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Research Article

A Bayesian Abstract Economy with a Measure Space of Agents

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We define the model of an abstract economy with differential (asymmetric) information and a measure space of agents. We generalize N. C. Yannelis's result (2007), considering that each agent is characterised by a random preference correspondence instead of having a random utility function. We establish two different equilibrium existence results.

1. Introduction

We define the model of an abstract economy with asymmetric information and a measure set of agents, each of which is characterized by a private information set, an action (strategy) correspondence, a random constraint correspondence, and a random preference correspondence. The preference correspondences need not be representable by utility functions. The equilibrium concept is an extension of the deterministic equilibrium. We also present the model of Yannelis (see [1]) in which the agents maximize their expected utilities. Our model is a generalization of Yannelis's model.

A purpose of this paper is to prove the existence of equilibrium for an abstract economy with differential information and a measure space of agents. The assumptions on correspondences refer to upper semicontinuity and measurable graph. We use in this paper several results on the continuity and measurability of the set of integrable selections from a Banach-valued correspondences.

The model of an abstract economy with differential (asymmetric) information captures the meaning of trades under uncertainty. All economic activity in a society is made under conditions of uncertainty (incomplete information). The asymmetric information in the Arrow-Debreu model was introduced by Radner [2]. In his model each agent has his own private information set which is described by a partition of an exogenously given set of states

of nature. The information partition of each agent generates a σ-algebra, and his net trades are measurable with respect to it (this σ-algebra). Thus, optimal choices reflect the private information of each agent.

The paper is organized as follows. In Section 2, some notational and terminological conventions are given. We also present, for the reader's convenience, some results on Bochner integration. In Section 3, Yannelis's expected utility model of differential information abstract economy and his main result in [1] are presented. Section 4 introduces our model, that is, the abstract economy with asymmetric information and a continuum of agents. Section 5 contains existence results for upper semicontinuous correspondences.

2. Mathematical Preliminaries

2.1. Notation and Definition

Throughout this paper, we will use the following notation:

(1) $\mathbb{R}_{++}$ denotes the set of strictly positive reals,

(2) $\mathrm{co}A$ denotes the convex hull of the set A,

(3) $\overline{\mathrm{co}}A$ denotes the closed convex hull of the set A,

(4) 2^A denotes the set of all nonempty subsets of the set A,

(5) if $A \subset X$, where X is a topological space, $\mathrm{cl}A$ denotes the closure of A.

For the reader's convenience, we review a few basic definitions and results from continuity and measurability of correspondences, Bochner integrable functions, and the integral of a correspondence.

Let X and Y be sets.

Definition 2.1. The *graph* of the correspondence $\Phi : X \to 2^Y$ is the set $G_\Phi = \{(x, y) \in X \times Y : y \in \Phi(x)\}$.

Let X, Y be topological spaces and let $\Phi : X \to 2^Y$ be a correspondence.

(1) Φ is said to be *upper semicontinuous* if for each $x \in X$ and each open set V in Y with $\Phi(x) \subset V$, there exists an open neighborhood U of x in X such that $\Phi(y) \subset V$ for each $y \in U$.

(2) Φ is said to be *lower semicontinuous* if for each $x \in X$ and each open set V in Y with $\Phi(x) \cap V \neq \emptyset$, there exists an open neighborhood U of x in X such that $\Phi(y) \cap V \neq \emptyset$ for each $y \in U$.

(3) Φ is said to have *open lower sections* if $\Phi^{-1}(y) := \{x \in X : y \in \Phi(x)\}$ is open in X for each $y \in Y$.

Lemma 2.2 (see [3]). *Let X and Y be two topological spaces and let A be an open subset of X. Suppose that $\Phi_1 : X \to 2^Y$, $\Phi_2 : X \to 2^Y$ are upper semicontinuous such that $\Phi_2(x) \subset \Phi_1(x)$ for all $x \in A$. Then the correspondence $\Phi : X \to 2^Y$ defined by*

$$\Phi(x) = \begin{cases} \Phi_1(x), & \text{if } x \notin A, \\ \Phi_2(x), & \text{if } x \in A \end{cases} \tag{2.1}$$

is also upper semicontinuous.

Let now $(\Omega, \mathcal{F}, \mu)$ be a complete, finite measure space, and let Y be a topological space.

Definition 2.3. (1) The correspondence $\Phi : \Omega \to 2^Y$ is said to have a *measurable graph* if $G_\Phi \in \mathcal{F} \otimes \beta(Y)$, where $\beta(Y)$ denotes the Borel σ-algebra on Y and $\otimes$ denotes the product σ-algebra.

(2) The correspondence $\Phi : \Omega \to 2^Y$ is said to be *lower measurable* if for every open subset V of Y, the set $\{\omega \in \Omega : \Phi(\omega) \cap V \neq \emptyset\}$ is an element of $\mathcal{F}$.

Recall (see Debreu [4, page 359]) that if $\Phi : \Omega \to 2^Y$ has a measurable graph, then Φ is lower measurable. Furthermore, if $\Phi(\cdot)$ is closed valued and lower measurable, then $\Phi : \Omega \to 2^Y$ has a measurable graph.

Lemma 2.4 (see [5]). *Let $\Phi_n : \Omega \to 2^Y$, $n = 1, 2, \ldots$ be a sequence of correspondences with measurable graphs. Then the correspondences $\bigcup_n \Phi_n$, $\bigcap_n \Phi_n$ and $Y \setminus \Phi_n$ have measurable graphs.*

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and Y be a Banach space.

It is known (see [6, Theorem 2, page 45]) that if $x : \Omega \to Y$ is a μ-measurable function, then x is Bochner integrable if only if $\int_\Omega \|x(\omega)\| d\mu(\omega) < \infty$.

It is denoted by $L_1(\mu, Y)$ the space of equivalence classes of Y-valued Bochner integrable functions $x : \Omega \to Y$ normed by $\|x\| = \int_\Omega \|x(\omega)\| d\mu(\omega)$.

Also it is known (see [6, page 50]) that $L_1(\mu, Y)$ is a Banach space.

We denote by S_Φ the set of all selections of the correspondence $\Phi : \Omega \to 2^Y$ that belong to the space $L_1(\mu, Y)$, that is,

$$S_\Phi = \{x \in L_1(\mu, Y) : x(\omega) \in \Phi(\omega) \mu\text{-a.e.}\}. \tag{2.2}$$

Definition 2.5 (see [7]). The *integral of correspondence* $\Phi : \Omega \to 2^Y$ is the set $\{\int_\Omega x(\omega) d\mu(\omega) : x \in S_\Phi\}$.

We will denote the above set by $\int \Phi(\omega) d\omega$ or simply $\int \Phi$.

Definition 2.6. The correspondence $\Phi : \Omega \to 2^Y$ is said to be *integrably bounded* if there exists a map $h \in L_1(\mu, R)$ such that $\sup\{\|x\| : x \in \Phi(\omega)\} \leq h(\omega) \mu$-a.e.

Moreover, note that if Ω is a complete measure space, Y is a separable Banach space and $\Phi : \Omega \to 2^Y$ is an integrably bounded, nonempty valued correspondence having a measurable graph; then by the Aumann measurable selection theorem we can conclude that S_Φ is nonempty and therefore $\int_\Omega \Phi(\omega) d\mu(\omega)$ is nonempty as well.

Let X be a topological space and let $\Phi : \Omega \times X \to 2^Y$ be a nonempty valued correspondence.

Definition 2.7. A function $f : \Omega \times X \to Y$ is said to be a *Carathéodory-type selection* from Φ if $f(\omega, x) \in \Phi(\omega, x)$ for all $(\omega, x) \in \Omega \times X$, $f(\cdot, x)$ is measurable for all $x \in X$ and let $f(\omega, \cdot)$ is continuous for all $\omega \in \Omega$.

The results below have been used in the proof of our theorems. For more details and further references see the paper quoted.

Theorem 2.8 (Projection theorem). *Let $(\Omega, \mathcal{F}, \mu)$ be a complete, finite measure space, and let Y be a complete separable metric space. If H belongs to $\mathcal{F} \otimes \beta(Y)$, its projection $Proj_\Omega(H)$ belongs to $\mathcal{F}$.*

Theorem 2.9 (Aumann measurable selection theorem [2]). *Let $(\Omega, \mathcal{F}, \mu)$ be a complete finite measure space, let Y be a complete, separable metric space, and let $\Phi : \Omega \to 2^Y$ be a nonempty valued correspondence with a measurable graph, that is, $G_\Phi \in \mathcal{F} \otimes \beta(Y)$. Then there is a measurable function $f : \Omega \to Y$ such that $f(\omega) \in \Phi(\omega)\mu$-a.e.*

Theorem 2.10 (Diestel's theorem [2, Theorem 3.1]). *Let $(\Omega, \mathcal{F}, \mu)$ be a complete finite measure space, let X be a separable Banach space, and let $\Phi : \Omega \to 2^Y$ be an integrably bounded, convex, weakly compact and nonempty valued correspondence. Then $S_\Phi = \{x \in L_1(\mu, Y) : x(\omega) \in \Phi(\omega)\mu\text{-a.e.}\}$ is weakly compact in $L_1(\mu, Y)$.*

Theorem 2.11 (Carathéodory-type selection theorem [5]). *Let $(\Omega, \mathcal{F}, \mu)$ be a complete measure space, let Z be a complete separable metric space, and let Y be a separable Banach space. Let $X : \Omega \to 2^Y$ be a correspondence with a measurable graph, that is, $G_X \in \mathcal{F} \otimes \beta(Y)$ and let $\Phi : \Omega \times Z \to 2^Y$ be a convex valued correspondence (possibly empty) with a meaurable graph, that is, $G_\Phi \in \mathcal{F} \otimes \beta(Z) \otimes \beta(Y)$ where $\beta(Y)$ and $\beta(Z)$ are the Borel σ-algebras of Y and Z, respectively.*

Suppose that

(1) *for each $\omega \in \Omega, \Phi(\omega, x) \subset X(\omega)$ for all $x \in Z$,*

(2) *for each $\omega \in \Omega$, $\Phi(\omega, \cdot)$ has open lower sections in Z; that is, for each $\omega \in \Omega$ and $y \in Y$, $\Phi^{-1}(\omega, y) = \{x \in Z : y \in \Phi(\omega, x)\}$ is open in Z,*

(3) *for each $(\omega, x) \in \Omega \times Z$, if $\Phi(\omega, x) \neq \emptyset$, then $\Phi(\omega, x)$ has a nonempty interior in $X(\omega)$.*

Let $U = \{(\omega, x) \in \Omega \times Z : \Phi(\omega, x) \neq \emptyset\}$ and for each $x \in Z$, $U^x = \{\omega \in \Omega : (\omega, x) \in U\}$ and for each $\omega \in \Omega$, $U^\omega = \{x \in Z : (\omega, x) \in U\}$. Then for each $x \in Z$, U^x is a measurable set in Ω and there exists a Caratheodory-type selection from $\Phi_{|U}$; that is, there exists a function $f : U \to Y$ such that $f(\omega, x) \in \Phi(\omega, x)$ for all $(\omega, x) \in U$, for each $x \in Z$, $f(\cdot, x)$ is measurable on U^x, and for each $\omega \in \Omega$, $f(\omega, \cdot)$ is continuous on U^ω. Moreover, $f(\cdot, \cdot)$ is jointly measurable.

Theorem 2.12 (u.s.c. lifting theorem [2]). *Let Y be a separable space, let $(\Omega, \mathcal{F}, \mu)$ be a complete finite measure space, and let $X : \Omega \to 2^Y$ be an integrably bounded, nonempty, convex valued correspondence such that for all $\omega \in \Omega$, $X(\omega)$ is a weakly compact, convex subset of Y. Denote by S_X the set $\{x \in L_1(\mu, Y) : x(\omega) \in X(\omega)\mu\text{-a.e.}\}$. Let $\Phi : \Omega \times S_X \to 2^Y$ be a nonempty, closed, convex valued correspondence such that $\Phi(\omega, x) \subset X(\omega)$ for all $(\omega, x) \in \Omega \times S_X^1$. Assume that for each fixed $x \in S_X$, $\Phi(\cdot, x)$ has a measurable graph and that for each fixed $\omega \in \Omega$, $\Phi(\omega, \cdot) : S_X \to 2^Y$ is u.s.c. in the sense that the set $\{x \in S_X : \Phi(\omega, x) \subset V\}$ is weakly open in S_X for every norm open subset V of Y. Define the correspondence $\Psi : S_X \to 2^{S_X}$ by*

$$\Psi(x) = \{y \in S_X : y(\omega) \in \Phi(\omega, x)\mu\text{-a.e.}\}. \tag{2.3}$$

Then Ψ is weakly u.s.c.; that is, the set $\{x \in S_X : \Psi(x) \subset V\}$ is weakly open in S_X for every weakly open subset V of S_X.

Theorem 2.13 (Measurability lifting theorem [8]). *Let Y and E be separable Banach spaces, and let (T, τ, υ) and $(\Omega, \mathcal{F}, \mu)$ be finite complete separable measure spaces. Let $\gamma : T \times \Omega \times E \to 2^Y$ be a nonempty valued correspondence. Suppose that for each $y \in E$, $\gamma(\cdot, \cdot, y)$ has a measurable graph. Define the correspondence $\varphi : \Omega \times E \to 2^{L_1(\mu, Y)}$ by*

$$\varphi(t, y) = \{x(t) \in L_1(\mu, Y) : x(t, \omega) \in \gamma(t, \omega, y)\mu\text{-a.e.}\}. \tag{2.4}$$

Then for each $y \in E$, $\varphi(\cdot, y)$ has a measurable graph.

3. A Bayesian Social Equilibrium Existence Theorem

We present Yannelis's model [1] of an abstract economy with asymmetric information and a continuum of agents. In this model, there is assigned to each agent, in addition to his/her random utility function, a private information set, which is a measurable partition of the exogenously given probability measure space (which describes the states of nature of the world).

Let $(\Omega, \mathcal{F}, \mu)$ be a complete finite measure space, where Ω denotes the set of states of nature of the world and the σ-algebra $\mathcal{F}$ denotes the set of events. Let Y be a separable Banach space whose dual has the RNP, denoting the commodity or strategy space.

Definition 3.1. A *Bayesian abstract economy* (or social system) with differential information and a measure space of agents (T, τ, υ) is a set $G = \{(X, u, A, \mathcal{F}_t, q_t),\ t \in T\}$, where

(1) $X : T \times \Omega \to 2^Y$ is the *random action (strategy)* correspondence, where $X(t, \omega) \subset Y$ is interpreted as the strategy set of agent t of the state of nature ω;

(2) for each fixed $(t, \omega) \in T \times \Omega$, $u(t, \omega, \cdot, \cdot) : L_1(\nu, Y) \times X(t, \omega) \to \mathbb{R}$ is the *random utility function*, where $u(t, \omega, x, x_t)$ is interpreted as the utility function of agent t, at the state of nature ω, using his/her strategy x_t and all other players use the joint strategy x;

(3) $A : T \times \Omega \times L_1(\upsilon, Y) \to 2^Y$ is the *random constraint correspondence* of agent t, where for all $(t, \omega, x) \in T \times \Omega \times L_1(\upsilon, Y)$, $A(t, \omega, x) \subset X(t, \omega)$, and $A(t, \omega, x)$ is interpreted as the constraint of agent t, when the state is ω and other agents use the joint strategy x;

(4) $\mathcal{F}_t$ is a sub-σ-algebra of $\mathcal{F}$ which denotes the *private information of agent* t;

(5) $q_t : \Omega \to \mathbb{R}_{++}$ is the prior of agent t, which is a Radon-Nikodym derivative such that $\int_\Omega q_t(\omega) d\mu(\omega) = 1$.

Let $S^1_{X_t} = \{y(t) \in L_1(\mu, X) : y(t, \cdot) : \Omega \to Y$ be $\mathcal{F}_t$-measurable and $y(t, \omega) \in X(t, \omega)\mu$-a.e.$\}$. Notice that $S^1_{X_t}$ is the set of all Bocner integrable and $\mathcal{F}_t$-measurable selections from the random strategy of agent t. In essence this is the set, out of which agent t will pick his/her optimal choices. In particular, an element x_t in $S^1_{X_t}$ is called a *strategy* for agent t. The typical element of $S^1_{X_t}$ is denoted by $\tilde{x}_t$ and that of $X(t, \omega)$ by $x_t(\omega)$. Let $S^1_X = \{\tilde{x} \in L_1(\upsilon, L_1(\mu, Y)) : \tilde{x}(t, \cdot) \in S^1_{X_t}\ \upsilon$-a.e.$\}$. An element of S^1_X will be a *joint strategy profile*.

It will be convenient to assume that Ω is a *countable* set and the σ-algebra $\mathcal{F}_t$ is generated by a countable partition Λ of Ω. For each $\omega \in \Omega$, let $E_t(\omega)$ in Λ denote the smallest set in $\mathcal{F}_t$ containing ω and assume that, for each t,

$$\int_{\omega' \in E_t(\omega)} q_t(\omega') d\mu(\omega') > 0. \tag{3.1}$$

Definition 3.2. For each $(t, \omega) \in T \times \Omega$, the interim expected utility of agent t, $U(t, \omega, \cdot, \cdot) : S^1_X \times X(t, \omega) \to \mathbb{R}$ is defined as

$$U(t, \omega, \tilde{x}, x_t) = \int_{\omega' \in E_t(\omega)} u(t, \omega', \tilde{x}(\omega'), x_t(\omega')) q_t(\omega' \mid E_t(\omega)) d\mu(\omega'), \tag{3.2}$$

where

$$q_t(\omega' \mid E_t(\omega)) = \begin{cases} 0, & \text{if } \omega' \notin E_t(\omega) \\ \dfrac{q_t(\omega')}{\int_{\overline{\omega} \in E_t(\omega)} q_t(\overline{\omega}) d\mu(\overline{\omega})}, & \text{if } \omega' \in E_t(\omega). \end{cases} \tag{3.3}$$

Definition 3.3. A *social equilibrium* for G is a strategy profile $\tilde{x}^* \in S_X^1$ such that υ-a.e.

(i) $\tilde{x}^*(t,\omega) \in A(t,\omega,\tilde{x}^*)\mu$-a.e.,

(ii) $U(t,\omega,\tilde{x},\tilde{x}^*(t,\omega)) = \max_{y \in A(t,\omega,\tilde{x}^*)} U(t,\omega,\tilde{x},y)\mu$-a.e.

The following theorem is the main result of Yannelis in [1].

Theorem 3.4. *Let G be a social system with asymmetric information satisfying (A.1)–(A.4). Then there exists a social equilibrium for G.*

One has the following assumptions:

(A.1)

(a) $X : T \times \Omega \to 2^Y$ *is a nonempty, convex, compact valued, and integrably bounded correspondence,*

(b) *for each $t \in T$, $X(t,\cdot) : \Omega \to 2^Y$ has an $\mathcal{F}_t$ measurable graph, that is, for every open subset V of Y, the set $G_X(t,\cdot) \in \mathcal{F}_t \times \beta(Y)$.*

(A.2)

(a) *for each $(t,\omega) \in T \times \Omega$, $u(t,\omega,\cdot,\cdot) : L_1(\upsilon,Y) \times X(t,\omega) \to \mathbb{R}$ is continuous where $L_1(\upsilon,Y)$ is endowed with the weak topology and $X(t,\omega)$ with the norm topology,*

(b) *for each fixed $(x,y) \in L_1(\upsilon,Y) \times Y$, $u(\cdot,\cdot,x,y) : T \times \Omega \to \mathbb{R}$ is a measurable function,*

(c) *for each $(t,\omega,x) \in T \times \Omega \times L_1(\upsilon,Y)$, $u(t,\omega,x,\cdot) : X(t,\omega) \to \mathbb{R}$ is concave,*

(d) *for each $t \in T$, $u(t,\cdot,\cdot,\cdot)$ is integrably bounded.*

(A.3)

(a) $A : T \times \Omega \times L_1(\upsilon,Y) \to 2^Y$ *has a measurable graph,*

(b) *for each $(t,\omega) \in T \times \Omega$, $A(t,\omega,\cdot) : L_1(\upsilon,Y) \times \to 2^Y$ is a continuous correspondence with closed, convex, and nonempty values.*

(A.4) *The correspondence $t \to S_{X_t}^1$ has a measurable graph.*

Remark 3.5. This theorem and its proof remain unchanged if the random constraint correspondence is defined as $A_i : \Omega \times L_X \to 2^Y$.

4. The Model

We will study the next model of the abstract economy.

Let $(\Omega, \mathcal{F}, \mu)$ be a complete finite measure space, where Ω denotes the set of states of nature of the world and the σ-algebra $\mathcal{F}$ denotes the set of events. Let Y be a separable Banach space whose dual has the RNP, denoting the commodity or strategy space.

Definition 4.1. A *Bayesian abstract economy* (or social system) with differential information and a measure space of agents (T, τ, υ) is a set $G = \{(X, \mathcal{F}_t, A, P),\ t \in T\}$, where

(1) $X : T \times \Omega \to 2^Y$ is the random action (strategy) correspondence, where, $X(t, \omega) \subset Y$ is interpreted as the strategy set of agent t of the state of nature ω;

(2) $\mathcal{F}_t$ is a sub-σ-algebra of $\mathcal{F}$ which denotes the private information of agent t;

(3) for each $t \in T$, $A(t, \cdot, \cdot) : \Omega \times S^1_X \to 2^Y$ is the random constraint correspondence of agent t, where for all $(t, \omega, x) \in T \times \Omega \times S^1_X$, $A(t, \omega, x) \subset X(t, \omega)$;

(4) for each $t \in T$, $P(t, \cdot, \cdot) : \Omega \times S^1_X \to 2^Y$ is the random preference correspondence of agent t, where for all $(t, \omega, x) \in T \times \Omega \times S^1_X$, $P(t, \omega, x) \subset X(t, \omega)$.

Definition 4.2. A Bayesian equilibrium for G is a strategy profile $\tilde{x}^* \in S^1_X$ such that for υ-a.e.

(i) $\tilde{x}^*(t, \omega) \in A(t, \omega, \tilde{x}^*)\mu$-a.e.,

(ii) $A(t, \omega, \tilde{x}^*) \cap P(t, \omega, \tilde{x}^*) = \emptyset\mu$-a.e.

Remark 4.3. This model of abstract economy is a generalization of Yannelis's model presented in Section 3, since for intern expected utilities U we can define the correspondence $P : T \times \Omega \times S^1_X \to 2^Y$ by $P(t, \omega, x) = \{y \in Y : U(t, \omega, x(t, \omega), y) > U(t, \omega, x(t, \omega)\mu\text{-a.e.}\}$.

5. Bayesian Equilibrium Existence Theorems

Now we establish an equilibrium existence theorem for Bayesian abstract economies with a measure space of agents and with upper semicontinuous correspondences. Our theorem generalizes Theorem 1 in [1].

Theorem 5.1. *Let (T, τ, υ) be a measure space of agents and let $G = \{(X, \mathcal{F}_t, A, P),\ t \in T\}$ be a Bayesian abstract economy satisfying (A.1)–(A.5). Then there exists a Bayesian equilibrium for G.*

(A.1)

(a) *$X : T \times \Omega \to 2^Y$ is a nonempty, convex, weakly compact-valued, and integrably bounded correspondence,*

(b) *for each fixed $t \in T$, $X(t, \cdot)$ has an $\mathcal{F}_t$-measurable graph, that is, for every open subset V of Y, the set $G_{X(t,\cdot)} \in \mathcal{F}_t \times \beta(Y)$.*

(A.2)

(a) *$A : T \times \Omega \times S^1_X \to 2^Y$ has a measurable graph,*

(b) *for each $(t, \omega) \in T \times \Omega$, $A(t, \omega, \cdot) : S^1_X \to 2^Y$ is an upper semicontinuous correspondence with closed, convex and nonempty values.*

(A.3)

(a) $P : T \times \Omega \times S_X^1 \to 2^Y$ *has a measurable graph,*

(b) *for each* $(t,\omega) \in T \times \Omega$, $P(t,\omega,\cdot) : S_X^1 \to 2^Y$ *is an upper semicontinuous correspondence with closed, convex, and nonempty values.*

(A.4) *The correspondence* $t \to S_{X_t}^1$ *has a measurable graph.*

(A.5)

(a) *for each* $\omega \in \Omega$, *for each* $x(t) \in S_X^1$, $x(t,\omega) \notin A(t,\omega,x) \cap P(t,\omega,x)$,

(b) *the set* $U^{(t,\omega)} = \{x \in S_X^1 : A(t,\omega,x) \cap P(t,\omega,x) \neq \emptyset\}$ *is weakly open in* S_X^1.

Proof. Define $\Phi : T \times \Omega \times S_X^1 \to 2^Y$ by $\Phi(t,\omega,x) = A(t,\omega,x) \cap P(t,\omega,x)$. We will prove first that S_X^1 is nonempty, convex, weakly compact.

Since $(\Omega, \mathcal{F}, \mu)$ is a complete finite measure space, Y is a separable Banach space and $X(t,\cdot) : \Omega \to 2^Y$ has measurable graph, and by Aumann's selection theorem it follows that there exists a function $f(t,\cdot) : \Omega \to Y$ such that $f(t,\omega) \in X(t,\omega)\mu$-a.e. Since $X(t,\cdot)$ is integrably bounded, we have that $f(t,\cdot) \in L_1(\mu, Y)$, hence S_{X_t} is nonempty, and $S_X = \prod_{t\in T} S_{X_t}$ is nonempty. $S_{X_t}^1$ is convex and S_X^1 is also convex. Since $X(t,\cdot) : \Omega \to 2^Y$ is integrably bounded and has convex weakly compact values, by Diestel's Theorem it follows that $S_{X_t}^1$ is a weakly compact subset of $L_1(\mu, Y)$ and so is S_X^1. We have that S_X^1 is a metrizable set as being a weakly compact subset of the separable Banach space $L_1(\upsilon, L_1(\mu, Y))$ (Dunford-Schwartz [9, page 434]).

Since all the values of the correspondence A are contained in the compact set $X(\cdot,\cdot)$ and A is closed and convex valued (hence weakly closed), it follows that A is weakly compact valued.

Then Φ is convex valued and for each $(t,\omega) \in T \times \Omega$, $\Phi(t,\omega,\cdot)$ is upper semicontinuous. We have that $\Phi(\cdot,\cdot,x)$ has a measurable graph for each $x \in S_X^1$. Let $U = \{(t,\omega,x) \in \Omega \times S_X^1 : \Phi(t,\omega,x) \neq \emptyset\}$. For each $x \in S_X^1$, let $U^x = \{(t,\omega) \in T \times \Omega : \Phi(t,\omega,x) \neq \emptyset\}$ and for each $\omega \in \Omega$, let $U^{(t,\omega)} = \{x \in S_X^1 : \Phi(t,\omega,x) \neq \emptyset\}$. Define $G : T \times \Omega \times S_X^1 \to 2^Y$ by

$$G(t,\omega,x) = \begin{cases} \Phi(t,\omega,x), & \text{if } (t,\omega,x) \in U, \\ A(t,\omega,x), & \text{if } (t,\omega,x) \notin U. \end{cases} \tag{5.1}$$

For each $x \in S_X^1$, the correspondence $G(\cdot,\cdot,x)$ has a measurable graph.

By assumption (A5)(b), the set $U^{(t,\omega)} = \{x \in S_X^1 : A(t,\omega,x) \cap P(t,\omega,x) \neq \emptyset\}$ is weakly open in S_X^1. For each $(t,\omega) \in T \times \Omega$, $G(t,\omega,\cdot) : S_X^1 \to 2^Y$ is upper semicontinuous.

Let V be weakly open in S_X^1 and $x \in S_X^1$:

$$\begin{aligned} W &= \left\{x \in S_X^1 : G(t,\omega,x) \subset V\right\} \\ &= \left\{x \in U^{(t,\omega)} : G(t,\omega,x) \subset V\right\} \cup \left\{x \in S_X^1 \setminus U^{(t,\omega)} : G(t,\omega,x) \subset V\right\} \\ &= \left\{x \in U^{(t,\omega)} : \Phi(t,\omega,x) \subset V\right\} \cup \left\{x \in S_X^1 : G(t,\omega,x) \subset V\right\}. \end{aligned} \tag{5.2}$$

W is an open set, because $U^{(t,\omega)}$ is open, $\Phi(t,\omega,\cdot)$ is an upper semicontinuous map on $U^{(t,\omega)}$, and the set $\{x \in S_X^1 : G(t,\omega,x) \subset V\}$ is open since $G(t,\omega,\cdot)$ is u.s.c. Moreover, G is convex and nonempty valued.

Define the correspondence $\varphi : T \times S_X^1 \to 2^{L_1(\mu,Y)}$ by $\varphi(t,\tilde{x}) = \{\tilde{y}(t) \in L_1(\mu,Y) : \tilde{y}(t,\omega) \in G(t,\omega,\tilde{x})\mu\text{-a.e.}\} \cap S_{X_t}^1$.

By the measurability lifting theorem (Theorem 2.13), the correspondence $t \to \{\tilde{y}(t) \in L_1(\mu,Y) : \tilde{y}(t,\omega) \in G(t,\omega,\tilde{x})\mu\text{-a.e.}\}$ has a measurable graph and so does $t \to S_{X_t}^1$ by (A.4). Thus, for each fixed $\tilde{x} \in S_X^1$, $\varphi(\cdot,\tilde{x})$ has a measurable graph.

Since for each fixed $\tilde{x} \in S_X^1$, $G(\cdot,\cdot,\tilde{x})$ has a measurable graph and it is nonempty valued, then by the Aumann measurable selection theorem, it admits a measurable selection and we can conclude that φ is nonempty valued. It follows by the u.s.c. lifting theorem that for each fixed t, $\varphi(t,\cdot)$ is weakly u.s.c.

Define $G' : S_X^1 \to 2^{S_X^1}$, by $G'(\tilde{x}) = \{\tilde{y} \in S_X^1 : \tilde{y}(t) \in \varphi(t,\tilde{x})\nu\text{-a.e.}\}$.

Another application of the u.s.c. lifting theorem enables us to conclude that G' is a weakly u.s.c. correspondence which is obviously convex valued (since φ is convex valued) and also nonempty valued (recall once more the Aumann measurable selection theorem and notice that the set S_X^1 is metrizable).

G' is an upper semicontinuous correspondence and has also nonempty convex closed values.

By Fan-Glicksberg's fixed-point theorem in [5], there exists $\tilde{x}^* \in S_X^1$ such that $\tilde{x}^* \in G'(\tilde{x}^*)$. It follows that $\tilde{x}^* \in S_X^1$ and $\tilde{x}^*(t) \in \varphi(t,\tilde{x}^*)\nu$-a.e. Thus, we have that $\tilde{x}^* \in S_X^1$ and $\tilde{x}^*(t,\omega) \in G(t,\omega,\tilde{x}^*)\mu$-a.e., ν-a.e.

By assumption (A.4)(a), it follows that $\tilde{x}^*(t,\omega) \notin (A \cap P)(t,\omega,\tilde{x}^*)$, then we have that $\tilde{x}^* \notin U$ and $\tilde{x}^*(t,\omega) \in A(t,\omega,\tilde{x}^*)$.

Therefore, for υ-a.e.

(1) $\tilde{x}^*(t,\omega) \in A(t,\omega,\tilde{x}^*)\mu$-a.e.,

(2) $A(t,\omega,\tilde{x}^*) \cap P(t,\omega,\tilde{x}^*) = \emptyset\mu$-a.e. □

If there exists a selector F for $A \cap P$ such that it has measurable graph and it is weakly upper semicontinuous in the third argument, we obtain the following theorem.

Theorem 5.2. *Let (T,τ,υ) be a measure space of agents and let $G = \{(X,\mathcal{F}_t,A,P),\ t \in T\}$ be a Bayesian abstract economy satisfying (A.1)–(A.5). Then there exists a Bayesian equilibrium for G.*

(A.1)

(a) *$X : T \times \Omega \to 2^Y$ is a nonempty, convex, weakly compact-valued, and integrably bounded correspondence,*

(b) *for each fixed $t \in T$, $X(t,\cdot)$ has an $\mathcal{F}_t$-measurable graph, that is, for every open subset V of Y, the set $G_X(t,\cdot) \in \mathcal{F}_t \times \beta(Y)$.*

(A.2)

(a) *$A : T \times \Omega \times S_X^1 \to 2^Y$ has a measurable graph,*

(b) *for each $(t,\omega) \in T \times \Omega$, $A(t,\omega,\cdot) : S_X^1 \to 2^Y$ is an upper semicontinuous correspondence with closed convex and nonempty values.*

(A.3)

(a) $P : T \times \Omega \times S_X^1 \to 2^Y$ *has a measurable graph,*

(b) *for each* $(t,\omega) \in T \times \Omega$, $P(t,\omega,\cdot) : S_X^1 \to 2^Y$ *is a upper semicontinuous correspondence with closed, convex, and nonempty values.*

(A.4)

(a) *for each* $(t,\omega) \in T \times \Omega$, *the set* $U^{(t,\omega)} = \{x \in S_X^1 : A(t,\omega,x) \cap P(\omega,x) = \emptyset \mu\text{-a.e.}\}$ *is open in* S_X^1,

(b) *for each* $\omega \in \Omega$, *for each* $x \in U$, $x(t,\omega) \notin A(t,\omega,x) \cap P(t,\omega,x)$.

(A.5)

(a) *there exists a selector* $F : S_X^1 \to 2^Y$ *for* $A \cap P : S_X^1 \to 2^Y$ *such that for each* $x \in S_X^1$, $F(\cdot,\cdot,x)$ *has measurable graph and for each* $(t,\omega) \in T \times \Omega$, $F(t,\omega,\cdot) : U^{(t,\omega)} \to 2^Y$ *is weakly upper semicontinuous with closed and convex values.*

Proof. Define $G : T \times \Omega \times S_X^1 \to 2^Y$ by

$$G(t,\omega,x) = \begin{cases} F(t,\omega,x) & \text{if } (t,\omega,x) \in U, \\ A(t,\omega,x) & \text{if } (t,\omega,x) \notin U. \end{cases} \tag{5.3}$$

For each $x \in S_X^1$, the correspondence $G(\cdot,\cdot,x)$ has a measurable graph.

By assumption (A4)(b), the set $U^{(t,\omega)} = \{x \in S_X^1 : A(t,\omega,x) \cap P(t,\omega,x) \neq \emptyset\}$ is weakly open in S_X^1. For each $(t,\omega) \in T \times \Omega$, $G(t,\omega,\cdot) : S_X^1 \to 2^Y$ is upper semicontinuous.

Let V be weakly open in S_X^1 and $x \in S_X^1$:

$$\begin{aligned} W &= \left\{x \in S_X^1 : G(t,\omega,x) \subset V\right\} \\ &= \left\{x \in U^{(t,\omega)} : G(t,\omega,x) \subset V\right\} \cup \left\{x \in S_X^1 \setminus U^{(t,\omega)} : G(t,\omega,x) \subset V\right\} \\ &= \left\{x \in U^{(t,\omega)} : F(t,\omega,x) \subset V\right\} \cup \left\{x \in S_X^1 : G(t,\omega,x) \subset V\right\}. \end{aligned} \tag{5.4}$$

W is an open set, because $U^{(t,\omega)}$ is open, $\Phi(t,\omega,\cdot)$ is a upper semicontinuous map on $U^{(t,\omega)}$, and the set $\{x \in S_X^1 : G(t,\omega,x) \subset V\}$ is open since $G(t,\omega,\cdot)$ is u.s.c. Moreover, G is convex and nonempty valued.

Define the correspondence $\varphi : T \times S_X^1 \to 2^{L_1(\mu,Y)}$ by $\varphi(t,\tilde{x}) = \{\tilde{y}(t) \in L_1(\mu,Y) : \tilde{y}(t,\omega) \in G(t,\omega,\tilde{x})\mu\text{-a.e.}\} \cap S_{X_t}^1$.

By the measurability lifting theorem (Theorem 2.13), the correspondence $t \to \{\tilde{y}(t) \in L_1(\mu,Y) : \tilde{y}(t,\omega) \in G(t,\omega,\tilde{x})\mu\text{-a.e.}\}$ has a measurable graph and so does $t \to S_{X_t}^1$ by (A.4). Thus, for each fixed $\tilde{x} \in S_X^1$, $\varphi(\cdot,\tilde{x})$ has a measurable graph.

Since for each fixed $\tilde{x} \in S_X^1$, $G(\cdot,\cdot,\tilde{x})$ has a measurable graph and it is nonempty valued, then by the Aumann measurable selection theorem, it admits a measurable selection and we can conclude that φ is nonempty valued. It follows by the u.s.c. lifting theorem that for each fixed t, $\varphi(t,\cdot)$ is weakly u.s.c.

Define $G' : S_X^1 \to 2^{S_X^1}$, by $G'(\tilde{x}) = \{\tilde{y} \in S_X^1 : \tilde{y}(t) \in \varphi(t,\tilde{x})\nu\text{-a.e.}\}$.

Another application of the u.s.c. lifting Theorem enables us to conclude that G' is a weakly u.s.c. correspondence which is obviously convex valued (since φ is convex valued) and also nonempty valued (recall once more the Aumann measurable selection theorem and notice that the set S_X^1 is metrizable).

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By assumption (A.4)(a), it follows that $\tilde{x}^*(t,\omega) \notin (A \cap P)(t,\omega,\tilde{x}^*)$, then we have that $\tilde{x}^* \notin U$ and $\tilde{x}^*(t,\omega) \in A(t,\omega,\tilde{x}^*)$.

Therefore, for υ-a.e.

(1) $\tilde{x}^*(t,\omega) \in A(t,\omega,\tilde{x}^*)\mu$-a.e.,

(2) $A(t,\omega,\tilde{x}^*) \cap P(t,\omega,\tilde{x}^*) = \emptyset\mu$-a.e. □

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