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We establish strong convergence theorems for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of two relatively nonexpansive mappings in a Banach space by using a new hybrid method. Moreover we apply our main results to obtain strong convergence for a maximal monotone operator and two nonexpansive mappings in a Hilbert space.

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1. Introduction

Let $E$ be a real Banach space with $\| \cdot \|$ and let $C$ be a nonempty closed convex subset of $E$. A mapping $T$ of $C$ into itself is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We use $F(T)$ to denote the set of fixed points of $T$; that is, $F(T) = \{x \in C : x = Tx\}$. A mapping $T$ of $C$ into itself is called quasinonexpansive if $F(T)$ is nonempty and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$. For two mappings $S$ and $T$ of $C$ into itself, Das and Debata [1] considered the following iteration scheme: $x_0 \in C$ and

$$x_{n+1} = \alpha_n S(\beta_n Tx_n + (1 - \beta_n)x_n) + (1 - \alpha_n)x_n, \quad n \geq 0, \tag{1.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. In this case of $S = T$, such an iteration process was considered by Ishikawa [2]; see also Mann [3]. Das and Debata [1] proved the strong convergence of the iterates $\{x_n\}$ defined by (1.1) in the case when $E$ is strictly convex and $S, T$ are quasinonexpansive mappings. Fixed point iteration processes for nonexpansive mappings in a Hilbert space and a Banach space including Das and Debata’s iteration and
Ishikawa’s iteration have been studied by many researchers to approximating a common fixed point of two mappings; see, for instance, Takahashi and Tamura [4].

Let \( A \) be a maximal monotone operator from \( E \) to \( E^* \), where \( E^* \) is the dual space of \( E \). It is well known that many problems in nonlinear analysis and optimization can be formulated as follows. Find a point \( u \in E \) satisfying

\[
0 \in Au.
\]

We denote by \( A^{-1}0 \) the set of all points \( u \in C \) such that \( 0 \in Au \). Such a problem contains numerous problems in economics, optimization, and physics. A well-known method to solve this problem is called the proximal point algorithm: \( x_0 \in E \) and

\[
x_{n+1} = J_{r_n}x_n, \quad n = 0, 1, 2, 3, \ldots,
\]

where \( \{ r_n \} \subset (0, \infty) \) and \( J_{r_n} \) are the resolvents of \( A \). Many researchers have studied this algorithm in a Hilbert space; see, for instance, [5–8] and in a Banach space; see, for instance, [9–11].

Next, we recall that for all \( x \in E \) and \( x^* \in E^* \), we denote the value of \( x^* \) at \( x \) by \( \langle x, x^* \rangle \). Then, the normalized duality mapping \( J \) on \( E \) is defined by

\[
Jx = \left\{ x^* \in E^*: \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \right\}, \quad \forall x \in E.
\]

We know that if \( E \) is smooth, then the duality mapping \( J \) is single valued. Next, we assume that \( E \) is a smooth Banach space and define the function \( \phi: E \times E \to \mathbb{R} \) by

\[
\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2, \quad \forall y, x \in E.
\]

A point \( u \in C \) is said to be an asymptotic fixed point of \( T \) [12] if \( C \) contains a sequence \( \{ x_n \} \) which converges weakly to \( u \) and \( \lim_{n \to \infty} \| x_n - Tx_n \| = 0 \). We denote the set of all asymptotic fixed points of \( T \) by \( \bar{F}(T) \). A mapping \( T: C \to C \) is said to be relatively nonexpansive [13–15] if \( \bar{F}(T) = F(T) \neq \emptyset \) and \( \phi(u, Tx) \leq \phi(u, x) \) for all \( u \in F(T) \) and \( x \in C \). The asymptotic behavior of a relatively nonexpansive mapping was studied in [13–15].

In 2004, Matsushita and Takahashi [15] proposed the following modification of Mann’s iteration for a relatively nonexpansive mapping by using the hybrid method in a Banach space. Four years later, Qin and Su [16] have adapted Matsushita and Takahashi’s idea [15] to modify Halpern’s iteration and Ishikawa’s iteration for a relatively nonexpansive mapping in a Banach space. In particular, in a Hilbert space Mann’s iteration, Halpern’s iteration, and Ishikawa’s iteration were considered by many researchers.

Very recently, Inoue et al. [17] proved the following strong convergence theorem for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a relatively nonexpansive mapping by using the hybrid method.

**Theorem 1.1** (Inoue et al. [17]). Let \( E \) be a uniformly convex and uniformly smooth Banach space and let \( C \) be a nonempty closed convex subset of \( E \). Let \( A \subset E \times E^* \) be a maximal monotone operator satisfying \( D(A) \subset C \) and let \( J_r = (I + rA)^{-1}J \) for all \( r > 0 \). Let \( S: C \to C \) be a relatively...
nonexpansive mapping such that \( F(S) \cap A^{-1}0 \neq \emptyset \). Let \( \{x_n\} \) be a sequence generated by \( x_0 = x \in C \) and

\[
\begin{align*}
    u_n &= J^{-1}(\beta_n Jx_n + (1 - \beta_n) JSJr_n x_n), \\
    C_n &= \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
    Q_n &= \{z \in C : (x_n - z, Jx_0 - Jx_n) \geq 0\}, \\
    x_{n+1} &= \Pi_{C_n \cap Q_n}x_0
\end{align*}
\]  
(1.6)

for all \( n \in \mathbb{N} \cup \{0\} \), where \( J \) is the duality mapping on \( E \), \( \{\beta_n\} \subset [0, 1] \), and \( \{r_n\} \subset [a, \infty) \) for some \( a > 0 \). If \( \lim \inf_{n \to \infty}(1 - \beta_n) > 0 \), then \( \{x_n\} \) converges strongly to \( \Pi_{F(S) \cap A^{-1}0}x_0 \), where \( \Pi_{F(S) \cap A^{-1}0} \) is the generalized projection of \( E \) onto \( F(S) \cap A^{-1}0 \).

The purpose of this paper is to employ the idea of Inoue et al. [17] and Das and Debata [1] to introduce a new hybrid method for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of two relatively nonexpansive mappings. We prove a strong convergence theorem of the new hybrid method. Moreover we apply our main results to obtain strong convergence for a maximal monotone operator and two nonexpansive mappings in a Hilbert space.

### 2. Preliminaries

Throughout this paper, all linear spaces are real. Let \( \mathbb{N} \) and \( \mathbb{R} \) be the sets of all positive integers and real numbers, respectively. Let \( E \) be a Banach space and let \( E^* \) be the dual space of \( E \). For a sequence \( \{x_n\} \) of \( E \) and a point \( x \in E \), the **weak** convergence of \( \{x_n\} \) to \( x \) and the **strong** convergence of \( \{x_n\} \) to \( x \) are denoted by \( x_n \rightharpoonup x \) and \( x_n \to x \), respectively.

Let \( S(E) \) be the unit sphere centered at the origin of \( E \). Then the space \( E \) is said to be **smooth** if the limit

\[
\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}
\]

exists for all \( x, y \in S(E) \). It is also said to be **uniformly smooth** if the limit exists uniformly in \( x, y \in S(E) \). A Banach space \( E \) is said to be **strictly convex** if \( \|(x + y)/2\| < 1 \) whenever \( x, y \in S(E) \) and \( x \neq y \). It is said to be **uniformly convex** if for each \( \epsilon \in (0, 2] \), there exists \( \delta > 0 \) such that \( \|(x + y)/2\| < 1 - \delta \) whenever \( x, y \in S(E) \) and \( \|x - y\| \geq \epsilon \). We know the following [18]:

(i) if \( E \) is smooth, then \( J \) is single-valued;
(ii) if \( E \) is reflexive, then \( J \) is onto;
(iii) if \( E \) is strictly convex, then \( J \) is one to one;
(iv) if \( E \) is strictly convex, then \( J \) is strictly monotone;
(v) if \( E \) is uniformly smooth, then \( J \) is uniformly norm-to-norm continuous on each bounded subset of \( E \).
A Banach space $E$ is said to have the Kadec-Klee property if for a sequence $\{x_n\}$ of $E$ satisfying that $x_n \to x$ and $\|x_n\| \to \|x\|$, $x_n \to x$. It is known that if $E$ is uniformly convex, then $E$ has the Kadec-Klee property; see [18, 19] for more details. Let $E$ be a smooth, strictly convex, and reflexive Banach space and let $C$ be a closed convex subset of $E$. Throughout this paper, define the function $\phi : E \times E \to \mathbb{R}$ by

$$
\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2, \quad \forall y, x \in E. \tag{2.2}
$$

Observe that, in a Hilbert space $H$, (2.2) reduces to $\phi(x, y) = \|x - y\|^2$, for all $x, y \in H$. It is obvious from the definition of the function $\phi$ that, for all $x, y \in E$,

1. $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$,
2. $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle$,
3. $\phi(x, y) = \langle x, Jx - Jy \rangle + \langle y - x, Jy \rangle \leq \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$.\)

Following Alber [20], the generalized projection $\Pi_C$ from $E$ onto $C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$; that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem

$$
\phi(\overline{x}, x) = \min_{y \in C} \phi(y, x). \tag{2.3}
$$

Existence and uniqueness of the operator $\Pi_C$ follows from the properties of the functional $\phi(y, x)$ and strict monotonicity of the mapping $J$. In a Hilbert space, $\Pi_C$ is the metric projection of $H$ onto $C$. We need the following lemmas for the proof of our main results.

**Lemma 2.1** (Kamimura and Takahashi [6]). Let $E$ be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences in $E$ such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n \to \infty} \phi(x_n, y_n) = 0$, then $\lim_{n \to \infty} \|x_n - y_n\| = 0$.

**Lemma 2.2** (Matsushita and Takahashi [15]). Let $C$ be a closed convex subset of a smooth, strictly convex, and reflexive Banach space $E$ and let $T$ be a relatively nonexpansive mapping from $C$ into itself. Then $F(T)$ is closed and convex.

**Lemma 2.3** (Alber [20] and Kamimura and Takahashi [6]). Let $C$ be a closed convex subset of a smooth, strictly convex, and reflexive Banach space, $x \in E$ and let $z \in C$. Then, $z = \Pi_C x$ if and only if $\langle y - z, Jx - Jz \rangle \leq 0$ for all $y \in C$.

**Lemma 2.4** (Alber [20] and Kamimura and Takahashi [6]). Let $C$ be a closed convex subset of a smooth, strictly convex, and reflexive Banach space. Then

$$
\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y), \quad \forall x \in C, \ y \in E. \tag{2.4}
$$

Let $E$ be a smooth, strictly convex, and reflexive Banach space, and let $A$ be a set-valued mapping from $E$ to $E^*$ with graph $G(A) = \{(x, x^*): x^* \in Ax\}$, domain $D(A) = \{z \in E : Az \neq \emptyset\}$, and range $R(A) = \bigcup\{Az : z \in D(A)\}$. We denote a set-valued operator $A$ from $E$ to $E^*$ by $A \in E \times E^*$. $A$ is said to be monotone if $\langle x - y, x^* - y^* \rangle \geq 0$, for all $(x, x^*), (y, y^*) \in A$. A monotone operator $A \in E \times E^*$ is said to be maximal monotone if its graph is not properly
contained in the graph of any other monotone operator. We know that if \( A \) is a maximal monotone operator, then \( A^{-1}0 = \{ z \in D(A) : 0 \in Az \} \) is closed and convex. The following theorem is well known.

**Lemma 2.5** (Rockafellar [21]). Let \( E \) be a smooth, strictly convex, and reflexive Banach space and let \( A \subset E \times E^* \) be a monotone operator. Then \( A \) is maximal if and only if \( R(\lambda + r\lambda) = E^* \) for all \( r > 0 \).

Let \( E \) be a smooth, strictly convex, and reflexive Banach space, let \( C \) be a nonempty closed convex subset of \( E \) and let \( A \subset E \times E^* \) be a monotone operator satisfying

\[
D(A) \subset C \subset J^{-1}(\bigcap_{r>0} R(\lambda + r\lambda)).
\]  

(2.5)

Then we can define the resolvent \( J_{\lambda} : C \to D(A) \) of \( A \) by

\[
J_{\lambda}x = \{ z \in D(A) : Jx \in Jz + rAz\}, \quad \forall x \in C.
\]  

(2.6)

We know that \( J_{\lambda}x \) consists of one point. For \( r > 0 \), the Yosida approximation \( A_{\lambda} : C \to E^* \) is defined by \( A_{\lambda}x = (Jx - J_{\lambda}x)/r \) for all \( x \in C \).

**Lemma 2.6** (Kohsaka and Takahashi [22]). Let \( E \) be a smooth, strictly convex, and reflexive Banach space, let \( C \) be a nonempty closed convex subset of \( E \) and let \( A \subset E \times E^* \) be a monotone operator satisfying

\[
D(A) \subset C \subset J^{-1}(\bigcap_{r>0} R(\lambda + r\lambda)).
\]  

(2.7)

Let \( r > 0 \) and let \( J_{\lambda} \) and \( A_{\lambda} \) be the resolvent and the Yosida approximation of \( A \), respectively. Then, the following hold:

(i) \( \phi(u, J_{\lambda}x) + \phi(J_{\lambda}x, x) \leq \phi(u, x), \) for all \( x \in C \), \( u \in A^{-1}0 \);

(ii) \( (J_{\lambda}x, A_{\lambda}x) \in A_{\lambda} \), for all \( x \in C \);

(iii) \( F(J_{\lambda}) = A^{-1}0 \).

**Lemma 2.7** (Zălinescu [23] and Xu [24]). Let \( E \) be a uniformly convex Banach space and let \( r > 0 \). Then there exists a strictly increasing, continuous, and convex function \( g : [0, \infty) \to [0, \infty) \) such that \( g(0) = 0 \) and

\[
\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x - y\|)
\]  

(2.8)

for all \( x, y \in B_r(0) \) and \( t \in [0,1] \), where \( B_r(0) = \{ z \in E : \|z\| \leq r \} \).
3. Main Results

In this section, we prove a strong convergence theorem for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of two relatively nonexpansive mappings in a Banach space by using the hybrid method.

Theorem 3.1. Let $E$ be a uniformly convex and uniformly smooth Banach space and let $C$ be a nonempty closed convex subset of $E$. Let $A \subset E \times E^*$ be a maximal monotone operator satisfying $D(A) \subset C$ and let $J_r = (I + rA)^{-1}J$ for all $r > 0$. Let $S$ and $T$ be relatively nonexpansive mappings from $C$ into itself such that $\Omega = F(S) \cap F(T) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$ and

$$
\begin{align*}
  u_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JTz_n), \\
  z_n &= J^{-1}(\beta_n Jx_n + (1 - \beta_n)JSJr_nx_n), \\
  C_n &= \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
  Q_n &= \{z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\
  x_{n+1} &= \Pi_{C_n \cap Q_n}x_0
\end{align*}
$$

for all $n \in \mathbb{N} \cup \{0\}$, where $J$ is the duality mapping on $E$, $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If $\lim inf_{n \to \infty}(1 - \alpha_n) > 0$ and $\lim inf_{n \to \infty}\beta_n(1 - \beta_n) > 0$, then $\{x_n\}$ converges strongly to $\Pi_{\Omega x_0}$, where $\Pi_{\Omega}$ is the generalized projection of $E$ onto $\Omega$.

Proof. We first show that $C_n$ and $Q_n$ are closed and convex for each $n \geq 0$. From the definitions of $C_n$ and $Q_n$, it is obvious that $C_n$ is closed and $Q_n$ is closed and convex for each $n \geq 0$. Next, we prove that $C_n$ is convex. Since $\phi(z, u_n) \leq \phi(z, x_n)$ is equivalent to

$$
0 \leq \|x_n\|^2 - \|u_n\|^2 - 2\langle z, Jx_n - Jx_n\rangle, \tag{3.2}
$$

which is affine in $z$, and hence $C_n$ is convex. So, $C_n \cap Q_n$ is a closed and convex subset of $E$ for all $n \geq 0$. Next, we show that $\Omega \subset C_n$ for all $n \geq 0$. Indeed, let $u \in \Omega$ and $y_n = J_{r_n}x_n$ for all $n \geq 0$. Since $J_{r_n}$ are relatively nonexpansive mappings, we have

$$
\begin{align*}
  \phi(u, z_n) &= \phi\left(u, J^{-1}(\beta_n Jx_n + (1 - \beta_n)JSy_n)\right) \\
  &= \|u\|^2 - 2\langle u, \beta_n Jx_n + (1 - \beta_n)JSy_n \rangle + \|\beta_n Jx_n + (1 - \beta_n)JSy_n\|^2 \\
  &\leq \|u\|^2 - 2\beta_n\langle u, Jx_n \rangle - 2(1 - \beta_n)\langle u, JSy_n \rangle + \beta_n\|x_n\|^2 + (1 - \beta_n)\|Sy_n\|^2 \\
  &= \beta_n\phi(u, x_n) + (1 - \beta_n)\phi(u, Sy_n) \\
  &\leq \beta_n\phi(u, x_n) + (1 - \beta_n)\phi(u, y_n) \\
  &= \beta_n\phi(u, x_n) + (1 - \beta_n)\phi(u, J_{r_n}x_n) \\
  &\leq \beta_n\phi(u, x_n) + (1 - \beta_n)\phi(u, x_n) \\
  &= \phi(u, x_n).
\end{align*}
$$

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It follows that

\[
\phi(u, u_n) = \phi \left( u, J^{-1}(\alpha_n J x_n + (1 - \alpha_n) JT z_n) \right)
\]

\[
= \|u\|^2 - 2\langle u, \alpha_n J x_n + (1 - \alpha_n) JT z_n \rangle + \|\alpha_n J x_n + (1 - \alpha_n) JT z_n\|^2
\]

\[
\leq \|u\|^2 - 2\alpha_n \langle u, J x_n \rangle - 2(1 - \alpha_n) \langle u, JT z_n \rangle + \alpha_n \|x_n\|^2 + (1 - \alpha_n) \|T z_n\|^2
\]

\[
= \alpha_n \phi(u, x_n) + (1 - \alpha_n) \phi(u, T z_n)
\]

\[
\leq \alpha_n \phi(u, x_n) + (1 - \alpha_n) \phi(u, z_n)
\]

\[
\leq \alpha_n \phi(u, x_n) + (1 - \alpha_n) \phi(u, x_n)
\]

\[
= \phi(u, x_n).
\]

So, \( u \in C_n \) for all \( n \geq 0 \), which implies that \( \Omega \subset C_n \). Next, we show that \( \Omega \subset Q_n \) for all \( n \geq 0 \). We prove by induction. For \( n = 0 \), we have \( \Omega \subset C = Q_0 \). Assume that \( \Omega \subset Q_n \). Since \( x_{n+1} \) is the projection of \( x_0 \) onto \( C_n \cap Q_n \), by Lemma 2.3 we have

\[
\langle x_{n+1} - z, J x_0 - J x_{n+1} \rangle \geq 0, \quad \forall z \in C_n \cap Q_n.
\]

(3.5)

As \( \Omega \subset C_n \cap Q_n \) by the induction assumptions, we have

\[
\langle x_{n+1} - z, J x_0 - J x_{n+1} \rangle \geq 0, \quad \forall z \in \Omega.
\]

(3.6)

This together with definition of \( Q_{n+1} \) implies that \( \Omega \subset Q_{n+1} \) and hence \( \Omega \subset Q_n \) for all \( n \geq 0 \). So, we have that \( \Omega \subset C_n \cap Q_n \) for all \( n \geq 0 \). This implies that \( \{x_n\} \) is well defined. From definition of \( Q_n \) that \( x_n = \Pi_{Q_n} x_0 \) and \( x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in C_n \cap Q_n \subset Q_n \), we have

\[
\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \geq 0.
\]

(3.7)

Therefore, \( \{\phi(x_n, x_0)\} \) is nondecreasing. It follows from Lemma 2.4 and \( x_n = \Pi_{Q_n} x_0 \) that

\[
\phi(x_n, x_0) = \phi(\Pi_{Q_n} x_0, x_0) \leq \phi(u, x_0) - \phi(u, \Pi_{Q_n} x_0) \leq \phi(u, x_0)
\]

(3.8)

for all \( u \in \Omega \subset Q_n \). Therefore, \( \{\phi(x_n, x_0)\} \) is bounded. Moreover, by definition of \( \phi \), we know that \( \{x_n\} \) is bounded. So, we have \( \{y_n\} \) and \( \{z_n\} \) are bounded. So, the limit of \( \{\phi(x_n, x_0)\} \) exists. From \( x_n = \Pi_{Q_n} x_0 \) and Lemma 2.4, we have

\[
\phi(x_{n+1}, x_n) = \phi(x_{n+1}, \Pi_{Q_n} x_0) \leq \phi(x_{n+1}, x_0) - \phi(\Pi_{Q_n} x_0, x_0) = \phi(x_{n+1}, x_0) - \phi(x_n, x_0)
\]

(3.9)
for all \( n \geq 0 \). This implies that \( \lim_{n \to \infty} \phi(x_{n+1}, x_n) = 0 \). From \( x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in C_n \), we have

\[
\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n).
\] (3.10)

Therefore, we have \( \lim_{n \to \infty} \phi(x_{n+1}, u_n) = 0 \).

Since \( \lim_{n \to \infty} \phi(x_{n+1}, x_n) = \lim_{n \to \infty} \phi(x_{n+1}, u_n) = 0 \) and \( E \) is uniformly convex and smooth, we have from Lemma 2.1 that

\[
\lim_{n \to \infty} \| x_{n+1} - x_n \| = \lim_{n \to \infty} \| x_{n+1} - u_n \| = 0.
\] (3.11)

So, we have \( \lim_{n \to \infty} \| x_n - u_n \| = 0 \). Since \( J \) is uniformly norm-to-norm continuous on bounded sets, we have

\[
\lim_{n \to \infty} \| Jx_{n+1} - Jx_n \| = \lim_{n \to \infty} \| Jx_{n+1} - Ju_n \| = \lim_{n \to \infty} \| Jx_n - Ju_n \| = 0.
\] (3.12)

On the other hand, we have

\[
\| Jx_{n+1} - Ju_n \| = \| Jx_{n+1} - \alpha_n Jx_n - (1 - \alpha_n) Jz_n \|
\]

\[
= \| \alpha_n (Jx_{n+1} - Jx_n) + (1 - \alpha_n) (Jx_{n+1} - Jz_n) \|
\]

\[
= \| (1 - \alpha_n) (Jx_{n+1} - Jz_n) - \alpha_n (Jx_n - Jx_{n+1}) \|
\]

\[
\geq (1 - \alpha_n) \| Jx_{n+1} - Jz_n \| - \alpha_n \| Jx_n - Jx_{n+1} \|.
\] (3.13)

This follows that

\[
\| Jx_{n+1} - Jz_n \| \leq \frac{1}{1 - \alpha_n} (\| Jx_{n+1} - Ju_n \| + \alpha_n \| Jx_n - Jx_{n+1} \|).
\] (3.14)

From (3.12) and \( \lim \inf_{n \to \infty} (1 - \alpha_n) > 0 \), we obtain that \( \lim_{n \to \infty} \| Jx_{n+1} - Jz_n \| = 0 \).

Since \( J^{-1} \) is uniformly norm-to-norm continuous on bounded sets, we have

\[
\lim_{n \to \infty} \| x_{n+1} - Tz_n \| = 0.
\] (3.15)

From

\[
\| x_n - Tz_n \| \leq \| x_n - x_{n+1} \| + \| x_{n+1} - Tz_n \|,
\] (3.16)

we have

\[
\lim_{n \to \infty} \| x_n - Tz_n \| = 0.
\] (3.17)
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Since \( \{x_n\} \) and \( \{y_n\} \) are bounded, we also obtain that \( \{Jx_n\} \) and \( \{JSy_n\} \) are bounded. So, there exists \( r > 0 \) such that \( \{Jx_n\}, \{JSy_n\} \subset B_r(0) \). Therefore Lemma 2.7 is applicable and we observe that

\[
\phi(u, z_n) = \phi\left( u, J^{-1}(\beta_n Jx_n + (1 - \beta_n) JSy_n) \right)
\]

\[
= \|u\|^2 - 2(\|u\|^2 - 2(\beta_n Jx_n + (1 - \beta_n) JSy_n) + \|\beta_n Jx_n + (1 - \beta_n) JSy_n\|^2
\]

\[
\leq \|u\|^2 - 2\phi(u, Jx_n) - 2(\beta_n Jx_n + (1 - \beta_n) JSy_n) + \beta_n \|x_n\|^2 + (1 - \beta_n) \|Sy_n\|^2
\]

\[
- \beta_n(1 - \beta_n)g(\|Jx_n - JSy_n\|)
\]

(3.18)

where \( g : [0, \infty) \to [0, \infty) \) is a continuous, strictly increasing, and convex function with \( g(0) = 0 \). That is

\[
\beta_n(1 - \beta_n)g(\|Jx_n - JSy_n\|) \leq \phi(u, x_n) - \phi(u, z_n).
\]

(3.19)

Let \( \|x_{n_k} - Sy_{n_k}\| \) be any subsequence of \( \|x_n - Sy_n\| \). Since \( \{x_{n_k}\} \) is bounded, there exists a subsequence \( \{x_{n_k'}\} \) of \( \{x_{n_k}\} \) such that

\[
\lim_{j \to \infty} \phi\left( u, x_{n_k'} \right) = \limsup_{k \to \infty} \phi(u, x_{n_k}) = a,
\]

(3.20)

where \( u \in \Omega \). By (2) and (3), we have

\[
\phi\left( u, x_{n_k'} \right) = \phi\left( u, Tz_{n_k'} \right) + \phi\left( Tz_{n_k'}, x_{n_k'} \right) + 2(\|u - Tz_{n_k'}\| JTx_{n_k'} - Jx_{n_k'})
\]

\[
\leq \phi(u, z_{n_k'}) + \|Tz_{n_k'}\| \|JTx_{n_k'} - Jx_{n_k'}\| + \|Tz_{n_k'} - x_{n_k'}\| \|x_{n_k'}\|
\]

\[
+ 2\|u - Tz_{n_k'}\| \|JTx_{n_k'} - Jx_{n_k'}\|.
\]

(3.21)
Since \( \lim_{n \to \infty} \| x_n - Tz_n \| = 0 \) and hence \( \lim_{n \to \infty} \| Jx_n - JTz_n \| = 0 \), it follows that

\[
a = \lim \inf_{j \to \infty} \phi(u, x_{n_j}) \leq \lim \inf_{j \to \infty} \phi(u, z_{n_j}).
\]  

(3.22)

We also have from (3.3) that

\[
\limsup_{j \to \infty} \phi(u, z_{n_j}) \leq \limsup_{j \to \infty} \phi(u, x_{n_j}) = a,
\]

(3.23)

and hence

\[
\lim_{j \to \infty} \phi(u, x_{n_j}) = \lim_{j \to \infty} \phi(u, z_{n_j}) = a.
\]

(3.24)

Since \( \liminf_{n \to \infty} \beta_n (1 - \beta_n) > 0 \), it follows from (3.19) that \( \lim_{j \to \infty} g(\| Jx_{n_j} - JSy_{n_j} \|) = 0 \). By properties of the function \( g \), we have \( \lim_{j \to \infty} \| Jx_{n_j} - JSy_{n_j} \| = 0 \). Since \( J^{-1} \) is also uniformly norm-to-norm continuous on bounded sets, we obtain \( \lim_{j \to \infty} \| x_{n_j} - S y_{n_j} \| = 0 \) and then

\[
\lim_{n \to \infty} \| x_n - S y_n \| = 0.
\]

(3.25)

So, we have \( \lim_{n \to \infty} \| Jx_n - JSy_n \| = 0 \). Since

\[
\| Jz_n - Jx_n \| = \| \beta_n Jx_n + (1 - \beta_n) JSy_n - Jx_n \|
= (1 - \beta_n) \| JSy_n - Jx_n \| \leq \| JSy_n - Jx_n \|,
\]

(3.26)

it follows that \( \lim_{n \to \infty} \| Jz_n - Jx_n \| = 0 \), and hence

\[
\lim_{n \to \infty} \| x_n - z_n \| = 0.
\]

(3.27)

From (3.3), we have

\[
\frac{1}{1 - \beta_n} (\phi(u, z_n) - \beta_n \phi(u, x_n)) \leq \phi(u, y_n).
\]

(3.28)

Using \( y_n = J_{r_n} x_n \) and Lemma 2.6, we have

\[
\phi(y_n, x_n) = \phi(J_{r_n} x_n, x_n) \leq \phi(u, x_n) - \phi(u, J_{r_n} x_n) = \phi(u, x_n) - \phi(u, y_n).
\]

(3.29)
It follows that

\[
\phi(y_n, x_n) \leq \phi(u, x_n) - \phi(u, y_n)
\]

\[
\leq \phi(u, x_n) - \frac{1}{1 - \beta_n} (\phi(u, z_n) - \beta_n \phi(u, x_n))
\]

\[
= \frac{1}{1 - \beta_n} (\phi(u, x_n) - \phi(u, z_n))
\]

\[
= \frac{1}{1 - \beta_n} \left( \|x_n\|^2 - \|z_n\|^2 - 2(u, Jx_n - Jz_n) \right)
\]

\[
\leq \frac{1}{1 - \beta_n} \left( \|x_n - z_n\| (\|x_n\| + \|z_n\|) + 2\|u\| \|Jx_n - Jz_n\| \right)
\]

\[
\leq \frac{1}{1 - \beta_n} (\|x_n - z_n\| (\|x_n\| + \|z_n\|) + 2\|u\| \|Jx_n - Jz_n\|).
\]  

(3.30)

Since \(\liminf_{n \to \infty} \beta_n (1 - \beta_n) > 0\), we have that \(\liminf_{n \to \infty} (1 - \beta_n) > 0\). So, we have \(\lim_{n \to \infty} \phi(y_n, x_n) = 0\). Since \(E\) is uniformly convex and smooth, we have from Lemma 2.1 that

\[
\lim_{n \to \infty} \|y_n - x_n\| = 0.
\]  

(3.31)

Since

\[
\|z_n - Tz_n\| \leq \|z_n - x_n\| + \|x_n - Tz_n\|,
\]

\[
\|y_n - Sy_n\| \leq \|y_n - x_n\| + \|x_n - Sy_n\|,
\]

from (3.17), (3.25), (3.27), and (3.31), we obtain that

\[
\lim_{n \to \infty} \|z_n - Tz_n\| = \lim_{n \to \infty} \|y_n - Sy_n\| = 0.
\]  

(3.33)

Since \(\{x_n\}\) is bounded, there exists a subsequence \(\{x_{n_k}\}\) of \(\{x_n\}\) such that \(x_{n_k} \to v\). From \(\lim_{n \to \infty} \|x_n - y_n\| = 0\) and \(\lim_{n \to \infty} \|x_n - z_n\| = 0\), we have \(y_{n_k} \to v\) and \(z_{n_k} \to v\). Since \(S\) and \(T\) are relatively nonexpansive, we have that \(v \in \tilde{F}(S) \cap \tilde{F}(T) = F(S) \cap F(T)\). Next, we show \(v \in A^{-1}0\). Since \(J\) is uniformly norm-to-norm continuous on bounded sets, from (3.31) we have

\[
\lim_{n \to \infty} \|Jx_n - Jy_n\| = 0.
\]  

(3.34)
From $r_n \geq a$, we have

$$\lim_{n \to \infty} \frac{1}{r_n} \| Jx_n - Jy_n \| = 0. \quad (3.35)$$

Therefore, we have

$$\lim_{n \to \infty} \| A_{r_n} x_n \| = \lim_{n \to \infty} \frac{1}{r_n} \| Jx_n - Jy_n \| = 0. \quad (3.36)$$

For $(p, p^*) \in A$, from the monotonicity of $A$, we have $\langle p - y_n, p^* - A_{r_n} x_n \rangle \geq 0$ for all $n \geq 0$. Replacing $n$ by $n_k$ and letting $k \to \infty$, we get $\langle p - v, p^* \rangle \geq 0$. From the maximality of $A$, we have $v \in A^{-1}0$, that is, $v \in \Omega$.

Finally, we show that $x_n \to \Pi_\Omega x_0$. Let $w = \Pi_\Omega x_0$. From $x_{n+1} = \Pi_{C_n \cap Q_n} x_0$ and $w \in \Omega \subset C_n \cap Q_n$, we obtain that

$$\phi(x_{n+1}, x_0) \leq \phi(w, x_0). \quad (3.37)$$

Since the norm is weakly lower semicontinuous, we have

$$\phi(v, x_0) = \| v \|^2 - 2 \langle v, Jx_0 \rangle + \| x_0 \|^2 \leq \liminf_{k \to \infty} \left( \| x_{n_k} \|^2 - 2 \langle x_{n_k}, Jx_0 \rangle + \| x_0 \|^2 \right) \quad (3.38)$$

$$= \liminf_{k \to \infty} \phi(x_{n_k}, x_0) \leq \limsup_{k \to \infty} \phi(x_{n_k}, x_0) \leq \phi(w, x_0).$$

From the definition of $\Pi_\Omega$, we obtain $v = w$. This implies that

$$\lim_{k \to \infty} \phi(x_{n_k}, x_0) = \phi(w, x_0). \quad (3.39)$$

Therefore we have

$$0 = \lim_{k \to \infty} \left( \phi(x_{n_k}, x_0) - \phi(w, x_0) \right)$$

$$= \lim_{k \to \infty} \left( \| x_{n_k} \|^2 - \| w \|^2 - 2 \langle x_{n_k} - w, Jx_0 \rangle \right) \quad (3.40)$$

$$= \lim_{k \to \infty} \left( \| x_{n_k} \|^2 - \| w \|^2 \right).$$

Since $E$ has the Kadec-Klee property, we obtain that $x_{n_k} \to w = \Pi_\Omega x_0$. Since $\{x_{n_k}\}$ is an arbitrary weakly convergent subsequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$. This completes the proof. \qed
As direct consequences of Theorem 3.1, we can obtain the following corollaries.

**Corollary 3.2.** Let $E$ be a uniformly convex and uniformly smooth Banach space and let $C$ be a nonempty closed convex subset of $E$. Let $A \subset E \times E^*$ be a maximal monotone operator satisfying $D(A) \subset C$ and let $J_r = (J + rA)^{-1}J$ for all $r > 0$. Let $T$ be a relatively nonexpansive mapping from $C$ into itself such that $\Omega = F(T) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$ and

$$u_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JTz_n),$$

$$z_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n)JTJ_r x_n),$$

$$C_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\},$$

$$Q_n = \{z \in C : (x_n - z, Jx_0 - Jx_n) \geq 0\},$$

$$x_{n+1} = \Pi_{C_n \cap Q_n} x_0$$

for all $n \in \mathbb{N} \cup \{0\}$, where $J$ is the duality mapping on $E$, $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$, and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If $\liminf_{n \to \infty} (1 - \alpha_n) > 0$ and $\liminf_{n \to \infty} \beta_n (1 - \beta_n) > 0$, then $\{x_n\}$ converges strongly to $\Pi_{\Omega} x_0$, where $\Pi_{\Omega}$ is the generalized projection of $E$ onto $\Omega$.

**Proof.** Putting $S = T$ in Theorem 3.1, we obtain Corollary 3.2.

**Corollary 3.3.** Let $E$ be a uniformly convex and uniformly smooth Banach space and let $C$ be a nonempty closed convex subset of $E$. Let $A \subset E \times E^*$ be a maximal monotone operator satisfying $D(A) \subset C$ and let $J_r = (J + rA)^{-1}J$ for all $r > 0$. Let $S : C \to C$ be a relatively nonexpansive mapping such that $\Omega = F(S) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$ and

$$u_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n)JSJ_r x_n),$$

$$C_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\},$$

$$Q_n = \{z \in C : (x_n - z, Jx_0 - Jx_n) \geq 0\},$$

$$x_{n+1} = \Pi_{C_n \cap Q_n} x_0$$

for all $n \in \mathbb{N} \cup \{0\}$, where $J$ is the duality mapping on $E$, $\{\beta_n\} \subset [0, 1]$, and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If $\liminf_{n \to \infty} \beta_n (1 - \beta_n) > 0$, then $\{x_n\}$ converges strongly to $\Pi_{\Omega} x_0$, where $\Pi_{\Omega}$ is the generalized projection of $E$ onto $\Omega$.

**Proof.** Putting $T = I$ and $\alpha_n = 0$ in Theorem 3.1, we obtain Corollary 3.3.

Let $E$ be a Banach space and let $f : E \to (-\infty, \infty]$ be a proper lower semicontinuous convex function. Define the subdifferential of $f$ as follows:

$$\partial f(x) = \{x^* \in E : f(y) \geq \langle y - x, x^* \rangle + f(x), \forall y \in E\}$$

for each $x \in E$. Then, we know that $\partial f$ is a maximal monotone operator; see [18] for more details.
Corollary 3.4. Let $E$ be a uniformly convex and uniformly smooth Banach space and let $C$ be a nonempty closed convex subset of $E$. Let $S$ and $T$ be relatively nonexpansive mappings from $C$ into itself such that $\Omega = F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$ and

$$
\begin{align*}
  u_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JT_n), \\
  z_n &= J^{-1}(\beta_n Jx_n + (1 - \beta_n) JSx_n), \\
  C_n &= \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
  Q_n &= \{z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\
  x_{n+1} &= \Pi_{C_n \cap Q_n} x_n
\end{align*}
$$

for all $n \in \mathbb{N} \cup \{0\}$, where $J$ is the duality mapping on $E$ and $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$. If $\liminf_{n \to \infty} (1 - \alpha_n) > 0$ and $\liminf_{n \to \infty} \beta_n (1 - \beta_n) > 0$, then $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$, where $\Pi_\Omega$ is the generalized projection of $E$ onto $\Omega$.

Proof. Set $A = \partial i_C$ in Theorem 3.1, where $i_C$ is the indicator function; that is,

$$
i_C(x) = \begin{cases} 0, & x \in C, \\ \infty, & \text{otherwise.} \end{cases}
$$

Then, we have that $A$ is a maximal monotone operator and $J_r = \Pi_C$ for $r > 0$. In fact, for any $x \in E$ and $r > 0$, we have from Lemma 2.3 that

$$
z = J_r x \iff z + r \partial i_C(z) \ni J_x$$

$$
\iff Jx - z \in \partial i_C(z)$$

$$
\iff i_C(y) \geq \left( y - z, \frac{Jx - Jz}{r} \right) + i_C(z), \quad \forall y \in E
$$

$$
\iff 0 \geq \langle y - z, Jx - Jz \rangle, \quad \forall y \in C
$$

$$
\iff z = \arg \min_{y \in C} \phi(y, x)
$$

$$
\iff z = \Pi_C x.
$$

So, from Theorem 3.1, we obtain Corollary 3.4.

4. Applications

In this section, we discuss the problem of strong convergence concerning a maximal monotone operator and two nonexpansive mappings in a Hilbert space. Using Theorem 3.1, we obtain the following results.
**Theorem 4.1.** Let $C$ be a nonempty closed convex subset of a Hilbert space $H$. Let $A \subset H \times H$ be a monotone operator satisfying $D(A) \subset C$ and let $J_r = (I + rA)^{-1}$ for all $r > 0$. Let $S$ and $T$ be nonexpansive mappings from $C$ into itself such that $\Omega = F(S) \cap F(T) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$ and

$$u_n = \alpha_n x_n + (1 - \alpha_n) Tz_n,$$
$$z_n = \beta_n x_n + (1 - \beta_n) Sf_{r_n} x_n,$$
$$C_n = \{z \in C : \|z - u_n\| \leq \|z - x_n\|\},$$
$$Q_n = \{z \in C : (x_n - z, x_0 - x_n) \geq 0\},$$
$$x_{n+1} = P_{C_n \cap Q_n} x_0$$

for all $n \in \mathbb{N} \cup \{0\}$, where $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If $\liminf_{n \to \infty} (1 - \alpha_n) > 0$ and $\liminf_{n \to \infty} \beta_n (1 - \beta_n) > 0$, then $\{x_n\}$ converges strongly to $P_\Omega x_0$, where $P_\Omega$ is the metric projection of $H$ onto $\Omega$.

**Proof.** We know that every nonexpansive mapping with a fixed point is a relatively nonexpansive one. We also know that $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$. Using Theorem 3.1, we are easily able to obtain the desired conclusion by putting $J = I$. This completes the proof. \qed

The following corollary follows from Theorem 4.1.

**Corollary 4.2.** Let $C$ be a nonempty closed convex subset of a Hilbert space $H$. Let $A \subset H \times H$ be a monotone operator satisfying $D(A) \subset C$ and let $J_r = (I + rA)^{-1}$ for all $r > 0$. Let $T$ be a nonexpansive mapping from $C$ into itself such that $\Omega = F(T) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$ and

$$u_n = \alpha_n x_n + (1 - \alpha_n) Tz_n,$$
$$z_n = \beta_n x_n + (1 - \beta_n) Tf_{r_n} x_n,$$
$$C_n = \{z \in C : \|z - u_n\| \leq \|z - x_n\|\},$$
$$Q_n = \{z \in C : (x_n - z, x_0 - x_n) \geq 0\},$$
$$x_{n+1} = P_{C_n \cap Q_n} x_0$$

for all $n \in \mathbb{N} \cup \{0\}$, where $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If $\liminf_{n \to \infty} (1 - \alpha_n) > 0$ and $\liminf_{n \to \infty} \beta_n (1 - \beta_n) > 0$, then $\{x_n\}$ converges strongly to $P_\Omega x_0$, where $P_\Omega$ is the metric projection of $H$ onto $\Omega$.

**Proof.** Putting $S = T$ in Theorem 4.1, we obtain Corollary 4.2. \qed

**Corollary 4.3.** Let $C$ be a nonempty closed convex subset of a Hilbert space $H$. Let $A \subset H \times H$ be a maximal monotone operator satisfying $D(A) \subset C$ and let $J_r = (I + rA)^{-1}$ for all $r > 0$. Let $S$ be
a nonexpansive mapping from $C$ into itself such that $\Omega = F(S) \cap A^{-1}0 \neq \emptyset$. Let \( \{x_n\} \) be a sequence generated by $x_0 \in C$ and

\[
\begin{aligned}
u_n &= \beta_n x_n + (1 - \beta_n) S r_n x_n, \\
c_n &= \{z \in C : \|z - u_n\| \leq \|z - x_n\|\}, \\
q_n &= \{z \in C : (x_n - z, x_0 - x_n) \geq 0\}, \\
x_{n+1} &= P_{c_n \cap q_n} x_0
\end{aligned}
\tag{4.3}
\]

for all $n \in \mathbb{N} \cup \{0\}$, where $\{\beta_n\} \subset [0,1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If $\inf_{n \to \infty} \beta_n(1 - \beta_n) > 0$ then $\{x_n\}$ converges strongly to $P_{\Omega} x_0$, where $P_{\Omega}$ is the metric projection of $H$ onto $\Omega$.

**Proof.** Putting $T = I$ and $\alpha_n = 0$ in Theorem 4.1, we obtain Corollary 4.3.

**Corollary 4.4.** Let $C$ be a nonempty closed convex subset of a Hilbert space $H$. Let $S$ and $T$ be nonexpansive mappings from $C$ into itself such that $\Omega = F(S) \cap F(T) \neq \emptyset$. Let \( \{x_n\} \) be a sequence generated by $x_0 = x \in C$ and

\[
\begin{aligned}
u_n &= \alpha_n x_n + (1 - \alpha_n) T z_n, \\
z_n &= \beta_n x_n + (1 - \beta_n) S x_n, \\
c_n &= \{z \in C : \|z - u_n\| \leq \|z - x_n\|\}, \\
q_n &= \{z \in C : (x_n - z, x_0 - x_n) \geq 0\}, \\
x_{n+1} &= P_{c_n \cap q_n} x_0
\end{aligned}
\tag{4.4}
\]

for all $n \in \mathbb{N} \cup \{0\}$, where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. If $\inf_{n \to \infty} (1 - \alpha_n) > 0$ and $\inf_{n \to \infty} \beta_n(1 - \beta_n) > 0$, then $\{x_n\}$ converges strongly to $P_{\Omega} x_0$, where $P_{\Omega}$ is the metric projection of $H$ onto $\Omega$.

**Proof.** Set $A = \partial i_C$ in Theorem 4.1, where $i_C$ is the indicator function; that is,

\[
i_C(x) = \begin{cases} 0, & x \in C, \\ \infty, & \text{otherwise.} \end{cases}
\tag{4.5}
\]

Then, we have that $A$ is a maximal monotone operator and $F_r = P_C$ for $r > 0$. In fact, for any $x \in E$ and $r > 0$, we have that

\[
\begin{aligned}z &= F_r x \iff z + r \partial i_C(z) \ni x \\
&\iff x - z \in r \partial i_C(z) \\
&\iff i_C(y) \geq \langle y - z, \frac{x - z}{r} \rangle + i_C(z), \quad \forall y \in E \\
&\iff 0 \geq (y - z, x - z), \quad \forall y \in C \\
&\iff z = P_C x.
\end{aligned}
\tag{4.6}
\]

So, from Theorem 4.1, we obtain Corollary 4.4. □
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