Research Article

On Approximate Euler Differential Equations

Soon-Mo Jung¹ and Seungwook Min²

¹ Mathematics Section, College of Science and Technology, Hong-Ik University,
 Chochiwon 339-701, South Korea

² Division of Computer Science, Sangmyung University, 7 Hongji-dong, Jongno-gu,
 Seoul 110-743, South Korea

Correspondence should be addressed to Soon-Mo Jung, smjung@hongik.ac.kr

Received 9 July 2009; Revised 7 September 2009; Accepted 16 September 2009

Recommended by Paul Eloe

We solve the inhomogeneous Euler differential equations of the form $x^2 y''(x) + \alpha x y'(x) + \beta y(x) = \sum_{m=0}^{\infty} a_m x^m$ and apply this result to the approximation of analytic functions of a special type by the solutions of Euler differential equations.

Copyright © 2009 S.-M. Jung and S. Min. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

The stability problem for functional equations starts from the famous talk of Ulam and the partial solution of Hyers to the Ulam's problem (see [1, 2]). Thereafter, Rassias [3] attempted to solve the stability problem of the Cauchy additive functional equation in a more general setting.

The stability concept introduced by Rassias' theorem significantly influenced a number of mathematicians to investigate the stability problems for various functional equations (see [4–10] and the references therein).

Assume that X and Y are a topological vector space and a normed space, respectively, and that I is an open subset of X . If for any function $f : I \rightarrow Y$ satisfying the differential inequality

$$\left\| a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) + h(x) \right\| \leq \varepsilon \quad (1.1)$$

for all $x \in I$ and for some $\varepsilon \geq 0$, there exists a solution $f_0 : I \rightarrow Y$ of the differential equation

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) + h(x) = 0 \quad (1.2)$$

such that $\|f(x) - f_0(x)\| \leq K(\varepsilon)$ for any $x \in I$, where $K(\varepsilon)$ is an expression of ε only, then we say that the above differential equation satisfies the Hyers-Ulam stability (or the local Hyers-Ulam stability if the domain I is not the whole space X). We may apply these terminologies for other differential equations. For more detailed definition of the Hyers-Ulam stability, refer to [1, 3, 5, 6, 8–11].

Obloza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations (see [12, 13]). Here, we will introduce a result of Alsina and Ger (see [14]): if a differentiable function $f : I \rightarrow \mathbb{R}$ is a solution of the differential inequality $|y'(x) - y(x)| \leq \varepsilon$, where I is an open subinterval of $\mathbb{R}$, then there exists a solution $f_0 : I \rightarrow \mathbb{R}$ of the differential equation $y'(x) = y(x)$ such that $|f(x) - f_0(x)| \leq 3\varepsilon$ for any $x \in I$.

This result of Alsina and Ger has been generalized by Takahasi, Miura, and Miyajima: they proved in [15] that the Hyers-Ulam stability holds for the Banach space-valued differential equation $y'(x) = \lambda y(x)$ (see also [16]).

Using the conventional power series method, the first author investigated the general solution of the inhomogeneous Hermite differential equation of the form

$$y''(x) - 2xy'(x) + 2\lambda y(x) = \sum_{m=0}^{\infty} a_m x^m \quad (1.3)$$

under some specific condition, where λ is a real number and the convergence radius of the power series is positive. This result was applied to prove that every analytic function can be approximated in a neighborhood of 0 by a Hermite function with an error bound expressed by $Cx^2e^{x^2}$ (see [17–20]).

In Section 2 of this paper, using power series method, we will investigate the general solution of the inhomogeneous Euler (or Cauchy) differential equation

$$x^2y''(x) + \alpha xy'(x) + \beta y(x) = \sum_{m=0}^{\infty} a_m x^m, \quad (1.4)$$

where α and β are fixed complex numbers and the coefficients a_m of the power series are given such that the radius of convergence is $\rho > 0$. Moreover, using the idea from [17–19], we will approximate some analytic functions by the solutions of Euler differential equations.

In this paper, $\mathbb{N}_0$ denotes the set of all nonnegative integers.

2. General Solution of Inhomogeneous Euler Equations

The second-order Euler differential equation

$$x^2y''(x) + \alpha xy'(x) + \beta y(x) = 0, \quad (2.1)$$

which is sometimes called the second-order Cauchy differential equation, is one of the most famous differential equations and frequently appears in applications.

The quadratic equation

$$m^2 + (\alpha - 1)m + \beta = 0 \quad (2.2)$$

is called the auxiliary equation of the Euler differential equation (2.1), and every solution of (2.1) is of the form

$$y_h(x) = \begin{cases} c_1 x^{m_1} + c_2 x^{m_2} & \text{if } m_1 \text{ and } m_2 \text{ are distinct roots of (2.2),} \\ (c_1 + c_2 \ln x) x^{(1-\alpha)/2} & \text{if } \frac{(1-\alpha)}{2} \text{ is a double root of (2.2),} \end{cases} \quad (2.3)$$

where c_1 and c_2 are complex constants (see [21, Section 2.7]).

Theorem 2.1. *Let α and β be complex constants such that no root of the auxiliary equation (2.2) is a nonnegative integer. If the radius of convergence of power series $\sum_{m=0}^{\infty} a_m x^m$ is at least $\rho > 0$, then every solution $y : (0, \rho) \rightarrow \mathbb{C}$ of the inhomogeneous Euler differential equation (1.4) can be expressed by*

$$y(x) = y_h(x) + \sum_{m=0}^{\infty} \frac{a_m x^m}{m^2 + (\alpha - 1)m + \beta} \quad (2.4)$$

for all $x \in (0, \rho)$, where $y_h(x)$ is a solution of the Euler differential equation (2.1).

Proof. Assume that $y : (0, \rho) \rightarrow \mathbb{C}$ is a function given by (2.4), where $y_h(x)$ is a solution of the homogeneous Euler differential equation (2.1). We first prove that the function $y_p(x)$, defined by $y(x) - y_h(x)$, satisfies the inhomogeneous equation (1.4).

Indeed, we have

$$\begin{aligned} x^2 y_p''(x) + \alpha x y_p'(x) + \beta y_p(x) &= \sum_{m=2}^{\infty} \frac{m(m-1)a_m x^m}{m^2 + (\alpha - 1)m + \beta} + \sum_{m=1}^{\infty} \frac{\alpha m a_m x^m}{m^2 + (\alpha - 1)m + \beta} \\ &\quad + \sum_{m=0}^{\infty} \frac{\beta a_m x^m}{m^2 + (\alpha - 1)m + \beta} \\ &= \sum_{m=0}^{\infty} a_m x^m, \end{aligned} \quad (2.5)$$

which proves that $y_p(x)$ is a particular solution of the inhomogeneous equation (1.4). Moreover, each power series appearing in the above equalities has the same radius of convergence as $\sum_{m=0}^{\infty} a_m x^m$ (which can be verified by using the ratio test). Since every solution to (1.4) can be expressed as a sum of a solution $y_h(x)$ of the homogeneous equation and a particular solution $y_p(x)$ of the inhomogeneous equation, every solution of (1.4) is certainly of the form (2.4).

We will now apply the ratio test to the power series in (2.4). Indeed, by setting $c_m = a_m / [m^2 + (\alpha - 1)m + \beta]$, we get

$$\left| \frac{c_{m+1}}{c_m} \right| = \left| \frac{m^2 + (\alpha - 1)m + \beta}{(m+1)^2 + (\alpha - 1)(m+1) + \beta} \right| \left| \frac{a_{m+1}}{a_m} \right|, \quad (2.6)$$

and hence we have

$$\limsup_{m \rightarrow \infty} \left| \frac{c_{m+1}}{c_m} \right| = \limsup_{m \rightarrow \infty} \left| \frac{a_{m+1}}{a_m} \right|, \quad (2.7)$$

which implies that the power series given in (2.4) has the same radius of convergence as power series $\sum_{m=0}^{\infty} a_m x^m$, which is at least ρ . That is, $y(x)$ given in (2.4) is well defined on its domain $(0, \rho)$. $\square$

3. Approximate Euler Differential Equations

In this section, assume that α and β are complex constants and ρ is a positive constant. For a given $K \geq 0$, we denote by $\mathcal{C}_K$ the set of all functions $y : (0, \rho) \rightarrow \mathbb{C}$ with the properties (a) and (b):

- (a) $y(x)$ is expressible by a power series $\sum_{m=0}^{\infty} b_m x^m$ whose radius of convergence is at least ρ ;
- (b) $\sum_{m=0}^{\infty} |a_m x^m| \leq K |\sum_{m=0}^{\infty} a_m x^m|$ for any $x \in (0, \rho)$, where we set $a_m = [m^2 + (\alpha - 1)m + \beta]b_m$ for $m \geq 0$.

Let $\{b_m\}$ be a sequence of positive real numbers such that the radius of convergence of the series $\sum_{m=0}^{\infty} b_m x^m$ is at least ρ , and let α and β satisfy either $\alpha \geq 1$ and $\beta > 0$ or $\beta \geq (\alpha - 1)^2/4$. If a function $y : (0, \rho) \rightarrow \mathbb{R}$ is defined by $y(x) = \sum_{m=0}^{\infty} b_m x^m$, then y certainly belongs to $\mathcal{C}_K$ with $K \geq 1$. So, the set $\mathcal{C}_K$ is not empty if $K \geq 1$. In particular, if ρ is small and K is large, then $\mathcal{C}_K$ is a large class of analytic functions $y : (0, \rho) \rightarrow \mathbb{C}$.

We will now solve the approximate Euler differential equations in a special class of analytic functions, $\mathcal{C}_K$.

Theorem 3.1. *Let α and β be complex constants such that no root of the auxiliary equation (2.2) is a nonnegative integer. If a function $y \in \mathcal{C}_K$ satisfies the differential inequality*

$$\left| x^2 y''(x) + \alpha x y'(x) + \beta y(x) \right| \leq \varepsilon \quad (3.1)$$

for all $x \in (0, \rho)$ and for some $\varepsilon \geq 0$, then there exists a solution $y_h : (0, \rho) \rightarrow \mathbb{C}$ of the Euler differential equation (2.1) such that

$$|y(x) - y_h(x)| \leq \left(\sup_{k \in \mathbb{N}_0} \frac{1}{|k^2 + (\alpha - 1)k + \beta|} \right) \frac{K\rho}{\rho - x} \varepsilon \quad (3.2)$$

for any $x \in (0, \rho)$.

Proof. Since y belongs to $\mathcal{C}_K$, it follows from (a) and (b) that

$$x^2 y''(x) + \alpha x y'(x) + \beta y(x) = \sum_{m=0}^{\infty} [m^2 + (\alpha - 1)m + \beta] b_m x^m = \sum_{m=0}^{\infty} a_m x^m \quad (3.3)$$

for all $x \in (0, \rho)$. By considering (3.1) and (3.3), we get

$$\left| \sum_{m=0}^{\infty} a_m x^m \right| \leq \varepsilon \quad (3.4)$$

for any $x \in (0, \rho)$. This inequality, together with (b), yields that

$$\sum_{m=0}^{\infty} |a_m x^m| \leq K \left| \sum_{m=0}^{\infty} a_m x^m \right| \leq K \varepsilon \quad (3.5)$$

for each $x \in (0, \rho)$.

Now, suppose that an arbitrary $x \in (0, \rho)$ is given. Then we can choose an arbitrary constant $\rho_0 \in (x, \rho)$. By Abel's formula (see [22, Theorem 6.30]), we have

$$\begin{aligned} \sum_{m=0}^n |a_m \rho_0^m| \frac{|x/\rho_0|^m}{|m^2 + (\alpha - 1)m + \beta|} &= \left(\sum_{m=0}^n |a_m \rho_0^m| \right) \frac{|x/\rho_0|^{n+1}}{|(n+1)^2 + (\alpha - 1)(n+1) + \beta|} \\ &+ \sum_{k=0}^n \left(\sum_{m=0}^k |a_m \rho_0^m| \right) \frac{|x/\rho_0|^k}{|k^2 + (\alpha - 1)k + \beta|} M(k; \rho_0; x), \end{aligned} \quad (3.6)$$

where we set

$$M(k; \rho_0; x) = 1 - \left| \frac{k^2 + (\alpha - 1)k + \beta}{(k+1)^2 + (\alpha - 1)(k+1) + \beta} \right| \left| \frac{x}{\rho_0} \right|. \quad (3.7)$$

Since $M(k; \rho_0; x) \leq 1$ for any $k \in \mathbb{N}_0$ and $\rho_0 \in (x, \rho)$, it follows from (3.5) and (3.6) that

$$\sum_{m=0}^n |a_m \rho_0^m| \frac{|x/\rho_0|^m}{|m^2 + (\alpha - 1)m + \beta|} \leq K \varepsilon \left(\frac{|x/\rho_0|^{n+1}}{|(n+1)^2 + (\alpha - 1)(n+1) + \beta|} + \sum_{k=0}^n \frac{|x/\rho_0|^k}{|k^2 + (\alpha - 1)k + \beta|} \right) \quad (3.8)$$

If we let $n \rightarrow \infty$ in the above inequality, then we obtain

$$\begin{aligned} \sum_{m=0}^{\infty} \frac{|a_m x^m|}{|m^2 + (\alpha - 1)m + \beta|} &\leq K \varepsilon \sum_{k=0}^{\infty} \frac{1}{|k^2 + (\alpha - 1)k + \beta|} \left| \frac{x}{\rho_0} \right|^k \\ &\leq \left(\sup_{k \in \mathbb{N}_0} \frac{1}{|k^2 + (\alpha - 1)k + \beta|} \right) \frac{K \rho_0}{\rho_0 - |x|} \varepsilon \end{aligned} \quad (3.9)$$

for all $x \in (0, \rho)$ and for any $\rho_0 \in (x, \rho)$. Since $\rho_0/(\rho_0 - x) \downarrow \rho/(\rho - x)$ as $\rho_0 \rightarrow \rho$, we get

$$\sum_{m=0}^{\infty} \frac{|a_m x^m|}{|m^2 + (\alpha - 1)m + \beta|} \leq \left(\sup_{k \in \mathbb{N}_0} \frac{1}{|k^2 + (\alpha - 1)k + \beta|} \right) \frac{K\rho}{\rho - x} \varepsilon \quad (3.10)$$

for every $x \in (0, \rho)$.

Finally, it follows from (3.3), Theorem 2.1, and (3.10) that there exists a solution y_h of the Euler differential equation (2.1) such that

$$\begin{aligned} |y(x) - y_h(x)| &\leq \left| \sum_{m=0}^{\infty} \frac{a_m x^m}{m^2 + (\alpha - 1)m + \beta} \right| \\ &\leq \sum_{m=0}^{\infty} \frac{|a_m x^m|}{|m^2 + (\alpha - 1)m + \beta|} \\ &\leq \left(\sup_{k \in \mathbb{N}_0} \frac{1}{|k^2 + (\alpha - 1)k + \beta|} \right) \frac{K\rho}{\rho - x} \varepsilon \end{aligned} \quad (3.11)$$

for any $x \in (0, \rho)$. □

4. An Example

We fix $\alpha = 0$, $\beta = 1/4$ and suppose that a small $\varepsilon > 0$ is given. We can choose a constant $0 < \rho < 1$ such that

$$\varepsilon = \sum_{m=1}^{\infty} \left(m - 1 + \frac{1}{4m} \right) \rho^m. \quad (4.1)$$

We will consider the function $y(x) = \ln(1 - x)$ which can be expressed by the power series $\sum_{m=1}^{\infty} (-x^m/m)$, whose radius of convergence is 1.

If we set $b_0 = 0$ and $b_m = -1/m$ for any $m \in \mathbb{N}$, then it follows from (b) that

$$a_m = \begin{cases} 0 & \text{for } m = 0, \\ -m + 1 - \frac{1}{4m} & \text{for } m \in \mathbb{N}. \end{cases} \quad (4.2)$$

Thus, we have

$$\sum_{m=0}^{\infty} |a_m x^m| = \sum_{m=1}^{\infty} \left(m - 1 + \frac{1}{4m} \right) x^m = \left| \sum_{m=0}^{\infty} a_m x^m \right| \quad (4.3)$$

for any $x \in (0, \rho)$, which enables us to choose $K = 1$. Thus, it holds that $y \in C_1$.

Moreover, we have

$$\begin{aligned} \left| x^2 y''(x) + \frac{1}{4} y(x) \right| &= \sum_{m=1}^{\infty} \left(m - 1 + \frac{1}{4m} \right) x^m \\ &\leq \sum_{m=1}^{\infty} \left(m - 1 + \frac{1}{4m} \right) \rho^m \\ &= \varepsilon \end{aligned} \quad (4.4)$$

for all $x \in (0, \rho)$. It now follows from Theorem 3.1 that there exist complex numbers A and B with

$$|\ln(1-x) - (A + B \ln x)\sqrt{x}| \leq \frac{4\rho}{\rho-x} \sum_{m=1}^{\infty} \left(m - 1 + \frac{1}{4m} \right) \rho^m \quad (4.5)$$

for any $x \in (0, \rho)$.

Acknowledgments

The authors would like to express their cordial thanks to the referees for their useful remarks and constructive comments which have improved the first version of this paper. This work was supported by National Research Foundation of Korea Grant funded by the Korean Government (no. 2009-0071206).

References

- [1] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.
- [2] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience, New York, NY, USA, 1960.
- [3] Th. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.
- [4] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64–66, 1950.
- [5] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, Singapore, 2002.
- [6] G. L. Forti, "Hyers-Ulam stability of functional equations in several variables," *Aequationes Mathematicae*, vol. 50, no. 1-2, pp. 143–190, 1995.
- [7] P. Găvruta, "A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings," *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431–436, 1994.
- [8] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Boston, Mass, USA, 1998.
- [9] D. H. Hyers and Th. M. Rassias, "Approximate homomorphisms," *Aequationes Mathematicae*, vol. 44, no. 2-3, pp. 125–153, 1992.
- [10] Th. M. Rassias, "On the stability of functional equations and a problem of Ulam," *Acta Applicandae Mathematicae*, vol. 62, no. 1, pp. 23–130, 2000.
- [11] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, Fla, USA, 2001.
- [12] M. Obłozza, "Hyers stability of the linear differential equation," *Rocznik Naukowo-Dydaktyczny. Prace Matematyczne*, no. 13, pp. 259–270, 1993.
- [13] M. Obłozza, "Connections between Hyers and Lyapunov stability of the ordinary differential equations," *Rocznik Naukowo-Dydaktyczny. Prace Matematyczne*, no. 14, pp. 141–146, 1997.

- [14] C. Alsina and R. Ger, "On some inequalities and stability results related to the exponential function," *Journal of Inequalities and Applications*, vol. 2, no. 4, pp. 373–380, 1998.
- [15] S.-E. Takahasi, T. Miura, and S. Miyajima, "On the Hyers-Ulam stability of the Banach space-valued differential equation $y' = \lambda y$," *Bulletin of the Korean Mathematical Society*, vol. 39, no. 2, pp. 309–315, 2002.
- [16] T. Miura, S.-M. Jung, and S.-E. Takahasi, "Hyers-Ulam-Rassias stability of the Banach space valued linear differential equations $y' = \lambda y$," *Journal of the Korean Mathematical Society*, vol. 41, no. 6, pp. 995–1005, 2004.
- [17] S.-M. Jung, "Legendre's differential equation and its Hyers-Ulam stability," *Abstract and Applied Analysis*, vol. 2007, Article ID 56419, 14 pages, 2007.
- [18] S.-M. Jung, "Approximation of analytic functions by Hermite functions," *Bulletin des Sciences Mathematiques*. In press.
- [19] S.-M. Jung, "Approximation of analytic functions by airy functions," *Integral Transforms and Special Functions*, vol. 19, no. 12, pp. 885–891, 2008.
- [20] S.-M. Jung, "An approximation property of exponential functions," *Acta Mathematica Hungarica*, vol. 124, no. 1-2, pp. 155–163, 2009.
- [21] E. Kreyszig, *Advanced Engineering Mathematics*, John Wiley & Sons, New York, NY, USA, 4th edition, 1979.
- [22] W. R. Wade, *An Introduction to Analysis*, Prentice-Hall, Upper Saddle River, NJ, USA, 2nd edition, 2000.