Research Article

Some New Wilker-Type Inequalities for Circular and Hyperbolic Functions

Ling Zhu

Department of Mathematics, Zhejiang Gongshang University, Hangzhou, Zhejiang 310018, China

Correspondence should be addressed to Ling Zhu, zhuling0571@163.com

Received 4 March 2009; Accepted 11 May 2009

Recommended by Ferhan Atici

In this paper, we give some new Wilker-type inequalities for circular and hyperbolic functions in exponential form by using generalizations of Cusa-Huygens inequality and Cusa-Huygens-type inequality.

Copyright © 2009 Ling Zhu. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

Wilker [1] proposed two open questions, the first of which was the following statement.

Problem 1. Let $0 < x < \pi/2$. Then

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 \quad (1.1)$$

holds.

Sumner et al. [2] proved inequality (1.1). Guo et al. [3] gave a new proof of inequality (1.1). Zhu [4, 5] showed two new simple proofs of Wilker's inequality above, respectively.

Recently, Wu and Srivastava [6] obtained Wilker-type inequality as follows:

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > 2, \quad 0 < x < \frac{\pi}{2}. \quad (1.2)$$

Baricz and Sandor [7] found that inequality (1.2) can be proved by using inequality (1.1).

On the other hand, in the form of inequality (1.1), Zhu [5] obtained the following Wilker type inequality:

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > 2, \quad x > 0. \quad (1.3)$$

In fact, we can obtain further results:

$$\begin{aligned} \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} &> \left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > 2, \quad 0 < x < \frac{\pi}{2}, \\ \left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} &> \left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x} > 2, \quad x > 0. \end{aligned} \quad (1.4)$$

In this note, we establish the following four new Wilker type inequalities in exponential form for circular and hyperbolic functions.

Theorem 1.1. *Let $0 < x < \pi/2$, $\alpha \in \mathbb{R}$ and $\alpha \neq 0$. Then*

(i) *when $\alpha > 0$, the inequality*

$$\left(\frac{\sin x}{x}\right)^{2\alpha} + \left(\frac{\tan x}{x}\right)^\alpha > \left(\frac{x}{\sin x}\right)^{2\alpha} + \left(\frac{x}{\tan x}\right)^\alpha \quad (1.5)$$

holds;

(ii) *when $\alpha < 0$, inequality (1.5) is reversed.*

Theorem 1.2. *Let $0 < x < \pi/2$ and $\alpha \geq 1$. Then the inequality*

$$\left(\frac{\sin x}{x}\right)^{2\alpha} + \left(\frac{\tan x}{x}\right)^\alpha > \left(\frac{x}{\sin x}\right)^{2\alpha} + \left(\frac{x}{\tan x}\right)^\alpha > 2 \quad (1.6)$$

holds.

Theorem 1.3. *Let $x > 0$, $\alpha \in \mathbb{R}$ and $\alpha \neq 0$. Then*

(i) *when $\alpha > 0$, the inequality*

$$\left(\frac{\sinh x}{x}\right)^{2\alpha} + \left(\frac{\tanh x}{x}\right)^\alpha > \left(\frac{x}{\sinh x}\right)^{2\alpha} + \left(\frac{x}{\tanh x}\right)^\alpha \quad (1.7)$$

holds;

(ii) *when $\alpha < 0$, inequality (1.7) is reversed.*

Theorem 1.4. Let $x > 0$ and $\alpha \geq 1$. Then the inequality

$$\left(\frac{\sinh x}{x}\right)^{2\alpha} + \left(\frac{\tanh x}{x}\right)^{\alpha} > \left(\frac{x}{\sinh x}\right)^{2\alpha} + \left(\frac{x}{\tanh x}\right)^{\alpha} > 2 \quad (1.8)$$

holds.

2. Lemmas

Lemma 2.1 (see [8–24]). Let $f, g : [a, b] \rightarrow \mathbb{R}$ be two continuous functions which are differentiable on (a, b) . Further, let $g' \neq 0$ on (a, b) . If f'/g' is increasing (or decreasing) on (a, b) , then the functions $(f(x) - f(b))/(g(x) - g(b))$ and $(f(x) - f(a))/(g(x) - g(a))$ are also increasing (or decreasing) on (a, b) .

Lemma 2.2 (see [25–27]). Let a_n and b_n ($n = 0, 1, 2, \dots$) be real numbers, and let the power series $A(t) = \sum_{n=0}^{\infty} a_n t^n$ and $B(t) = \sum_{n=0}^{\infty} b_n t^n$ be convergent for $|t| < R$. If $b_n > 0$ for $n = 0, 1, 2, \dots$, and if a_n/b_n is strictly increasing (or decreasing) for $n = 0, 1, 2, \dots$, then the function $A(t)/B(t)$ is strictly increasing (or decreasing) on $(0, R)$.

Lemma 2.3 (see [28, 29]). Let $|x| < \pi$, then the inequality

$$\frac{x}{\sin x} = 1 + \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| x^{2n} \quad (2.1)$$

holds.

Lemma 2.4. Let $|x| < \pi$, then the inequality

$$\frac{1}{\sin^2 x} = \csc^2 x = \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| (2n-1) x^{2n-2} \quad (2.2)$$

holds.

Proof. The following power series expansion can be found in [30, 1.3.1.4 (3)]

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n-1}, \quad |x| < \pi. \quad (2.3)$$

Then

$$\frac{1}{\sin^2 x} = \csc^2 x = -(\cot x)' = \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| (2n-1) x^{2n-2}, \quad |x| < \pi. \quad (2.4)$$

□

Lemma 2.5 (see [5, 31]). *Let $0 < x < \pi/2$. Then the inequality*

$$\left(\frac{\sin x}{x}\right)^3 > \cos x \quad (2.5)$$

holds.

Lemma 2.6 (see [5, 31, 32]). *Let $x > 0$. Then the inequality*

$$\left(\frac{\sinh x}{x}\right)^3 > \cosh x \quad (2.6)$$

holds.

Lemma 2.7. *Let $0 < x < \pi/2$. Then the function $G(\alpha) = ((\sin x/x)^{2\alpha} + (\tan x/x)^\alpha) / ((x/\sin x)^{2\alpha} + (x/\tan x)^\alpha)$ increases as α increases on $(-\infty, +\infty)$.*

Lemma 2.8. *Let $x > 0$. Then the function $H(\alpha) = ((\sinh x/x)^{2\alpha} + (\tanh x/x)^\alpha) / ((x/\sinh x)^{2\alpha} + (x/\tanh x)^\alpha)$ increases as α increases on $(-\infty, +\infty)$.*

Lemma 2.9 (a generalization of Cusa-Huygens inequality). *Let $0 < x < \pi/2$ and $\alpha \geq 1$. Then the inequality*

$$2\left(\frac{x}{\sin x}\right)^\alpha + \left(\frac{x}{\tan x}\right)^\alpha > 3 \quad (2.7)$$

or

$$\left(\frac{\sin x}{x}\right)^\alpha < \frac{2}{3} + \frac{1}{3}(\cos x)^\alpha \quad (2.8)$$

holds.

Lemma 2.10 (a generalization of Cusa-Huygens type inequality). *Let $x > 0$ and $\alpha \geq 1$. Then the inequality*

$$2\left(\frac{x}{\sinh x}\right)^\alpha + \left(\frac{x}{\tanh x}\right)^\alpha > 3 \quad (2.9)$$

or

$$\left(\frac{\sinh x}{x}\right)^\alpha < \frac{2}{3} + \frac{1}{3}(\cosh x)^\alpha \quad (2.10)$$

holds.

3. Proofs of Lemma 2.7 and Theorem 1.1

Proof of Lemma 2.7. Direct calculation yields $G'(\alpha) = J(\alpha)/[(x/\sin x)^{2\alpha} + (x/\tan x)^\alpha]^2$, where

$$\begin{aligned}
 J(\alpha) &= \left[\left(\frac{\tan x}{x} \right)^\alpha \left(\frac{x}{\sin x} \right)^{2\alpha} - \left(\frac{x}{\tan x} \right)^\alpha \left(\frac{\sin x}{x} \right)^{2\alpha} + 2 \right] \log \frac{\tan x}{x} \\
 &\quad + 2 \left[\left(\frac{\tan x}{x} \right)^\alpha \left(\frac{x}{\sin x} \right)^{2\alpha} - \left(\frac{x}{\tan x} \right)^\alpha \left(\frac{\sin x}{x} \right)^{2\alpha} - 2 \right] \log \frac{x}{\sin x} \\
 &= \left[\left(\frac{2x}{\sin 2x} \right)^\alpha - \left(\frac{\sin 2x}{2x} \right)^\alpha + 2 \right] \log \frac{\tan x}{x} + 2 \left[\left(\frac{2x}{\sin 2x} \right)^\alpha - \left(\frac{\sin 2x}{2x} \right)^\alpha - 2 \right] \log \frac{x}{\sin x} \\
 &= \log \left[\left(\frac{\tan x}{x} \right)^{(2x/\sin 2x)^\alpha - (\sin 2x/2x)^\alpha + 2} \left(\frac{x^2}{\sin^2 x} \right)^{(2x/\sin 2x)^\alpha - (\sin 2x/2x)^\alpha - 2} \right] \\
 &= \log \left[\left(\frac{2x}{\sin 2x} \right)^{(2x/\sin 2x)^\alpha - (\sin 2x/2x)^\alpha} \left(\left(\frac{\sin x}{x} \right)^3 \frac{1}{\cos x} \right)^2 \right].
 \end{aligned} \tag{3.1}$$

First, we have $[(\sin x/x)^3(1/\cos x)]^2 > 1$ by Lemma 2.5. Second, when letting $2x/\sin 2x = t$ for $0 < x < \pi/2$, we have $t > 1$, and $t^\alpha - t^{-\alpha} > 0$ for $\alpha > 0$, so $t^{t^\alpha - t^{-\alpha}} > 1$ and $(2x/\sin 2x)^{(2x/\sin 2x)^\alpha - (\sin 2x/2x)^\alpha} [(\sin x/x)^3(1/\cos x)]^2 > 1$. Thus $J(\alpha) > 0$ and $G'(\alpha) > 0$. The proof of Lemma 2.7 is complete. $\square$

Proof of Theorem 1.1. From Lemma 2.7 we have $G(\alpha) > G(0) = 1$ for $\alpha > 0$. That is, (1.5) holds. At the same time, we have $G(\alpha) < G(0) = 1$ for $\alpha < 0$. That is, (1.5) is reversed. $\square$

4. Proofs of Lemma 2.9 and Theorem 1.2

Proof of Lemma 2.9. Let $F(x) = ((\sin x/x)^\alpha - 1)/((\cos x)^\alpha - 1) =: f(x)/g(x)$, where $f(x) = (\sin x/x)^\alpha - 1$, and $g(x) = (\cos x)^\alpha - 1$. Then

$$k(x) = \frac{f'(x)}{g'(x)} = \left(\frac{\sin x}{x \cos x} \right)^{\alpha-1} \frac{\sin x - x \cos x}{x^2 \sin x}, \quad k'(x) = \left(\frac{\sin x}{x \cos x} \right)^{\alpha-2} \frac{u(x)}{x^4 \sin x \cos^2 x}, \tag{4.1}$$

where

$$\begin{aligned}
 u(x) &= (\alpha - 1)(x - \sin x \cos x)(\sin x - x \cos x) + \cos x(x^2 - 2 \sin^2 x + x \sin x \cos x) \\
 &= (x \sin x - \sin^2 x \cos x - x^2 \cos x + x \cos^2 x \sin x) \alpha \\
 &\quad - (x \sin x + \sin^2 x \cos x - 2x^2 \cos x) \\
 &= (x \sin x - \sin^2 x \cos x - x^2 \cos x + x \cos^2 x \sin x)(\alpha - G(x)),
 \end{aligned} \tag{4.2}$$

where $G(x) = (x \sin x + \sin^2 x \cos x - 2x^2 \cos x) / (x \sin x - \sin^2 x \cos x - x^2 \cos x + x \cos^2 x \sin x)$. Then

$$G(x) = \frac{2x / \sin 2x + 1 - 2x^2 / \sin^2 x}{2x / \sin 2x - 1 - (x / \sin x)^2 + x \cot x} := \frac{A(x)}{B(x)}, \quad (4.3)$$

where $A(x) = 2x / \sin 2x + 1 - (2x^2 / \sin^2 x)$, and $B(x) = 2x / \sin 2x - 1 - (x / \sin x)^2 + x \cot x$. By (2.1), (2.2), and (2.3), we have

$$\begin{aligned} A(x) &= 1 + \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| (2x)^{2n} + 1 - 2 \left(1 + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| (2n-1)x^{2n} \right) \\ &= \sum_{n=1}^{\infty} \frac{(2^{2n} - 4n)2^{2n}}{(2n)!} |B_{2n}| x^{2n} = \sum_{n=2}^{\infty} \frac{(2^{2n} - 4n)2^{2n}}{(2n)!} |B_{2n}| x^{2n} =: \sum_{n=2}^{\infty} a_n x^{2n}, \\ B(x) &= 1 + \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| (2x)^{2n} - 1 - \left(1 + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| (2n-1)x^{2n} \right) + 1 - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n} \\ &= \sum_{n=1}^{\infty} \frac{(2^{2n} - 2n - 2)2^{2n}}{(2n)!} |B_{2n}| x^{2n} = \sum_{n=2}^{\infty} \frac{(2^{2n} - 2n - 2)2^{2n}}{(2n)!} |B_{2n}| x^{2n} =: \sum_{n=2}^{\infty} b_n x^{2n}, \end{aligned} \quad (4.4)$$

where $a_n = ((2^{2n} - 4n)2^{2n} / (2n)!) |B_{2n}|$ and $b_n = ((2^{2n} - 2n - 2)2^{2n} / (2n)!) |B_{2n}| > 0$.

When setting $c_n = a_n / b_n$, we have that $c_n = (2^{2n} - 4n) / (2^{2n} - 2n - 2)$ is increasing for $n = 2, 3, \dots$, $A(x)/B(x)$ is increasing from $(0, \pi/2)$ onto $(4/5, 1)$ by Lemma 2.2. When $\alpha \geq 1$, we have $u(x) \geq 0$. So $k(x)$ is increasing on $(0, \pi/2)$. This leads to that $f'(x)/g'(x)$ is increasing on $(0, \pi/2)$. Thus $F(x) = f(x)/g(x) = (f(x) - f(0^+)) / (g(x) - g(0^+))$ is increasing on $(0, \pi/2)$ by Lemma 2.1. At the same time, $\lim_{x \rightarrow 0^+} F(x) = 1/3$. So the proof of Lemma 2.9 is complete. $\square$

Proof of Theorem 1.2. From Theorem 1.1, when $\alpha \geq 1$ we have

$$\left(\frac{\sin x}{x} \right)^{2\alpha} + \left(\frac{\tan x}{x} \right)^{\alpha} > \left(\frac{x}{\sin x} \right)^{2\alpha} + \left(\frac{x}{\tan x} \right)^{\alpha}. \quad (4.5)$$

On the other hand, when $\alpha \geq 1$ we can obtain

$$1 + \left(\frac{x}{\sin x} \right)^{2\alpha} + \left(\frac{x}{\tan x} \right)^{\alpha} \geq 2 \left(\frac{x}{\sin x} \right)^{\alpha} + \left(\frac{x}{\tan x} \right)^{\alpha} > 3 \quad (4.6)$$

by the arithmetic mean-geometric mean inequality and Lemma 2.9. So

$$\left(\frac{x}{\sin x} \right)^{2\alpha} + \left(\frac{x}{\tan x} \right)^{\alpha} > 2 \quad (4.7)$$

holds.

Combining (4.5) and (4.7) gives (1.6). $\square$

5. Proofs of Lemma 2.8 and Theorem 1.3

Proof of Lemma 2.8. Direct calculation yields $H'(\alpha) = I(\alpha)/[(x/\sinh x)^{2\alpha} + (x/\tanh x)^\alpha]^2$, where

$$\begin{aligned} I(\alpha) &= \left[\left(\frac{\tanh x}{x} \right)^\alpha \left(\frac{x}{\sinh x} \right)^{2\alpha} + \left(\frac{x}{\tanh x} \right)^\alpha \left(\frac{\sinh x}{x} \right)^{2\alpha} + 2 \right] \log \frac{\tanh x}{x} \\ &\quad + 2 \left[\left(\frac{\tanh x}{x} \right)^\alpha \left(\frac{x}{\sinh x} \right)^{2\alpha} + \left(\frac{x}{\tanh x} \right)^\alpha \left(\frac{\sinh x}{x} \right)^{2\alpha} + 2 \right] \log \frac{\sinh x}{x} \\ &= \left[\left(\frac{\tanh x}{x} \right)^\alpha \left(\frac{x}{\sinh x} \right)^{2\alpha} + \left(\frac{x}{\tanh x} \right)^\alpha \left(\frac{\sinh x}{x} \right)^{2\alpha} + 2 \right] \log \left[\left(\frac{\sinh x}{x} \right)^3 \frac{1}{\cosh x} \right]. \end{aligned} \quad (5.1)$$

First, $(\tanh x/x)^\alpha (x/\sinh x)^{2\alpha} + (x/\tanh x)^\alpha (\sinh x/x)^{2\alpha} + 2 > 0$ for $x > 0$. Second, we have $\log[(\sinh x/x)^3(1/\cosh x)] > 0$ by Lemma 2.6. Thus $I(\alpha) > 0$ and $H'(\alpha) > 0$. The proof of Lemma 2.8 is complete. $\square$

Proof of Theorem 1.3. From Lemma 2.8 we have $H(\alpha) > H(0) = 1$ for $\alpha > 0$. That is, (1.7) holds. At the same time, we have $H(\alpha) < H(0) = 1$ for $\alpha < 0$. That is, (1.7) is reversed. $\square$

6. Proofs of Lemma 2.10 and Theorem 1.4

Proof of Lemma 2.10. Let $Q(x) = ((\sinh x/x)^\alpha - 1)/((\cosh x)^\alpha - 1) =: f(x)/g(x)$, where $f(x) = (\sinh x/x)^\alpha - 1$, and $g(x) = (\cosh x)^\alpha - 1$. Then

$$k(x) =: \frac{f'(x)}{g'(x)} = \left(\frac{\sinh x}{x \cosh x} \right)^{\alpha-1} \frac{x \cosh x - \sinh x}{x^2 \sinh x} =: \left(\frac{\sinh x}{x \cosh x} \right)^{\alpha-1} \frac{A(x)}{B(x)}, \quad (6.1)$$

where $A(x) = x \cosh x - \sinh x$ and $B(x) = x^2 \sinh x$. Since

$$\begin{aligned} A(x) &= x \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty} \frac{(2n)x^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(2n+2)x^{2n+3}}{(2n+3)!} =: \sum_{n=0}^{\infty} a_n x^{2n+3}, \\ B(x) &= \sum_{n=0}^{\infty} \frac{x^{2n+3}}{(2n+1)!} =: \sum_{n=0}^{\infty} b_n x^{2n+3}, \end{aligned} \quad (6.2)$$

where $a_n = (2n+2)/(2n+3)!$ and $b_n = 1/(2n+1)!$.

When setting $c_n = a_n/b_n$, we have $c_n = 1/(2n+3)$ is decreasing for $n = 0, 1, 2, \dots$, $A(x)/B(x)$ is decreasing on $(0, +\infty)$ by Lemma 2.2. At the same time, the function $(\tanh x/x)^{\alpha-1}$ is decreasing on $(0, +\infty)$ when $\alpha \geq 1$. By (6.1), we obtain that $k(x) = f'(x)/g'(x)$ is decreasing on $(0, +\infty)$. Thus $Q(x) = f(x)/g(x) = (f(x) - f(0^+))/(g(x) - g(0^+))$ is decreasing on $(0, +\infty)$ by Lemma 2.1. At the same time, $\lim_{x \rightarrow 0^+} Q(x) = 1/3$. So the proof of Lemma 2.10 is complete. $\square$

Proof of Theorem 1.4. By the same way as Theorem 1.2, we can prove Theorem 1.4. $\square$

7. Open Problem

In this section, we pose the following open problem: find the respective largest range of α such that the inequalities (1.6) and (1.8) hold.

References

- [1] J. B. Wilker, "E3306," *The American Mathematical Monthly*, vol. 96, no. 1, p. 55, 1989.
- [2] J. S. Sumner, A. A. Jagers, M. Vowe, and J. Anglesio, "Inequalities involving trigonometric functions," *The American Mathematical Monthly*, vol. 98, no. 3, pp. 264–267, 1991.
- [3] B.-N. Guo, B.-M. Qiao, F. Qi, and W. Li, "On new proofs of Wilker's inequalities involving trigonometric functions," *Mathematical Inequalities & Applications*, vol. 6, no. 1, pp. 19–22, 2003.
- [4] L. Zhu, "A new simple proof of Wilker's inequality," *Mathematical Inequalities & Applications*, vol. 8, no. 4, pp. 749–750, 2005.
- [5] L. Zhu, "On Wilker-type inequalities," *Mathematical Inequalities & Applications*, vol. 10, no. 4, pp. 727–731, 2007.
- [6] S.-H. Wu and H. M. Srivastava, "A weighted and exponential generalization of Wilker's inequality and its applications," *Integral Transforms and Special Functions*, vol. 18, no. 7-8, pp. 529–535, 2007.
- [7] A. Baricz and J. Sandor, "Extensions of the generalized Wilker inequality to Bessel functions," *Journal of Mathematical Inequalities*, vol. 2, no. 3, pp. 397–406, 2008.
- [8] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, "Inequalities for quasiconformal mappings in space," *Pacific Journal of Mathematics*, vol. 160, no. 1, pp. 1–18, 1993.
- [9] G. D. Anderson, S.-L. Qiu, M. K. Vamanamurthy, and M. Vuorinen, "Generalized elliptic integrals and modular equations," *Pacific Journal of Mathematics*, vol. 192, no. 1, pp. 1–37, 2000.
- [10] I. Pinelis, "L'Hospital type results for monotonicity, with applications," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 3, no. 1, article 5, pp. 1–5, 2002.
- [11] I. Pinelis, "'Non-strict' l'Hospital-type rules for monotonicity: intervals of constancy," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 8, no. 1, article 14, pp. 1–8, 2007.
- [12] L. Zhu, "Sharpening Jordan's inequality and the Yang Le inequality," *Applied Mathematics Letters*, vol. 19, no. 3, pp. 240–243, 2006.
- [13] L. Zhu, "Sharpening Jordan's inequality and Yang Le inequality. II," *Applied Mathematics Letters*, vol. 19, no. 9, pp. 990–994, 2006.
- [14] L. Zhu, "Sharpening of Jordan's inequalities and its applications," *Mathematical Inequalities & Applications*, vol. 9, no. 1, pp. 103–106, 2006.
- [15] L. Zhu, "Some improvements and generalizations of Jordan's inequality and Yang Le inequality," in *Inequalities and Applications*, Th. M. Rassias and D. Andrica, Eds., CLUJ University Press, Cluj-Napoca, Romania, 2008.
- [16] L. Zhu, "A general refinement of Jordan-type inequality," *Computers & Mathematics with Applications*, vol. 55, no. 11, pp. 2498–2505, 2008.
- [17] F. Qi, D.-W. Niu, J. Cao, and S. X. Chen, "A general generalization of Jordan's inequality and a refinement of L. Yang's inequality," *RGMLA Research Report Collection*, vol. 10, no. 3, supplement, 2007.
- [18] S. Wu and L. Debnath, "A new generalized and sharp version of Jordan's inequality and its applications to the improvement of the Yang Le inequality," *Applied Mathematics Letters*, vol. 19, no. 12, pp. 1378–1384, 2006.
- [19] S. Wu and L. Debnath, "A new generalized and sharp version of Jordan's inequality and its applications to the improvement of the Yang Le inequality. II," *Applied Mathematics Letters*, vol. 20, no. 5, pp. 532–538, 2007.
- [20] D.-W. Niu, Z.-H. Huo, J. Cao, and F. Qi, "A general refinement of Jordan's inequality and a refinement of L. Yang's inequality," *Integral Transforms and Special Functions*, vol. 19, no. 3-4, pp. 157–164, 2008.
- [21] S.-H. Wu, H. M. Srivastava, and L. Debnath, "Some refined families of Jordan-type inequalities and their applications," *Integral Transforms and Special Functions*, vol. 19, no. 3-4, pp. 183–193, 2008.
- [22] S. Wu and L. Debnath, "A generalization of L'Hospital-type rules for monotonicity and its application," *Applied Mathematics Letters*, vol. 22, no. 2, pp. 284–290, 2009.

- [23] S. Wu and L. Debnath, "Jordan-type inequalities for differentiable functions and their applications," *Applied Mathematics Letters*, vol. 21, no. 8, pp. 803–809, 2008.
- [24] S. Wu and L. Debnath, "A generalization of L'Hôpital-type rules for monotonicity and its application," *Applied Mathematics Letters*, vol. 22, no. 2, pp. 284–290, 2009.
- [25] M. Biernacki and J. Krzyż, "On the monotonicity of certain functionals in the theory of analytic functions," *Annales Universitatis Mariae Curie-Skłodowska*, vol. 9, pp. 135–147, 1955.
- [26] S. Ponnusamy and M. Vuorinen, "Asymptotic expansions and inequalities for hypergeometric functions," *Mathematika*, vol. 44, no. 2, pp. 278–301, 1997.
- [27] H. Alzer and S.-L. Qiu, "Monotonicity theorems and inequalities for the complete elliptic integrals," *Journal of Computational and Applied Mathematics*, vol. 172, no. 2, pp. 289–312, 2004.
- [28] J.-L. Li, "An identity related to Jordan's inequality," *International Journal of Mathematics and Mathematical Sciences*, vol. 2006, Article ID 76782, 6 pages, 2006.
- [29] S.-H. Wu and H. M. Srivastava, "A further refinement of Wilker's inequality," *Integral Transforms and Special Functions*, vol. 19, no. 9-10, pp. 757–765, 2008.
- [30] A. Jeffrey, *Handbook of Mathematical Formulas and Integrals*, Elsevier Academic Press, San Diego, Calif, USA, 3rd edition, 2004.
- [31] D. S. Mitrinović, *Analytic Inequalities*, Springer, New York, NY, USA, 1970.
- [32] J. C. Kuang, *Applied Inequalities*, Shangdong Science and Technology Press, Jinan, China, 3rd edition, 2004.