Research Article

Some Computational Formulas for D -Nörlund Numbers

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The author establishes some identities involving the D numbers, Bernoulli numbers, and central factorial numbers of the first kind. A generating function and several computational formulas for D -Nörlund numbers are also presented.

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1. Introduction and Results

The Bernoulli polynomials $B_n^{(k)}(x)$ of order k , for any integer k , may be defined by (see [1–5])

$$\left(\frac{t}{e^t - 1}\right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}, \quad |t| < 2\pi. \quad (1.1)$$

The numbers $B_n^{(k)} = B_n^{(k)}(0)$ are the Bernoulli numbers of order k , $B_n^{(1)} = B_n$ are the ordinary Bernoulli numbers (see [2, 6, 7]). By (1.1), we can get (see [4, page 145])

$$\frac{d}{dx} B_n^{(k)}(x) = n B_{n-1}^{(k)}(x), \quad (1.2)$$

$$B_n^{(k+1)}(x) = \frac{k-n}{k} B_n^{(k)}(x) + (x-k) \frac{n}{k} B_{n-1}^{(k)}(x), \quad (1.3)$$

$$B_n^{(k+1)}(x+1) = \frac{nx}{k} B_{n-1}^{(k)}(x) - \frac{n-k}{k} B_n^{(k)}(x), \quad (1.4)$$

where $n \in \mathbb{N}$, with $\mathbb{N}$ being the set of positive integers.

The numbers $B_n^{(n)}$ are called the Nörlund numbers (see [4, 8]). A generating function for the Nörlund numbers $B_n^{(n)}$ is (see [4, page 150])

$$\frac{t}{(1+t)\log(1+t)} = \sum_{n=0}^{\infty} B_n^{(n)} \frac{t^n}{n!}. \quad (1.5)$$

The D numbers $D_{2n}^{(k)}$ may be defined by (see [4, 5])

$$(t \csc t)^k = \sum_{n=0}^{\infty} (-1)^n D_{2n}^{(k)} \frac{t^{2n}}{(2n)!}, \quad |t| < \pi. \quad (1.6)$$

By (1.1), (1.6), and note that $\csc t = 2i/(e^{it} - e^{-it})$ (where $i^2 = -1$), we can get

$$D_{2n}^{(k)} = 4^n B_{2n}^{(k)} \left(\frac{k}{2}\right). \quad (1.7)$$

Taking $k = 1, 2$ in (1.7), and note that $B_{2n}^{(1)}(1/2) = (2^{1-2n} - 1)B_{2n}$, $B_{2n}^{(2)}(1) = (1 - 2n)B_{2n}$ (see [4, pages 22 and 145]), we have

$$D_{2n}^{(1)} = (2 - 2^{2n})B_{2n}, \quad D_{2n}^{(2)} = 4^n(1 - 2n)B_{2n}. \quad (1.8)$$

The numbers $D_{2n}^{(2n)}$ are called the D -Nörlund numbers. These numbers $D_{2n}^{(2n)}$ and $D_{2n}^{(2n-1)}$ have many important applications. For example (see [4, page 246])

$$\int_0^{\pi/2} \frac{\sin t}{t} dt = \sum_{n=0}^{\infty} \frac{(-1)^n D_{2n}^{(2n)}}{(2n+1)!}, \quad \int_0^{\pi/2} \frac{\sin t}{t} dt = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} D_{2n}^{(2n-1)}}{2^{2n}(2n-1)(n!)^2}, \quad (1.9)$$

$$\frac{2}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} D_{2n}^{(2n-1)}}{(2n-1)(2n)!}. \quad (1.10)$$

We now turn to the central factorial numbers $t(n, k)$ of the first kind, which are usually defined by (see [9–12])

$$x\left(x + \frac{n}{2} - 1\right)\left(x + \frac{n}{2} - 2\right) \cdots \left(x + \frac{n}{2} - n + 1\right) = \sum_{k=0}^n t(n, k) x^k, \quad (1.11)$$

or by means of the following generating function:

$$\left(2 \log \left(\frac{x}{2} + \sqrt{1 + \frac{x^2}{4}} \right)\right)^k = k! \sum_{n=k}^{\infty} t(n, k) \frac{x^n}{n!}. \quad (1.12)$$

It follows from (1.11) or (1.12) that

$$t(n, k) = t(n-2, k-2) - \frac{1}{4}(n-2)^2 t(n-2, k), \quad (1.13)$$

and that

$$\begin{aligned} t(n, 0) &= \delta_{n,0} \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}), & t(n, n) &= 1 \quad (n \in \mathbb{N}), \\ t(n, k) &= 0 \quad (n+k \text{ odd}), & t(n, k) &= 0 \quad (k > n \text{ or } k < 0), \end{aligned} \quad (1.14)$$

where $\delta_{m,n}$ denotes the Kronecker symbol.

By (1.13), we have

$$t(2n+1, 1) = \frac{(-1)^n (2n)!}{4^{2n}} \binom{2n}{n}, \quad t(2n+2, 2) = (-1)^n (n!)^2 \quad (n \in \mathbb{N}_0), \quad (1.15)$$

$$t(2n+2, 4) = (-1)^{n+1} (n!)^2 \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \right) \quad (n \in \mathbb{N}), \quad (1.16)$$

$$t(2n+1, 3) = \frac{(-1)^{n-1} (2n)!}{4^{2n-1}} \binom{2n}{n} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n-1)^2} \right) \quad (n \in \mathbb{N}). \quad (1.17)$$

The main purpose of this paper is to prove some identities involving D numbers, Bernoulli numbers, and central factorial numbers of the first kind and obtain a generating function and several computational formulas for the D -Nörlund numbers. That is, we will prove the following main conclusion.

Theorem 1.1. *Let $n \in \mathbb{N}$, $k \in \mathbb{N} \setminus \{1\}$. Then*

$$D_{2n}^{(k)} = \frac{(2n-k+2)(2n-k+1)}{(k-2)(k-1)} D_{2n}^{(k-2)} - \frac{2n(2n-1)(k-2)}{k-1} D_{2n-2}^{(k-2)}. \quad (1.18)$$

Remark 1.2. By (1.18), we may immediately deduce the following (see [4, page 147]):

$$D_{2n}^{(2n+1)} = \frac{(-1)^n (2n)!}{4^n} \binom{2n}{n}, \quad D_{2n}^{(2n+2)} = \frac{(-1)^n 4^n}{2n+1} (n!)^2. \quad (1.19)$$

Theorem 1.3. *Let $n \geq k$ ($n, k \in \mathbb{N}_0$). Then*

$$D_{2n-2k}^{(2n+1)} = \frac{4^{n-k}}{\binom{2n}{2k}} t(2n+1, 2k+1), \quad (1.20)$$

$$D_{2n-2k}^{(2n)} = \frac{4^{n-k}}{\binom{2n-1}{2k-1}} t(2n, 2k) \quad (k \geq 1). \quad (1.21)$$

Remark 1.4. By (1.20) and (1.17), we may immediately deduce the following:

$$D_{2n}^{(2n+3)} = \frac{(-1)^n (2n)!}{2 \cdot 4^{2n}} \binom{2n+2}{n+1} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n+1)^2} \right). \quad (1.22)$$

Theorem 1.5. *Let $n \in \mathbb{N}_0$. Then*

$$\sum_{j=0}^n \frac{(-1)^j}{4^j (2j+1)} \binom{2j}{j} \frac{D_{2n-2j}^{(2n-2j)}}{(2n-2j)!} = \frac{(-1)^n}{4^n} \binom{2n}{n}, \quad (1.23)$$

so one finds $D_0^{(0)} = 1, D_2^{(2)} = -2/3, D_4^{(4)} = 88/15, D_6^{(6)} = -3056/21, D_8^{(8)} = 319616/45, D_{10}^{(10)} = -18940160/33, \dots$

By (1.23), and note that

$$\begin{aligned} \log(t + \sqrt{1+t^2}) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n (2n+1)} \binom{2n}{n} t^{2n+1} \quad (|t| < 1), \\ \frac{1}{\sqrt{1+t^2}} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} \binom{2n}{n} t^{2n} \quad (|t| < 1), \end{aligned} \quad (1.24)$$

one may immediately deduce the following Corollary 1.6.

Corollary 1.6. *Let $n \in \mathbb{N}_0$. Then*

$$\sum_{n=0}^{\infty} D_{2n}^{(2n)} \frac{t^{2n}}{(2n)!} = \frac{t}{\sqrt{1+t^2} \log(t + \sqrt{1+t^2})} \quad (|t| < 1). \quad (1.25)$$

Theorem 1.7. *Let $n \in \mathbb{N}$. Then*

(i)

$$D_{2n}^{(2n)} = \frac{(-1)^n (2n)!}{4^n} \binom{2n}{n} + n \cdot 4^n \sum_{j=1}^n \frac{(1 - 2^{1-2j})}{j} t(2n, 2j) B_{2j} \quad (1.26)$$

(ii)

$$D_{2n}^{(2n)} = \frac{(-1)^n 4^n (n!)^2}{2n+1} + n \cdot 4^n \sum_{j=1}^n \frac{2j-1}{j} t(2n, 2j) B_{2j} \quad (1.27)$$

(iii)

$$D_{2n}^{(2n)} = \sum_{j=0}^n \frac{4^{n-j}}{2j+1} t(2n+1, 2j+1). \quad (1.28)$$

Theorem 1.8. *Let $n \in \mathbb{N}_0$. Then*

(i)

$$\sum_{j=0}^n \frac{2-4^j}{4^j} t(2n+1, 2j+1) B_{2j} = \frac{(-1)^n (n!)^2}{2n+1} \quad (1.29)$$

(ii)

$$\sum_{j=0}^n (1-2j) t(2n+1, 2j+1) B_{2j} = \frac{(-1)^n (2n)!}{2 \cdot 4^{2n}} \binom{2n+2}{n+1} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n+1)^2} \right). \quad (1.30)$$

2. Proof of the Theorems

Proof of Theorem 1.1. By (1.4) and (1.3), we have

$$\begin{aligned} B_{2n}^{(k)}(x+1) &= \frac{2nx}{k-1} B_{2n-1}^{(k-1)}(x) - \frac{2n-k+1}{k-1} B_{2n}^{(k-1)}(x) \\ &= \frac{2nx}{k-1} \left(\frac{k-2n-1}{k-2} B_{2n-1}^{(k-2)}(x) + (x-k+2) \frac{2n-1}{k-2} B_{2n-2}^{(k-2)}(x) \right) \\ &\quad - \frac{2n-k+1}{k-1} \left(\frac{k-2n-2}{k-2} B_{2n}^{(k-2)}(x) + (x-k+2) \frac{2n}{k-2} B_{2n-1}^{(k-2)}(x) \right) \\ &= \frac{(2n-k+1)(2n-k+2)}{(k-1)(k-2)} B_{2n}^{(k-2)}(x) + \frac{2n(2n-1)}{(k-1)(k-2)} x(x-k+2) B_{2n-2}^{(k-2)}(x) \\ &\quad - \frac{2n(2n-k+1)}{(k-1)(k-2)} (2x-k+2) B_{2n-1}^{(k-2)}(x). \end{aligned} \quad (2.1)$$

Setting $x = (k-2)/2$ in (2.1), we get

$$B_{2n}^{(k)}\left(\frac{k}{2}\right) = \frac{(2n-k+1)(2n-k+2)}{(k-1)(k-2)} B_{2n}^{(k-2)}\left(\frac{k-2}{2}\right) - \frac{2n(2n-1)(k-2)}{4(k-1)} B_{2n-2}^{(k-2)}\left(\frac{k-2}{2}\right). \quad (2.2)$$

By (2.2) and (1.7), we immediately obtain (1.18). This completes the proof of Theorem 1.1. $\square$

Proof of Theorem 1.3. By the usage of Theorem 1.1 and (1.13). $\square$

Proof of Theorem 1.5. Note the identity (see [4, page 203])

$$\begin{aligned} B_{2n+1}^{(k)}\left(x + \frac{k}{2}\right) \\ = \sum_{j=0}^n \binom{2n+1}{2j+1} \frac{D_{2n-2j}^{(k-2j-1)}}{2^{2n-2j}} x \left(x^2 - \left(\frac{1}{2}\right)^2\right) \left(x^2 - \left(\frac{3}{2}\right)^2\right) \cdots \left(x^2 - \left(\frac{2j-1}{2}\right)^2\right), \end{aligned} \quad (2.3)$$

we have

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{B_{2n+1}^{(2n+1)}(x + (2n+1)/2)}{x} &= \frac{1}{4^n} \sum_{j=0}^n \binom{2n+1}{2j+1} D_{2n-2j}^{(2n-2j)} (-1)^j 1^2 \cdot 3^2 \cdots (2j-1)^2 \\ &= \frac{(2n+1)!}{4^n} \sum_{j=0}^n \frac{(-1)^j}{4^j (2j+1)} \binom{2j}{j} \frac{D_{2n-2j}^{(2n-2j)}}{(2n-2j)!}. \end{aligned} \quad (2.4)$$

By (2.4) and (1.2), we have

$$\lim_{x \rightarrow 0} (2n+1) B_{2n}^{(2n+1)}\left(x + \frac{2n+1}{2}\right) = \frac{(2n+1)!}{4^n} \sum_{j=0}^n \frac{(-1)^j}{4^j (2j+1)} \binom{2j}{j} \frac{D_{2n-2j}^{(2n-2j)}}{(2n-2j)!}, \quad (2.5)$$

that is,

$$B_{2n}^{(2n+1)}\left(\frac{2n+1}{2}\right) = \frac{(2n)!}{4^n} \sum_{j=0}^n \frac{(-1)^j}{4^j (2j+1)} \binom{2j}{j} \frac{D_{2n-2j}^{(2n-2j)}}{(2n-2j)!}. \quad (2.6)$$

By (2.6) and (1.7), we have

$$D_{2n}^{(2n+1)} = (2n)! \sum_{j=0}^n \frac{(-1)^j}{4^j (2j+1)} \binom{2j}{j} \frac{D_{2n-2j}^{(2n-2j)}}{(2n-2j)!}. \quad (2.7)$$

By (2.7) and (1.19), we immediately obtain (1.23). This completes the proof of Theorem 1.5. $\square$

Proof of Theorem 1.7. By (1.6), we have

$$D_{2n}^{(k)} = \sum_{j=0}^n \binom{2n}{2j} D_{2n-2j}^{(k-l)} D_{2j}^{(l)}, \quad (2.8)$$

where l is an integer.

Setting $k = 2n + 1, l = 1$ in (2.8), and note that $D_0^{(1)} = 1$, we have

$$D_{2n}^{(2n+1)} = \sum_{j=0}^n \binom{2n}{2j} D_{2n-2j}^{(2n)} D_{2j}^{(1)} = D_{2n}^{(2n)} + \sum_{j=1}^n \binom{2n}{2j} D_{2n-2j}^{(2n)} D_{2j}^{(1)}. \quad (2.9)$$

By (2.9), (1.19), (1.8), and (1.21), we immediately obtain (1.26).

Setting $k = 2n + 2, l = 2$ in (2.8), and note that $D_0^{(2)} = 1$, we have

$$D_{2n}^{(2n+2)} = \sum_{j=0}^n \binom{2n}{2j} D_{2n-2j}^{(2n)} D_{2j}^{(2)} = D_{2n}^{(2n)} + \sum_{j=1}^n \binom{2n}{2j} D_{2n-2j}^{(2n)} D_{2j}^{(2)}. \quad (2.10)$$

By (2.10), (1.19), (1.8), and (1.21), we immediately obtain (1.27).

Setting $k = 2n, l = -1$ in (2.8), and note that (1.20) and $D_{2j}^{(-1)} = 1/(2j + 1)$, we immediately obtain (1.18). This completes the proof of Theorem 1.7. $\square$

Proof of Theorem 1.8. Setting $k = 2n + 2, l = 1$ in (2.8), and note (1.19), (1.20), and (1.8), we immediately obtain (1.29).

Setting $k = 2n + 3, l = 2$ in (2.8), and note (1.22), (1.20), and (1.8), we immediately obtain (1.30). This completes the proof of Theorem 1.8. $\square$

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