the part of $H$ and $K$ on $X_1$, respectively.

**Proof.** Suppose that, for any $x \in X_1$, $2$-ACP$(H, K; Cx)$ has a unique solution $u(\cdot, ; Cx) \in C^1([0, \infty) \times [0, \infty), X)$. For $x \in X_1$ and $0 < s, t < \infty$, define $W_1(s, t)x := u(s, t; Cx)$.

From the uniqueness of solution $W_1(s, t)$ is a well-defined and linear operator on $X_1$ and

$$W_1(0, 0)x = u(0, 0; x) = Cx.$$ \hspace{1cm} (3.9)

By uniqueness of solutions one can see that

$$W_1(s + s', t + t')C_1 = W_1(s, t)W_1(s', t').$$ \hspace{1cm} (3.10)

We are going to show that $W_1(s, t)$ is a bounded operator on $(X_1, \| \cdot \|)$. Let $0 < s, t < \infty$. Define the mapping $\phi_{s,t} : X_1 \to C([0, s] \times [0, t], X_1)$ by $\phi_{s,t}x = W_1(\cdot, \cdot)x = u(\cdot, ; Cx)$. Obviously $\phi_{s,t}$ is linear. We claim that this mapping is closed. Suppose that $x_n \in X_1$, $x_n \to x$ and $u(\cdot, ; Cx_n) = \phi_{s,t}(x_n) \to y$ in $C([0, s] \times [0, t], X_1)$ with its usual supremum norm. From the Cauchy problem we know that

$$u(\mu, \nu; Cx_n) = Cx_n + \int_0^\mu Hu(\eta, \nu; Cx_n)d\eta, \quad \int_0^\nu Ku(\mu, \eta; Cx_n)d\eta.$$ \hspace{1cm} (3.11)

Letting $n \to \infty$, we obtain

$$y(\mu, \nu) = Cx + \int_0^\mu H y(\eta, \nu)d\eta, \quad \int_0^\nu Ky(\mu, \eta)d\eta.$$ \hspace{1cm} (3.12)
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for any \((\mu, \nu) \in [0, s] \times [0, t]\). Now define \(\tilde{y}\) on \([0, \infty) \times [0, \infty)\) by

\[
\tilde{y}(\mu, \nu) = \begin{cases} 
Cy(\mu, \nu), & 0 \leq \mu \leq s, \ 0 \leq \nu \leq t, \\
W_1(0, \nu - t)y(\mu, t), & 0 \leq \mu \leq s, \ t < \nu < \infty, \\
W_1(\mu - s, 0)y(s, \nu), & s < \mu < \infty, \ 0 \leq \nu \leq t, \\
W_1(\mu - s, \nu - t)y(s, t), & s < \mu < \infty, \ t < \nu < \infty.
\end{cases} 
\] (3.13)

One can see that \(\tilde{y}\) is a solution of the \(2-ACP(H, K; C^2x)\). Indeed from (3.12)

\[
\tilde{y}(0, 0) = Cy(0, 0) = C^2x. \] (3.14)

Also (3.12) and the fact that \(C\) commutes with \(H\) and \(K\) imply that

\[
\frac{\partial}{\partial \mu} \tilde{y}(\mu, \nu) = \begin{cases} 
Hy(\mu, \nu), & 0 \leq \mu \leq s, \ 0 \leq \nu \leq t, \\
HW_1(0, \nu - t)y(\mu, t), & 0 \leq \mu \leq s, \ t < \nu < \infty, \\
HW_1(\mu - s, 0)y(s, \nu), & s < \mu < \infty, \ 0 \leq \nu \leq t, \\
HW_1(\mu - s, \nu - t)y(s, t), & s < \mu < \infty, \ t < \nu < \infty,
\end{cases} 
\] (3.15)

\[= H\tilde{y}(\mu, \nu).\]

Similarly

\[
\frac{\partial}{\partial \nu} \tilde{y}(\mu, \nu) = K\tilde{y}(\mu, \nu). \] (3.16)

Uniqueness of the solution implies that

\[\tilde{y}(\cdot, \cdot) = u(\cdot, \cdot; Cx^3) = W_1(\cdot, \cdot)Cx = CW_1(\cdot, \cdot)x. \] (3.17)

In particular for \(0 \leq \mu \leq s\) and \(0 \leq \nu \leq s\),

\[Cy(\mu, \nu) = \tilde{y}(\mu, \nu) = CW_1(\mu, \nu)x = C\phi_{s,t}(x)(\mu, \nu). \] (3.18)

The fact that \(C\) is injective implies that \(y = \phi_{s,t}(x)\), which shows that \(\phi_{s,t}\) is a continuous operator.

By the Closed Graph Theorem \(\phi_{s,t}\) is a continuous operator from Banach space \(X_1\) into the Banach space \(C([0, s] \times [0, t], X_1)\). So if \(x_n \to x\) in \(X_1\), then \(\phi_{s,t}(x_n) \to \phi_{s,t}(x)\) in \(C([0, s] \times [0, t], X_1)\); thus for each \((\mu, \nu) \in [0, s] \times [0, t]\),

\[W_1(s, t)x_n = \phi_{s,t}(x_n)(\mu, \nu) \to \phi_{s,t}(x)(\mu, \nu) = W_1(\mu, \nu)x. \] (3.19)
But $s$ and $t$ were arbitrary; hence $W_1(\mu, \nu)$ is continuous for any $\mu, \nu \in [0, \infty)$. Also for every $x \in X_1$, $W_1(\cdot, \cdot)x = \phi_{s,t}(x)$ is continuous on $[0, s] \times [0, t]$; that is, $W_1(\cdot, \cdot)$ is strongly continuous family of operators.

Now let $(A, B)$ be its infinitesimal generator and $x \in D(A)$, then

$$\left\| A \right\| = \lim_{s \to 0} \frac{W_1(s,0)x - C_1x}{s} = C_1Ax,$$

(3.20)

which implies that $\lim_{s \to 0}((W_1(s,0)x - Cx)/s) = CAx$, but $D(A) \subseteq D(H)$

$$\lim_{s \to 0} \frac{W_1(s,0)x - Cx}{s} = \lim_{s \to 0} \frac{u(s,0;Cx) - Cx}{s} = \frac{\partial}{\partial s}u(0,0;Cx)$$

$$= HCx$$

$$= CHx.$$

Hence $CHx = CAx$. The injectivity of $C$ implies that $Hx = Ax \in X_1 \cap D(H) \cap D(K)$. Thus $x \in D(K) \cap D(H^2) \cap D(KH) = D(H_1)$ and $H_1x = Ax$. This shows that $A$ is a restriction of $H_1$. Similarly one can see that $B$ is a restriction of $K_1$, which completes the proof. \(\Box\)

We conclude this section with a simple example as an application of our discussion. Consider the following sequence of initial value problems:

$$\frac{\partial}{\partial s}u_n(s,t) = nu_n(s,t),$$

$$\frac{\partial}{\partial t}u_n(s,t) = n^2u_n(s,t), \quad n \in \mathbb{N},$$

$$u_n(0,0) = e^{-n^2}q_n.$$

(3.22)

Suppose that $X = c_0$, the space of all complex sequences in $\mathbb{C}$ which vanish at infinity. Now define linear operators $H$ and $K$ in $X$ and operator $C$ on $X$ as follows:

$$H(x_n)_{n \in \mathbb{N}} = (nx_n)_{n \in \mathbb{N}}, \quad K(x_n)_{n \in \mathbb{N}} = (n^2x_n)_{n \in \mathbb{N}}, \quad C(x_n)_{n \in \mathbb{N}} = (e^{-n^2}x_n)_{n \in \mathbb{N}}.$$  

(3.23)

Using these operators the initial value problem (3.22) can be rewrite as follows:

$$\frac{\partial}{\partial s}u(s,t) = Hu(s,t),$$

$$\frac{\partial}{\partial t}u(s,t) = Ku(s,t),$$

$$u(0,0) = Cq.$$  

(3.24)
where \( u(s,t) = (u_n(s,t))_{n \in \mathbb{N}} \) and \( q = (q_n)_{n \in \mathbb{N}} \). One can easily see that \((H,K)\) is the generator of the following two-parameter \(C\)-semigroup:

\[
W(s,t)(x_n)_{n \in \mathbb{N}} = (e^{n^2(t-1)+sn}x_n)_{n \in \mathbb{N}} \tag{3.25}
\]

on \( X \). Hence for every \( q = (q_n)_{n \in \mathbb{N}} \in D(H) \cap D(K) \), by Theorem 3.1, the abstract Cauchy problem (3.24) has the unique solution

\[
u(s,t) = W(s,t)q = (e^{n^2(t-1)+sn}q_n)_{n \in \mathbb{N}}. \tag{3.26}\]

This implies that for each \( n \in \mathbb{N} \), \( u_n(s,t) = e^{n^2(t-1)+tn}q_n \) is a solution of (3.22).

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### References


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