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*Research Article*

# Permanence of Periodic Predator-Prey System with General Nonlinear Functional Response and Stage Structure for Both Predator and Prey

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We study the permanence of periodic predator-prey system with general nonlinear functional responses and stage structure for both predator and prey and obtain that the predator and the prey species are permanent.



## 1. Introduction

An important and ubiquitous problem in predator-prey theory and related topics in mathematical ecology concerns the long-term coexistence of species. In the natural world there are many species whose individual members have a life history that takes them through two stages: immature and mature. In particular, we have in mind mammalian populations and some amphibious animals, which exhibit these two stages. Recently, nonautonomous systems with a stage structure have been considered in [1–16]; in particular periodic predator-prey systems with a stage structure were discussed in [3, 4, 7, 13, 14].

Already, in [3], Cui and Song proposed the following predator-prey model with stage structure for prey:

$$\begin{aligned}
\dot{x}_1(t) &= a(t)x_2(t) - b(t)x_1(t) - d(t)x_1^2(t) - p(t)x_1(t)y(t),\\
\dot{x}_2(t) &= c(t)x_1(t) - f(t)x_2^2(t),\\
\dot{y}(t) &= y(t)\big(-g(t) + h(t)x_1(t) - q(t)y(t)\big).
\end{aligned} \tag{1.1}$$

They obtained a set of sufficient and necessary condition which guarantee the permanence of the system.

In [4], Cui and Takeuchi considered the following periodic predator-prey system with a stage structure:

$$\begin{aligned}
\dot{x}_1(t) &= a(t)x_2(t) - b(t)x_1(t) - d(t)x_1^2(t) - p(t)\phi(t,x_1)x_1y(t),\\
\dot{x}_2(t) &= c(t)x_1(t) - f(t)x_2^2(t),\\
\dot{y}(t) &= y(t)\big(-g(t) + h(t)\phi(t,x_1)y(t) - q(t)y(t)\big),
\end{aligned} \tag{1.2}$$

where

$$0 < \phi(t,x_1) < L, \quad \frac{\partial}{\partial x_1}\big(\phi(t,x_1)x_1\big) \geq 0 \quad (x_1 > 0). \tag{1.3}$$

Recently, Huang et al. [7] studied the following periodic stage-structured three-species predator-prey system with Holling IV and Beddington-DeAngelis functional response:

$$\begin{aligned}
\dot{x}_1(t) &= a(t)x_2(t) - b(t)x_1(t) - d(t)x_1^2(t) - \frac{h_1(t)x_1(t)}{k_1(t) + x_1^2(t)}y_1(t),\\
\dot{x}_2(t) &= c(t)x_1(t) - f(t)x_2^2(t) - \frac{h_2(t)x_2(t)}{k_2(t) + m(t)x_2(t) + n(t)y_2(t)}y_2(t),\\
\dot{y}_1(t) &= y_1(t)\left(-q_1(t) + \frac{p_1(t)x_1(t)}{k_1(t) + x_1^2(t)} - g_1(t)y_1(t)\right),\\
\dot{y}_2(t) &= y_2(t)\left(-q_2(t) + \frac{p_2(t)x_2(t)}{k_2(t) + m(t)x_2(t) + n(t)y_2(t)} - g_2(t)y_2(t)\right),
\end{aligned} \tag{1.4}$$

where $a(t)$, $b(t)$, $c(t)$, $d(t)$, $f(t)$, $m(t), n(t)$, $h_i(t)$, $k_i(t)$, $p_i(t)$, $q_i(t)$, and $g_i(t)$ $(i = 1,2)$ are all continuous positive $\omega$-periodic functions; $x_1(t)$ and $x_2(t)$ denote the density of immature and mature prey species at time $t$, respectively; $y_1(t)$ represents the density of the predator that preys on immature prey; $y_1(t)$ represents the density of the other predator that preys on mature prey at time $t$.

It is assumed in the classical predator-prey model that each individual predator admits the same ability to attack prey and each individual prey admits the same risk to be attacked by predator. This assumption seems not to be realistic for many animals. On the other hand, predator-prey systems where only immature individuals are consumed by their predator are well known in nature. One example is described in [9], where the Chinese fire-bellied newt, which is unable to prey upon the mature rana chensinensis, can only prey on its immature.

To the best of the authors' knowledge, for the nonautonomous periodic case of predator-prey systems with stage structure for both predator and prey, whether one could obtain the permanence of the system or not is still an open problem.

Motivated by the above question, we consider the following periodic predator-prey system with general nonlinear functional responses and stage structure for both predator and prey:

$$\begin{aligned}
\dot{x}_1(t) &= a_1(t)x_2(t) - b_1(t)x_1(t) - d_1(t)x_1^2(t) - g(x_1(t))y_2(t),\\
\dot{x}_2(t) &= c_1(t)x_1(t) - f_1(t)x_2^2(t),\\
\dot{y}_1(t) &= a_2(t)y_2(t) - b_2(t)y_1(t) - d_2(t)y_1^2(t) + k(t)g(x_1(t))y_1(t),\\
\dot{y}_2(t) &= c_2(t)y_1(t) - f_2(t)y_2^2(t),
\end{aligned} \tag{1.5}$$

where $a_i(t), b_i(t), c_i(t), d_i(t), f_i(t), i = 1,2$, and $k(t)$ are all continuous positive $\omega$-periodic functions. Here $x_1(t)$ and $x_2(t)$ denote the density of immature and mature prey species, respectively, and $y_1(t)$ and $y_2(t)$ denote the density of immature and mature predator species, respectively. The function $g(x)$ is assumed to satisfy the following assumptions which has been studied in detail by Georgescu and Morosanu in [17].

(G) $g(x)$ of class $C^1$ is increasing on $R_+, g(0) = 0$, and such that $x \mapsto g(x)/x$ is decreasing on $R_+, |g'(x)| \le L$ for $x \in R_+$, where $L \ge 0$.

Note that hypothesis (G) is satisfied if function $g(x)$ represents Holling type II functional response, that is, $g(x) = ax/(1+bx)$, in which $a$ is the search rate of the resource and of the intermediate consumer, and $b$ represents the corresponding clearance rate, that is, search rate multiplied by the (supposedly constant) handling time.

The aim of this paper is, by further developing the analysis technique of Cui and Song [3] and Cui and Takeuchi [4], to obtain a set of sufficient and necessary conditions which ensure the permanence of the system (1.5). The rest of the paper is arranged as follows. In Section 2, we introduce some lemmas and then state the main result of this paper. The result is proved in Section 3.

## 2. Main Results

*Definition 2.1.* The system

$$\dot{x} = F(t,x), \quad x \in R^n \tag{2.1}$$

is said to be permanent if there exists a compact set $K$ in the interior of $R_+^n = \{(x_1, x_2, \ldots, x_n) \in R^n \mid x_i \ge 0,\ i = 1,2,\ldots,n\}$, such that all solutions starting in the interior of $R_+^n$ ultimately enter $K$ and remain in $K$.

**Lemma 2.2** (see [6]). *If $a(t), b(t), c(t), d(t)$, and $f(t)$ are all $\omega$-periodic, then system*

$$\begin{aligned}
\dot{x}_1(t) &= a(t)x_2(t) - b(t)x_1(t) - d(t)x_1^2(t),\\
\dot{x}_2(t) &= c(t)x_1(t) - f(t)x_2^2(t)
\end{aligned} \tag{2.2}$$

*has a positive $\omega$-periodic solution $(x_1^*(t), x_2^*(t))$ which is globally asymptotically stable with respect to $R_+^2 = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$.*

**Theorem 2.3.** *System* (1.5) *is permanent.*

## 3. Proof of the Main Results

We need the following propositions to prove Theorem 2.3. The hypotheses of the lemmas and theorems of the preceding section are assumed to hold in what follows.

**Proposition 3.1.** *There exists a positive constant $M_x$ such that*

$$\limsup_{t\to\infty} x_i(t) \le M_x, \quad i = 1,2. \tag{3.1}$$

*Proof.* Obviously, $R_4^+$ is a positively invariant set of system (1.5). Given any positive solution $(x_1(t), x_2(t), y_1(t), y_2(t))$ of (1.5), from the first and second equations of system(1.5), we have

$$\begin{aligned} \dot{x}_1(t) &\le a_1(t)x_2(t) - b_1(t)x_1(t) - d_1(t)x_1^2(t), \\ \dot{x}_2(t) &\le c_1(t)x_1(t) - f_1(t)x_2^2(t). \end{aligned} \tag{3.2}$$

By Lemma 2.2, the following auxiliary equation

$$\begin{aligned} \dot{u}_1(t) &= a_1(t)u_2(t) - b_1(t)u_1(t) - d_1(t)u_1^2(t), \\ \dot{u}_2(t) &= c_1(t)u_1(t) - f_1(t)u_2^2(t) \end{aligned} \tag{3.3}$$

has a globally asymptotically stable positive $\omega$-periodic solution $(x_1^*(t), x_2^*(t))$. Let $(u_1(t), u_2(t))$ be the solution of (3.3) with $u_i(0) = x_i(0)$. By comparison, we then have

$$x_i(t) \le u_i(t), \quad i = 1,2 \tag{3.4}$$

for $t \ge 0$. From the global attractivity of $(x_1^*(t), x_2^*(t))$, for any positive $\varepsilon > 0$ small enough, there exists a $T_0 > 0$ such that

$$\left|u_i(t) - x_i^*(t)\right| < \varepsilon, \quad t \ge T_0 \tag{3.5}$$

(3.4) combined with (3.5) leads to

$$x_i(t) < x_i^*(t) + \varepsilon, \quad t > T_0. \tag{3.6}$$

Let $M_x = \max_{0\le t\le\omega}\{x_i^*(t) + \varepsilon : i = 1,2\}$, then we have

$$\limsup_{t\to\infty} x_i(t) \le M_x. \tag{3.7}$$

This completes the proof of Proposition 3.1. □

**Proposition 3.2.** *There exists a positive constant $m_y$ such that*

$$\liminf_{t\to\infty} y_i(t) \ge m_y, \quad i = 1,2. \tag{3.8}$$

*Proof.* Given any positive solution $(x_1(t), x_2(t), y_1(t), y_2(t))$ of (1.5), from the third and fourth equations of system (1.5), we have

$$\begin{aligned} \dot{y}_1(t) &\geq a_2(t)y_2(t) - b_2(t)y_1(t) - d_2(t)y_1^2(t), \\ \dot{y}_2(t) &\geq c_2(t)y_1(t) - f_2(t)y_2^2(t). \end{aligned} \tag{3.9}$$

By Lemma 2.2, the following auxiliary equation

$$\begin{aligned} \dot{v}_1(t) &= a_2(t)v_2(t) - b_2(t)v_1(t) - d_2(t)v_1^2(t), \\ \dot{v}_2(t) &= c_2(t)v_1(t) - f_2(t)v_2^2(t) \end{aligned} \tag{3.10}$$

has a globally asymptotically stable positive $\omega$-periodic solution $(y_1^*(t), y_2^*(t))$. Let $(v_1(t), v_2(t))$ be the solution of (3.10) with $v_i(0) = y_i(0)$. By comparison, we then have

$$y_i(t) \geq v_i(t), \qquad i = 1, 2 \tag{3.11}$$

for $t \geq 0$. From the global attractivity of $(y_1^*(t), y_2^*(t))$, for any positive $\varepsilon > 0$ small enough $(\varepsilon < \min_{0\leq t\leq\omega}\{y_i^*(t) : i = 1, 2\})$, there exists a $T_1 > 0$ such that

$$\left|v_i(t) - y_i^*(t)\right| < \varepsilon, \quad t \geq T_1 \tag{3.12}$$

(3.11) combined with (3.12) leads to

$$y_i(t) > y_i^*(t) - \varepsilon, \quad t > T_1. \tag{3.13}$$

Let $m_y = \min_{0\leq t\leq\omega}\{y_i^*(t) - \varepsilon : i = 1, 2\}$, then we have

$$\liminf_{t\to\infty} y_i(t) \geq m_y. \tag{3.14}$$

This completes the proof of Proposition 3.2. □

**Proposition 3.3.** *There exists a positive constant $M_y$ such that*

$$\limsup_{t\to+\infty} y_i(t) \leq M_y, \quad i = 1, 2. \tag{3.15}$$

*Proof.* Given any positive solution $(x_1(t), x_2(t), y_1(t), y_2(t))$ of (1.5), by Proposition 3.1, there exists a $T_2 > 0$ such that

$$0 < x_i(t) \leq M_x, \quad t > T_2. \tag{3.16}$$

From the third and fourth equations of system (1.5), we have

$$\begin{aligned} \dot{y}_1(t) &\le a_2(t)y_2(t) - \big(b_2(t) - k(t)g(M_x)\big)y_1(t) - d_2(t)y_1^2(t), \\ \dot{y}_2(t) &\le c_2(t)y_1(t) - f_2(t)y_2^2(t). \end{aligned} \tag{3.17}$$

By Lemma 2.2, the following auxiliary equation

$$\begin{aligned} \dot{v}_1(t) &= a_2(t)v_2(t) - \big(b_2(t) - k(t)g(M_x)\big)v_1(t) - d_2(t)v_1^2(t), \\ \dot{v}_2(t) &= c_2(t)v_1(t) - f_2(t)v_2^2(t) \end{aligned} \tag{3.18}$$

has a globally asymptotically stable positive $\omega$-periodic solution $(\widetilde{y}_1^*(t), \widetilde{y}_2^*(t))$. Let $(v_1(t), v_2(t))$ be the solution of (3.18) with $v_i(0) = y_i(0)$. By comparison, we then have

$$y_i(t) \le v_i(t), \quad i = 1, 2 \tag{3.19}$$

for $t \ge 0$. From the global attractivity of $(\widetilde{y}_1^*(t), \widetilde{y}_2^*(t))$, for above given positive $\varepsilon > 0$, there exists a $T_3 > 0$ such that

$$\left|v_i(t) - \widetilde{y}_i^*(t)\right| < \varepsilon, \quad t \ge T_3 \tag{3.20}$$

(3.19) combined with (3.20) leads to

$$y_i(t) < \widetilde{y}_i^*(t) + \varepsilon, \quad t > T_3. \tag{3.21}$$

Let $M_y = \max_{0 \le t \le \omega}\{\widetilde{y}_i^*(t) + \varepsilon : i = 1, 2\}$, then we have

$$\limsup_{t \to \infty} y_i(t) \le M_y. \tag{3.22}$$

This completes the proof of Proposition 3.3. □

**Proposition 3.4.** *There exists a positive constant $m_x < M_x$, such that*

$$\liminf_{t \to \infty} x_i(t) \ge m_x, \quad i = 1, 2. \tag{3.23}$$

*Proof.* By Propositions 3.1 and 3.3, there exists a $T_4 > 0$ such that

$$0 < x_i(t) \le M_x; \quad 0 < y_i(t) \le M_y; t > T_4. \tag{3.24}$$

From the first and second equations of system (1.5), we have

$$\begin{aligned} \dot{x}_1(t) &\geq a_1(t)x_2(t) - \left(b_1(t) + \frac{g(x_1(t))}{x_1(t)}M_y\right)x_1(t) - d_1(t)x_1^2(t), \\ \dot{x}_2(t) &\geq c_1(t)x_1(t) - f_1(t)x_2^2(t). \end{aligned} \tag{3.25}$$

Since $g(x)$ is assumed to satisfy the assumptions (G), by the differential mean value theorem, we have

$$\frac{g(x_1(t))}{x_1(t)} = \frac{g(x_1(t)) - g(0)}{x_1(t) - 0} = g'(\xi) \leq L, \quad 0 < \xi < x_1(t). \tag{3.26}$$

From (3.25) and (3.26), one has

$$\begin{aligned} \dot{x}_1(t) &\geq a_1(t)x_2(t) - \left(b_1(t) + LM_y\right)x_1(t) - d_1(t)x_1^2(t), \\ \dot{x}_2(t) &\geq c_1(t)x_1(t) - f_1(t)x_2^2(t). \end{aligned} \tag{3.27}$$

By Lemma 2.2, the following auxiliary equation:

$$\begin{aligned} \dot{u}_1(t) &= a_1(t)u_2(t) - \left(b_1(t) + LM_y\right)u_1(t) - d_1(t)u_1^2(t), \\ \dot{u}_2(t) &= c_1(t)u_1(t) - f_1(t)u_2^2(t) \end{aligned} \tag{3.28}$$

has a globally asymptotically stable positive $\omega$-periodic solution $(\tilde{x}_1^*(t), \tilde{x}_2^*(t))$. Let $(u_1(t), u_2(t))$ be the solution of (3.27) with $u_i(0) = x_i(0)$. By comparison, we then have

$$x_i(t) \geq u_i(t), \quad i = 1, 2 \tag{3.29}$$

for $t \geq 0$. From the global attractivity of $(\tilde{x}_1^*(t), \tilde{x}_2^*(t))$, for any positive $\varepsilon > 0$ small enough $(\varepsilon < \min_{0\leq t\leq\omega}\{\tilde{x}_i^*(t) : i = 1, 2\})$, there exists a $T_5 > 0$ such that

$$\left|u_i(t) - \tilde{x}_i^*(t)\right| < \varepsilon, \quad t \geq T_5 \tag{3.30}$$

(3.29) combined with (3.30) leads to

$$x_i(t) > \tilde{x}_i^*(t) - \varepsilon, \quad t > T_5. \tag{3.31}$$

Let $m_x = \min_{0\leq t\leq\omega}\{\tilde{x}_i^*(t) - \varepsilon : i = 1, 2\}$, then we have

$$\liminf_{t\to\infty} x_i(t) \geq m_x. \tag{3.32}$$

This completes the proof of Proposition 3.4. □

*Proof of Theorem 2.3.* By Propositions 3.1, 3.2, 3.3, and 3.4, system (1.5) is permanent. This completes the proof of Theorem 2.3. □

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