Research Article

Multiplicity Results for p -Laplacian with Critical Nonlinearity of Concave-Convex Type and Sign-Changing Weight Functions

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The multiple results of positive solutions for the following quasilinear elliptic equation: $-\Delta_p u = \lambda f(x)|u|^{q-2}u + g(x)|u|^{p^*-2}u$ in Ω , $u = 0$ on $\partial\Omega$, are established. Here, $0 \in \Omega$ is a bounded smooth domain in $\mathbb{R}^N$, Δ_p denotes the p -Laplacian operator, $1 \leq q < p < N$, $p^* = Np/(N-p)$, λ is a positive real parameter, and f, g are continuous functions on $\overline{\Omega}$ which are somewhere positive but which may change sign on Ω . The study is based on the extraction of Palais-Smale sequences in the Nehari manifold.

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1. Introduction

In this paper, we study the multiple results of positive solutions for the following quasilinear elliptic equation:

$$\begin{aligned} -\Delta_p u &= \lambda f(x)|u|^{q-2}u + g(x)|u|^{p^*-2}u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{E_{\lambda f, g}}$$

where $\lambda > 0$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p -Laplacian, $0 \in \Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $1 < q < p < N$, $p^* = Np/(N-p)$ is the so-called critical Sobolev exponent and the weight functions f, g are satisfying the following conditions:

- (f1) $f \in C(\overline{\Omega})$ and $f^+ = \max\{f, 0\} \not\equiv 0$;
- (f2) there exist $\beta_0, \rho_0 > 0$ and $x_0 \in \Omega$ such that $B(x_0, 2\rho_0) \subset \Omega$ and $f(x) \geq \beta_0$ for all $x \in B(x_0, 2\rho_0)$. Without loss of generality, we assume that $x_0 = 0$,
- (g1) $g \in C(\overline{\Omega})$ and $g^+ = \max\{g, 0\} \not\equiv 0$;
- (g2) $|g^+|_\infty = g(0) = \max_{x \in \overline{\Omega}} g(x)$;

- (g3) $g(x) > 0$ for all $x \in B(0, 2\rho_0)$;
 (g4) there exists $\beta > N/(p-1)$ such that

$$g(x) = g(0) + o(|x|^\beta) \quad \text{as } x \rightarrow 0. \quad (1.1)$$

For the weight functions $f \equiv g \equiv 1$, $(E_{\lambda f, g})$ has been studied extensively. Historically, the role played by such concave-convex nonlinearities in producing multiple solutions was investigated first in the work [1]. They studied the following semilinear elliptic equation:

$$\begin{aligned} -\Delta u &= \lambda u^{q-1} + u^{2^*-1} \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (1.2)$$

for $1 < q < 2$ and showed the existence of $\lambda_0 > 0$ such that (1.2) admits at least two solutions for all $\lambda \in (0, \lambda_0)$ and no solution for $\lambda > \lambda_0$. Subsequently, in the work [2, 3], the corresponding quasilinear version has been studied

$$\begin{aligned} -\Delta_p u &= \lambda u^{q-1} + u^{p^*-1} \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (1.3)$$

where $1 < p < N$ and $1 < q < p$. They obtained results similar to the results of [1] above, but only for some ranges of the exponents p and q . We summarize their results in what follows.

Theorem 1.1 (see [2, 3]). *Assume that either $2N/(N+2) < p < 3$ or $p > 3, p > q > p^* - 2/(p-1)$. Then there exists $\lambda_0 > 0$ such that (1.3) admits at least two solutions for all $\lambda \in (0, \lambda_0)$ and no solution for $\lambda > \lambda_0$.*

It is possible to get complete multiplicity result for problem (1.3) if Ω is taken to be a ball in $\mathbb{R}^N$. Prashanth and Sreenadh [4] have studied (1.3) in the unit ball $B^N(0; 1)$ in $\mathbb{R}^N$ and obtained the following results.

Theorem 1.2 (see [4]). *Let $\Omega = B^N(0; 1), 1 < p < N, 1 < q < p$. Then there exists $\lambda_0 > 0$ such that (1.3) admits at least two solutions for all $\lambda \in (0, \lambda_0)$ and no solution for $\lambda > \lambda_0$. Additionally, if $1 < p < 2$, then (1.3) admits exactly two solutions for all small $\lambda > 0$.*

For $p = 2$, Tang [5] has studied the exact multiplicity about the following semilinear elliptic equation:

$$\begin{aligned} -\Delta u &= \lambda u^{q-1} + u^{r-1} \quad \text{in } B^N(0; 1), \\ u &> 0 \quad \text{in } B^N(0; 1), \\ u &= 0 \quad \text{on } \partial B^N(0; 1), \end{aligned} \quad (1.4)$$

where $1 < q < 2 < r \leq 2N/(N-2)$ and $N \geq 3$. We also mention his result below.

Theorem 1.3 (see [5]). *There exists $\lambda_0 > 0$ such that (1.4) admits exactly two solutions for $\lambda \in (0, \lambda_0)$, exactly one solution for $\lambda = \lambda_0$, and no solution for $\lambda > \lambda_0$.*

To proceed, we make some motivations of the present paper. Recently, in [6] the author has considered (1.2) with subcritical nonlinearity of concave-convex type, $g \equiv 1$, and f is a continuous function which changes sign in $\overline{\Omega}$, and showed the existence of $\lambda_0 > 0$ such that (1.2) admits at least two solutions for all $\lambda \in (0, \lambda_0)$ via the extraction of Palais-Smale sequences in the Nehari manifold. In a recent work [7], the author extended the results of [6] to the quasilinear case with the more general weight functions f, g but also having subcritical nonlinearity of concave-convex type. In the present paper, we continue the study of [7] by considering critical nonlinearity of concave-convex type and sign-changing weight functions f, g .

In this paper, we use a variational method involving the Nehari manifold to prove the multiplicity of positive solutions. The Nehari method has been used also in [8] to prove the existence of multiple for a singular elliptic problem. The existence of at least one solution can be obtained by using the same arguments as in the subcritical case [7]. The existence of a second solution needs different arguments due to the lack of compactness of the Palais-Smale sequences. For what, we need additional assumptions (f2) and (g2) to prove the compactness of the extraction of Palais-Smale sequences in the Nehari manifold (see Theorem 4.4). The multiplicity result is proved only for the parameter $\lambda \in (0, (q/p)\Lambda_1)$ (see Theorem 1.5) but for all $1 < p < N$ and $1 \leq q < p$. This is not the case in the papers referred [2, 3] where the multiplicity is global but not with the full range of p, q and with the weight functions $f \equiv g \equiv 1$. Finally, we mention a recent contribution on p -Laplacian equation with changing sign nonlinearity by Figuereido et al. [9] which gives the global multiplicity but not with the full range of p and q . The method used in the paper by Figuereido et al. is similar to the method introduced in [1].

In order to represent our main results, we need to define the following constant Λ_1 . Set

$$\Lambda_1 = \left(\frac{p-q}{(p^*-q)|g^+|_\infty} \right)^{(p-q)/(p^*-p)} \left(\frac{p^*-p}{(p^*-q)|f^+|_\infty} \right) |\Omega|^{(q-p^*)/p^*} S^{(N/p)-(N/p^2)q+(q/p)} > 0, \quad (1.5)$$

where $|\Omega|$ is the Lebesgue measure of Ω and S is the best Sobolev constant (see (2.2)).

Theorem 1.4. *Assume (f1) and (g1) hold. If $\lambda \in (0, \Lambda_1)$, then $(E_{\lambda f, g})$ admits at least one positive solution $u_\lambda \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$.*

Theorem 1.5. *Assume that (f1)-(f2) and (g1)-(g4) hold. If $\lambda \in (0, (q/p)\Lambda_1)$, then $(E_{\lambda f, g})$ admits at least two positive solutions $u_\lambda, U_\lambda \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$.*

This paper is organized as follows. In Section 2, we give some preliminaries and some properties of Nehari manifold. In Sections 3 and 4, we complete proofs of Theorems 1.4 and 1.5.

2. Preliminaries and Nehari Manifold

Throughout this paper, (f1) and (g1) will be assumed. The dual space of a Banach space E will be denoted by E^{-1} . $W_0^{1,p}(\Omega)$ denotes the standard Sobolev space with the following

norm:

$$\|u\|^p = \int_{\Omega} |\nabla u|^p dx. \quad (2.1)$$

$W_0^{1,p}(\Omega)$ with the norm $\|\cdot\|$ is simply denoted by W . We denote the norm in $L^p(\Omega)$ by $|\cdot|_p$ and the norm in $L^p(\mathbb{R}^N)$ by $|\cdot|_{L^p(\mathbb{R}^N)}$. $|\Omega|$ is the Lebesgue measure of Ω . $B(x, r)$ is a ball centered at x with radius r . $O(\varepsilon^t)$ denotes $|O(\varepsilon^t)|/\varepsilon^t \leq C$, $o(\varepsilon^t)$ denotes $|o(\varepsilon^t)|/\varepsilon^t \rightarrow 0$ as $\varepsilon \rightarrow 0$, and $o_n(1)$ denotes $o_n(1) \rightarrow 0$ as $n \rightarrow \infty$. C, C_i will denote various positive constants; the exact values of which are not important. S is the best Sobolev embedding constant defined by

$$S = \inf_{u \in W \setminus \{0\}} \frac{|\nabla u|_p^p}{|u|_{p^*}^p}. \quad (2.2)$$

Definition 2.1. Let $c \in \mathbb{R}$, E be a Banach space and $I \in C^1(E, \mathbb{R})$.

- (i) $\{u_n\}$ is a $(PS)_c$ -sequence in E for I if $I(u_n) = c + o_n(1)$ and $I'(u_n) = o_n(1)$ strongly in E^{-1} as $n \rightarrow \infty$.
- (ii) We say that I satisfies the $(PS)_c$ condition if any $(PS)_c$ -sequence $\{u_n\}$ in E for I has a convergent subsequence.

Associated with $(E_{\lambda f, g})$, we consider the energy functional J_{λ} in W , for each $u \in W$,

$$J_{\lambda}(u) = \frac{1}{p} \|u\|^p - \frac{\lambda}{q} \int_{\Omega} f |u|^q dx - \frac{1}{p^*} \int_{\Omega} g |u|^{p^*} dx. \quad (2.3)$$

It is well known that J_{λ} is of C^1 in W and the solutions of $(E_{\lambda f, g})$ are the critical points of the energy functional J_{λ} (see Rabinowitz [10]).

As the energy functional J_{λ} is not bounded below on W , it is useful to consider the functional on the Nehari manifold

$$\mathcal{N}_{\lambda} = \{u \in W \setminus \{0\} : \langle J'_{\lambda}(u), u \rangle = 0\}. \quad (2.4)$$

Thus, $u \in \mathcal{N}_{\lambda}$ if and only if

$$\langle J'_{\lambda}(u), u \rangle = \|u\|^p - \lambda \int_{\Omega} f |u|^q dx - \int_{\Omega} g |u|^{p^*} dx = 0. \quad (2.5)$$

Note that $\mathcal{N}_{\lambda}$ contains every nonzero solution of $(E_{\lambda f, g})$. Moreover, we have the following results.

Lemma 2.2. *The energy functional J_{λ} is coercive and bounded below on $\mathcal{N}_{\lambda}$.*

Proof. If $u \in \mathcal{N}_\lambda$, then by (f1), (2.5), and the Hölder inequality and the Sobolev embedding theorem we have

$$J_\lambda(u) = \frac{p^* - p}{p^* p} \|u\|^p - \lambda \left(\frac{p^* - q}{p^* q} \right) \int_{\Omega} f |u|^q dx \quad (2.6)$$

$$\geq \frac{1}{N} \|u\|^p - \lambda \left(\frac{p^* - q}{p^* q} \right) S^{-q/p} |\Omega|^{(p^* - q)/p^*} \|u\|^q \|f^+\|_{\infty}. \quad (2.7)$$

Thus, J_λ is coercive and bounded below on $\mathcal{N}_\lambda$. $\square$

Define

$$\psi_\lambda(u) = \langle J'_\lambda(u), u \rangle. \quad (2.8)$$

Then for $u \in \mathcal{N}_\lambda$,

$$\langle \psi'_\lambda(u), u \rangle = p \|u\|^p - \lambda q \int_{\Omega} f |u|^q dx - p^* \int_{\Omega} g |u|^{p^*} dx \quad (2.9)$$

$$= (p - q) \|u\|^p - (p^* - q) \int_{\Omega} g |u|^{p^*} dx \quad (2.10)$$

$$= \lambda (p^* - q) \int_{\Omega} f |u|^q dx - (p^* - p) \|u\|^p. \quad (2.11)$$

Similar to the method used in Tarantello [11], we split $\mathcal{N}_\lambda$ into three parts:

$$\begin{aligned} \mathcal{N}_\lambda^+ &= \{u \in \mathcal{N}_\lambda : \langle \psi'_\lambda(u), u \rangle > 0\}, \\ \mathcal{N}_\lambda^0 &= \{u \in \mathcal{N}_\lambda : \langle \psi'_\lambda(u), u \rangle = 0\}, \\ \mathcal{N}_\lambda^- &= \{u \in \mathcal{N}_\lambda : \langle \psi'_\lambda(u), u \rangle < 0\}. \end{aligned} \quad (2.12)$$

Then, we have the following results.

Lemma 2.3. Assume that u_λ is a local minimizer for J_λ on $\mathcal{N}_\lambda$ and $u_\lambda \notin \mathcal{N}_\lambda^0$. Then $J'_\lambda(u_\lambda) = 0$ in W^{-1} .

Proof. Our proof is almost the same as that in Brown and Zhang [12, Theorem 2.3] (or see Binding et al. [13]). $\square$

Lemma 2.4. One has the following.

- (i) If $u \in \mathcal{N}_\lambda^+$, then $\int_{\Omega} f |u|^q dx > 0$.
- (ii) If $u \in \mathcal{N}_\lambda^0$, then $\int_{\Omega} f |u|^q dx > 0$ and $\int_{\Omega} g |u|^{p^*} dx > 0$.
- (iii) If $u \in \mathcal{N}_\lambda^-$, then $\int_{\Omega} g |u|^{p^*} dx > 0$.

Proof. The proof is immediate from (2.10) and (2.11). $\square$

Moreover, we have the following result.

Lemma 2.5. *If $\lambda \in (0, \Lambda_1)$, then $\mathcal{N}_\lambda^0 = \emptyset$ where Λ_1 is the same as in (1.5).*

Proof. Suppose otherwise that there exists $\lambda \in (0, \Lambda_1)$ such that $\mathcal{N}_\lambda^0 \neq \emptyset$. Then by (2.10) and (2.11), for $u \in \mathcal{N}_\lambda^0$, we have

$$\begin{aligned} \|u\|^p &= \frac{p^* - q}{p - q} \int_{\Omega} g|u|^{p^*} dx, \\ \|u\|^p &= \lambda \frac{p^* - q}{p^* - p} \int_{\Omega} f|u|^q dx. \end{aligned} \quad (2.13)$$

Moreover, by (f1), (g1), and the Hölder inequality and the Sobolev embedding theorem, we have

$$\begin{aligned} \|u\| &\geq \left(\frac{p - q}{(p^* - q)|g^+|_{\infty}} S^{p^*/p} \right)^{1/(p^* - p)}, \\ \|u\| &\leq \left[\lambda \frac{p^* - q}{p^* - p} S^{-q/p} |\Omega|^{(p^* - q)/p^*} |f^+|_{\infty} \right]^{1/(p - q)}. \end{aligned} \quad (2.14)$$

This implies

$$\lambda \geq \left(\frac{p - q}{(p^* - q)|g^+|_{\infty}} \right)^{(p - q)/(p^* - p)} \left(\frac{p^* - p}{(p^* - q)|f^+|_{\infty}} \right) |\Omega|^{(q - p^*)/p^*} S^{(N/p) - (N/p^2)q + (q/p)} = \Lambda_1, \quad (2.15)$$

which is a contradiction. Thus, we can conclude that if $\lambda \in (0, \Lambda_1)$, we have $\mathcal{N}_\lambda^0 = \emptyset$. $\square$

By Lemma 2.5, we write $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$ and define

$$\alpha_\lambda = \inf_{u \in \mathcal{N}_\lambda} J_\lambda(u), \quad \alpha_\lambda^+ = \inf_{u \in \mathcal{N}_\lambda^+} J_\lambda(u), \quad \alpha_\lambda^- = \inf_{u \in \mathcal{N}_\lambda^-} J_\lambda(u). \quad (2.16)$$

Then we get the following result.

Theorem 2.6. (i) *If $\lambda \in (0, \Lambda_1)$ and $u \in \mathcal{N}_\lambda^+$, then one has $J_\lambda(u) < 0$ and $\alpha_\lambda \leq \alpha_\lambda^+ < 0$.*

(ii) *If $\lambda \in (0, (q/p)\Lambda_1)$, then $\alpha_\lambda^- > d_0$ for some positive constant d_0 depending on $\lambda, p, q, N, S, |f^+|_{\infty}, |g^+|_{\infty}$, and $|\Omega|$.*

Proof. (i) Let $u \in \mathcal{N}_\lambda^+$. By (2.10), we have

$$\frac{p - q}{p^* - q} \|u\|^p > \int_{\Omega} g|u|^{p^*} dx, \quad (2.17)$$

and so

$$\begin{aligned}
 J_\lambda(u) &= \left(\frac{1}{p} - \frac{1}{q}\right) \|u\|^p + \left(\frac{1}{q} - \frac{1}{p^*}\right) \int_{\Omega} g|u|^{p^*} dx \\
 &< \left[\left(\frac{1}{p} - \frac{1}{q}\right) + \left(\frac{1}{q} - \frac{1}{p^*}\right) \frac{p-q}{p^*-q} \right] \|u\|^p \\
 &= -\frac{p-q}{qN} \|u\|^p < 0.
 \end{aligned} \tag{2.18}$$

Therefore, from the definition of α_λ , α_λ^+ , we can deduce that $\alpha_\lambda \leq \alpha_\lambda^+ < 0$.

(ii) Let $u \in \mathcal{N}_\lambda^-$. By (2.10), we have

$$\frac{p-q}{p^*-q} \|u\|^p < \int_{\Omega} g|u|^{p^*} dx. \tag{2.19}$$

Moreover, by (g1) and the Sobolev embedding theorem, we have

$$\int_{\Omega} g|u|^{p^*} dx \leq S^{-p^*/p} \|u\|^{p^*} |g^+|_{\infty}. \tag{2.20}$$

This implies

$$\|u\| > \left(\frac{p-q}{(p^*-q)|g^+|_{\infty}} \right)^{1/(p^*-p)} S^{N/p^2}, \quad \forall u \in \mathcal{N}_\lambda^-. \tag{2.21}$$

By (2.7) in the proof of Lemma 2.2, we have

$$\begin{aligned}
 J_\lambda(u) &\geq \|u\|^q \left[\frac{p^*-p}{p^*p} \|u\|^{p-q} - \lambda S^{-q/p} \frac{p^*-q}{p^*q} |\Omega|^{(p^*-q)/p^*} |f^+|_{\infty} \right] \\
 &> \left(\frac{p-q}{(p^*-q)|g^+|_{\infty}} \right)^{q/(p^*-p)} S^{qN/p^2} \\
 &\quad \times \left[\frac{p^*-p}{p^*p} S^{(p-q)N/p^2} \left(\frac{p-q}{(p^*-q)|g^+|_{\infty}} \right)^{(p-q)/(p^*-p)} - \lambda S^{-q/p} \frac{p^*-q}{p^*q} |\Omega|^{(p^*-q)/p^*} |f^+|_{\infty} \right].
 \end{aligned} \tag{2.22}$$

Thus, if $\lambda \in (0, (q/p)\Lambda_1)$, then

$$J_\lambda(u) > d_0, \quad \forall u \in \mathcal{N}_\lambda^-, \tag{2.23}$$

for some positive constant $d_0 = d_0(\lambda, p, q, N, S, |f^+|_{\infty}, |g^+|_{\infty}, |\Omega|)$. This completes the proof. $\square$

For each $u \in W$ with $\int_{\Omega} g|u|^{p^*} dx > 0$, we write

$$t_{\max} = \left(\frac{(p-q)\|u\|^p}{(p^*-q)\int_{\Omega} g|u|^{p^*} dx} \right)^{1/(p^*-p)} > 0. \quad (2.24)$$

Then the following lemma holds.

Lemma 2.7. *Let $\lambda \in (0, \Lambda_1)$. For each $u \in W$ with $\int_{\Omega} g|u|^{p^*} dx > 0$, one has the following:*

(i) *if $\int_{\Omega} f|u|^q dx \leq 0$, then there exists a unique $t^- > t_{\max}$ such that $t^-u \in \mathcal{N}_{\lambda}^-$ and*

$$J_{\lambda}(t^-u) = \sup_{t \geq 0} J_{\lambda}(tu), \quad (2.25)$$

(ii) *if $\int_{\Omega} f|u|^q dx > 0$, then there exists unique $0 < t^+ < t_{\max} < t^-$ such that $t^+u \in \mathcal{N}_{\lambda}^+$, $t^-u \in \mathcal{N}_{\lambda}^-$, and*

$$J_{\lambda}(t^+u) = \inf_{0 \leq t \leq t_{\max}} J_{\lambda}(tu); \quad J_{\lambda}(t^-u) = \sup_{t \geq 0} J_{\lambda}(tu). \quad (2.26)$$

Proof. Fix $u \in W$ with $\int_{\Omega} g|u|^{p^*} dx > 0$. Let

$$k(t) = t^{p-q}\|u\|^p - t^{p^*-q} \int_{\Omega} g|u|^{p^*} dx \quad \text{for } t \geq 0. \quad (2.27)$$

It is clear that $k(0) = 0$, $k(t) \rightarrow -\infty$ as $t \rightarrow \infty$. From

$$k'(t) = (p-q)t^{p-q-1}\|u\|^p - (p^*-q)t^{p^*-q-1} \int_{\Omega} g|u|^{p^*} dx, \quad (2.28)$$

we can deduce that $k'(t) = 0$ at $t = t_{\max}$, $k'(t) > 0$ for $t \in (0, t_{\max})$ and $k'(t) < 0$ for $t \in (t_{\max}, \infty)$. Then $k(t)$ that achieves its maximum at $t_{\max}$ is increasing for $t \in [0, t_{\max})$ and decreasing for $t \in (t_{\max}, \infty)$. Moreover,

$$\begin{aligned} k(t_{\max}) &= \left(\frac{(p-q)\|u\|^p}{(p^*-q)\int_{\Omega} g|u|^{p^*} dx} \right)^{(p-q)/(p^*-p)} \|u\|^p \\ &\quad - \left(\frac{(p-q)\|u\|^p}{(p^*-q)\int_{\Omega} g|u|^{p^*} dx} \right)^{(p^*-q)/(p^*-p)} \int_{\Omega} g|u|^{p^*} dx \\ &= \|u\|^q \left[\left(\frac{p-q}{p^*-q} \right)^{(p-q)/(p^*-p)} - \left(\frac{p-q}{p^*-q} \right)^{(p^*-q)/(p^*-p)} \right] \left(\frac{\|u\|^{p^*}}{\int_{\Omega} g|u|^{p^*} dx} \right)^{(p-q)/(p^*-p)} \\ &\geq \|u\|^q \left(\frac{p^*-p}{p^*-q} \right) \left(\frac{p-q}{(p^*-q)|g^+|_{\infty}} S^{p^*/p} \right)^{(p-q)/(p^*-p)}. \end{aligned} \quad (2.29)$$

(i) We have $\int_{\Omega} f|u|^q dx \leq 0$. There exists a unique $t^- > t_{\max}$ such that $k(t^-) = \lambda \int_{\Omega} f|u|^q dx$ and $k'(t^-) < 0$. Now,

$$\begin{aligned}
& (p-q)(t^-)^p \|u\|^p - (p^*-q)(t^-)^p \int_{\Omega} g|u|^{p^*} dx \\
&= (t^-)^{1+q} \left[(p-q)(t^-)^{p-q-1} \|u\|^p - (p^*-q)(t^-)^{p^*-q-1} \int_{\Omega} g|u|^{p^*} dx \right] \\
&= (t^-)^{1+q} k'(t^-) < 0, \\
\langle J'_\lambda(t^-u), t^-u \rangle &= (t^-)^p \|u\|^p - (t^-)^{p^*} \int_{\Omega} g|u|^{p^*} dx - (t^-)^q \lambda \int_{\Omega} f|u|^q dx \\
&= (t^-)^q \left[k(t^-) - \lambda \int_{\Omega} f|u|^q dx \right] = 0.
\end{aligned} \tag{2.30}$$

Then we have that $t^-u \in \mathcal{N}_\lambda^-$. For $t > t_{\max}$, we have

$$\begin{aligned}
& (p-q)\|tu\|^p - (p^*-q) \int_{\Omega} g|tu|^{p^*} < 0, \quad \frac{d^2}{dt^2} J_\lambda(tu) < 0, \\
\frac{d}{dt} J_\lambda(tu) &= t^{p-1} \|u\|^p - t^{p^*-1} \int_{\Omega} g|u|^{p^*} dx - t^{q-1} \lambda \int_{\Omega} f|u|^q dx \\
&= 0 \quad \text{for } t = t^-.
\end{aligned} \tag{2.31}$$

Thus, $J_\lambda(t^-u) = \sup_{t \geq 0} J_\lambda(tu)$.

(ii) We have $\int_{\Omega} f|u|^q dx > 0$. By (2.29) and

$$\begin{aligned}
k(0) &= 0 < \lambda \int_{\Omega} f|u|^q dx \\
&\leq \lambda S^{-q/p} |\Omega|^{(p^*-q)/p^*} \|u\|^q \|f^+\|_\infty \\
&< \|u\|^q \left(\frac{p^*-p}{p^*-q} \right) \left(\frac{p-q}{(p^*-q) \|g^+\|_\infty} S^{p^*/p} \right)^{(p-q)/(p^*-p)} \\
&\leq k(t_{\max}) \quad \text{for } \lambda \in (0, \Lambda_1),
\end{aligned} \tag{2.32}$$

there are unique t^+ and t^- such that $0 < t^+ < t_{\max} < t^-$,

$$\begin{aligned}
k(t^+) &= \lambda \int_{\Omega} f|u|^q dx = k(t^-), \\
k'(t^+) &> 0 > k'(t^-).
\end{aligned} \tag{2.33}$$

We have $t^+u \in \mathcal{N}_\lambda^+$, $t^-u \in \mathcal{N}_\lambda^-$, and $J_\lambda(t^-u) \geq J_\lambda(tu) \geq J_\lambda(t^+u)$ for each $t \in [t^+, t^-]$ and $J_\lambda(t^+u) \leq J_\lambda(tu)$ for each $t \in [0, t^+]$. Thus,

$$J_\lambda(t^+u) = \inf_{0 \leq t \leq t_{\max}} J_\lambda(tu), \quad J_\lambda(t^-u) = \sup_{t \geq 0} J_\lambda(tu). \quad (2.34)$$

This completes the proof. $\square$

3. Proof of Theorem 1.4

First, we will use the idea of Tarantello [11] to get the following results.

Lemma 3.1. *If $\lambda \in (0, \Lambda_1)$, then for each $u \in \mathcal{N}_\lambda$, there exist $\epsilon > 0$ and a differentiable function $\xi : B(0; \epsilon) \subset W \rightarrow \mathbb{R}^+$ such that $\xi(0) = 1$, the function $\xi(v)(u - v) \in \mathcal{N}_\lambda$, and*

$$\langle \xi'(0), v \rangle = \frac{p \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \, dx - \lambda q \int_\Omega f |u|^{q-2} uv \, dx - p^* \int_\Omega g |u|^{p^*-2} uv \, dx}{(p - q) \|u\|^p - (p^* - q) \int_\Omega g |u|^{p^*} \, dx} \quad (3.1)$$

for all $v \in W$.

Proof. For $u \in \mathcal{N}_\lambda$, define a function $F : \mathbb{R} \times W \rightarrow \mathbb{R}$ by

$$\begin{aligned} F_u(\xi, w) &= \langle J'_\lambda(\xi(u - w)), \xi(u - w) \rangle \\ &= \xi^p \int_\Omega |\nabla(u - w)|^p \, dx - \xi^q \lambda \int_\Omega f |u - w|^{q-2} (u - w) \, dx \\ &\quad - \xi^{p^*} \int_\Omega g |u - w|^{p^*-2} (u - w) \, dx. \end{aligned} \quad (3.2)$$

Then $F_u(1, 0) = \langle J'_\lambda(u), u \rangle = 0$ and

$$\begin{aligned} \frac{d}{d\xi} F_u(1, 0) &= p \|u\|^p - \lambda q \int_\Omega f |u|^{q-2} u \, dx - p^* \int_\Omega g |u|^{p^*-2} u \, dx \\ &= (p - q) \|u\|^p - (p^* - q) \int_\Omega g |u|^{p^*} \, dx \neq 0. \end{aligned} \quad (3.3)$$

According to the implicit function theorem, there exist $\epsilon > 0$ and a differentiable function $\xi : B(0; \epsilon) \subset W \rightarrow \mathbb{R}$ such that $\xi(0) = 1$,

$$\langle \xi'(0), v \rangle = \frac{p \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \, dx - \lambda q \int_\Omega f |u|^{q-2} uv \, dx - p^* \int_\Omega g |u|^{p^*-2} uv \, dx}{(p - q) \|u\|^p - (p^* - q) \int_\Omega g |u|^{p^*} \, dx}, \quad (3.4)$$

$$F_u(\xi(v), v) = 0, \quad \forall v \in B(0; \epsilon),$$

which is equivalent to

$$\langle J'_\lambda(\xi(v)(u-v)), \xi(v)(u-v) \rangle = 0, \quad \forall v \in B(0; \epsilon), \quad (3.5)$$

that is, $\xi(v)(u-v) \in \mathcal{N}_\lambda$. $\square$

Lemma 3.2. *Let $\lambda \in (0, \Lambda_1)$, then for each $u \in \mathcal{N}_\lambda^-$, there exist $\epsilon > 0$ and a differentiable function $\xi^- : B(0; \epsilon) \subset W \rightarrow \mathbb{R}^+$ such that $\xi^-(0) = 1$, the function $\xi^-(v)(u-v) \in \mathcal{N}_\lambda^-$, and*

$$\langle (\xi^-)'(0), v \rangle = \frac{p \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \, dx - \lambda q \int_\Omega f |u|^{q-2} u v \, dx - p^* \int_\Omega g |u|^{p^*-2} u v \, dx}{(p-q) \|u\|^p - (p^*-q) \int_\Omega g |u|^{p^*} \, dx} \quad (3.6)$$

for all $v \in W$.

Proof. Similar to the argument in Lemma 3.1, there exist $\epsilon > 0$ and a differentiable function $\xi^- : B(0; \epsilon) \subset W \rightarrow \mathbb{R}$ such that $\xi^-(0) = 1$ and $\xi^-(v)(u-v) \in \mathcal{N}_\lambda$ for all $v \in B(0; \epsilon)$. Since

$$\langle \psi'_\lambda(u), u \rangle = (p-q) \|u\|^p - (p^*-q) \int_\Omega g |u|^{p^*} \, dx < 0. \quad (3.7)$$

Thus, by the continuity of the function ξ^- , we have

$$\begin{aligned} \langle \psi'_\lambda(\xi^-(v)(u-v)), \xi^-(v)(u-v) \rangle &= (p-q) \|\xi^-(v)(u-v)\|^p \\ &\quad - (p^*-q) \int_\Omega g |\xi^-(v)(u-v)|^{p^*} \, dx < 0, \end{aligned} \quad (3.8)$$

if ϵ sufficiently small, this implies that $\xi^-(v)(u-v) \in \mathcal{N}_\lambda^-$. $\square$

Proposition 3.3. (i) *If $\lambda \in (0, \Lambda_1)$, then there exists a $(PS)_{\alpha_\lambda}$ -sequence $\{u_n\} \subset \mathcal{N}_\lambda$ in W for J_λ .*
(ii) *If $\lambda \in (0, (q/p)\Lambda_1)$, then there exists a $(PS)_{\alpha_\lambda^-}$ -sequence $\{u_n\} \subset \mathcal{N}_\lambda^-$ in W for J_λ .*

Proof. (i) By Lemma 2.2 and the Ekeland variational principle [14], there exists a minimizing sequence $\{u_n\} \subset \mathcal{N}_\lambda$ such that

$$\begin{aligned} J_\lambda(u_n) &< \alpha_\lambda + \frac{1}{n}, \\ J_\lambda(u_n) &< J_\lambda(w) + \frac{1}{n} \|w - u_n\| \quad \text{for each } w \in \mathcal{N}_\lambda. \end{aligned} \quad (3.9)$$

By $\alpha_\lambda < 0$ and taking n large, we have

$$\begin{aligned} J_\lambda(u_n) &= \left(\frac{1}{p} - \frac{1}{p^*} \right) \|u_n\|^p - \left(\frac{1}{q} - \frac{1}{p^*} \right) \lambda \int_\Omega f |u_n|^q \, dx \\ &< \alpha_\lambda + \frac{1}{n} < \frac{\alpha_\lambda}{p}. \end{aligned} \quad (3.10)$$

From (2.7), (3.10), $\alpha_\lambda < 0$, and the Hölder inequality, we deduce that

$$|f^+|_\infty \lambda S^{-q/p} |\Omega|^{(p^*-q)/p^*} \|u_n\|^q \geq \lambda \int_\Omega f |u_n|^q dx > \frac{-p^* q}{p(p^* - q)} \alpha_\lambda > 0. \quad (3.11)$$

Consequently, $u_n \neq 0$ and putting together (3.10), (3.11), and the Hölder inequality, we obtain

$$\begin{aligned} \|u_n\| &> \left[\frac{-p^* q}{p \lambda (p^* - q) |f^+|_\infty} \alpha_\lambda S^{q/p} |\Omega|^{(q-p^*)/p^*} \right]^{1/q}, \\ \|u_n\| &< \left[\frac{p(p^* - q)}{q(p^* - p)} \lambda S^{-q/p} |\Omega|^{(p^*-q)/p^*} |f^+|_\infty \right]^{1/(p-q)}. \end{aligned} \quad (3.12)$$

Now, we show that

$$\|J'_\lambda(u_n)\|_{W^{-1}} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \quad (3.13)$$

Apply Lemma 3.1 with u_n to obtain the functions $\xi_n : B(0; \epsilon_n) \rightarrow \mathbb{R}^+$ for some $\epsilon_n > 0$, such that $\xi_n(w)(u_n - w) \in \mathcal{N}_\lambda$. Choose $0 < \rho < \epsilon_n$. Let $u \in W$ with $u \neq 0$ and let $w_\rho = \rho u / \|u\|$. We set $\eta_\rho = \xi_n(w_\rho)(u_n - w_\rho)$. Since $\eta_\rho \in \mathcal{N}_\lambda$, we deduce from (3.9) that

$$J_\lambda(\eta_\rho) - J_\lambda(u_n) \geq -\frac{1}{n} \|\eta_\rho - u_n\|, \quad (3.14)$$

and by the mean value theorem, we have

$$\langle J'_\lambda(u_n), \eta_\rho - u_n \rangle + o(\|\eta_\rho - u_n\|) \geq -\frac{1}{n} \|\eta_\rho - u_n\|. \quad (3.15)$$

Thus,

$$\langle J'_\lambda(u_n), -w_\rho \rangle + (\xi_n(w_\rho) - 1) \langle J'_\lambda(u_n), (u_n - w_\rho) \rangle \geq -\frac{1}{n} \|\eta_\rho - u_n\| + o(\|\eta_\rho - u_n\|). \quad (3.16)$$

Since $\xi_n(w_\rho)(u_n - w_\rho) \in \mathcal{N}_\lambda$ and (3.16) it follows that

$$-\rho \left\langle J'_\lambda(u_n), \frac{u}{\|u\|} \right\rangle + (\xi_n(w_\rho) - 1) \langle J'_\lambda(u_n) - J'_\lambda(\eta_\rho), (u_n - w_\rho) \rangle \geq -\frac{1}{n} \|\eta_\rho - u_n\| + o(\|\eta_\rho - u_n\|). \quad (3.17)$$

Thus,

$$\begin{aligned} \left\langle J'_\lambda(u_n), \frac{u}{\|u\|} \right\rangle &\leq \frac{\|\eta_\rho - u_n\|}{n\rho} + \frac{o(\|\eta_\rho - u_n\|)}{\rho} \\ &\quad + \frac{(\xi_n(w_\rho) - 1)}{\rho} \langle J'_\lambda(u_n) - J'_\lambda(\eta_\rho), (u_n - w_\rho) \rangle. \end{aligned} \quad (3.18)$$

Since $\|\eta_\rho - u_n\| \leq \rho\xi_n(w_\rho) + |\xi_n(w_\rho) - 1|\|u_n\|$ and

$$\lim_{\rho \rightarrow 0} \frac{|\xi_n(w_\rho) - 1|}{\rho} \leq \|\xi'_n(0)\|, \quad (3.19)$$

if we let $\rho \rightarrow 0$ in (3.18) for a fixed n , then by (3.12) we can find a constant $C > 0$, independent of ρ , such that

$$\left\langle J'_\lambda(u_n), \frac{u}{\|u\|} \right\rangle \leq \frac{C}{n} (1 + \|\xi'_n(0)\|). \quad (3.20)$$

The proof will be complete once we show that $\|\xi'_n(0)\|$ is uniformly bounded in n . By (3.1), (3.12), $(f1)$, $(g1)$, and the Hölder inequality and the Sobolev embedding theorem, we have

$$\langle \xi'_n(0), v \rangle \leq \frac{b\|v\|}{\left| (p-q)\|u_n\|^p - (p^*-q) \int_\Omega g|u_n|^{p^*} dx \right|} \quad \text{for some } b > 0. \quad (3.21)$$

We only need to show that

$$\left| (p-q)\|u_n\|^p - (p^*-q) \int_\Omega g|u_n|^{p^*} dx \right| > C \quad (3.22)$$

for some $C > 0$ and n large enough. We argue by contradiction. Assume that there exists a subsequence $\{u_n\}$ such that

$$(p-q)\|u_n\|^p - (p^*-q) \int_\Omega g|u_n|^{p^*} dx = o_n(1). \quad (3.23)$$

By (3.23) and the fact that $u_n \in \mathcal{N}_\lambda$, we get

$$\begin{aligned} \|u_n\|^p &= \frac{p^*-q}{p-q} \int_\Omega g|u_n|^{p^*} dx + o_n(1), \\ \|u_n\|^p &= \lambda \frac{p^*-q}{p^*-p} \int_\Omega f|u_n|^q dx + o_n(1). \end{aligned} \quad (3.24)$$

Moreover, by (f1), (g1), and the Hölder inequality and the Sobolev embedding theorem, we have

$$\begin{aligned} \|u_n\| &\geq \left[\frac{p-q}{(p^*-q)|g^+|_\infty} S^{p^*/p} \right]^{1/(p^*-p)} + o_n(1), \\ \|u_n\| &\leq \left[\lambda \frac{(p^*-q)|f^+|_\infty}{p^*-p} S^{-q/p} |\Omega|^{(p^*-q)/p^*} \right]^{1/(p-q)} + o_n(1). \end{aligned} \quad (3.25)$$

This implies $\lambda \geq \Lambda_1$ which is a contradiction. We obtain

$$\left\langle J'_\lambda(u_n), \frac{u}{\|u\|} \right\rangle \leq \frac{C}{n}. \quad (3.26)$$

This completes the proof of (i).

(ii) Similarly, by using Lemma 3.2, we can prove (ii). We will omit detailed proof here. $\square$

Now, we establish the existence of a local minimum for J_λ on $\mathcal{N}_\lambda^+$.

Theorem 3.4. *If $\lambda \in (0, \Lambda_1)$, then J_λ has a minimizer u_λ in $\mathcal{N}_\lambda^+$ and it satisfies that*

- (i) $J_\lambda(u_\lambda) = \alpha_\lambda = \alpha_\lambda^+$;
- (ii) u_λ is a positive solution of $(E_{\lambda f, g})$ in $C^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$.

Proof. By Proposition 3.3(i), there exists a minimizing sequence $\{u_n\}$ for J_λ on $\mathcal{N}_\lambda$ such that

$$J_\lambda(u_n) = \alpha_\lambda + o_n(1), \quad J'_\lambda(u_n) = o_n(1) \quad \text{in } W^{-1}. \quad (3.27)$$

Since J_λ is coercive on $\mathcal{N}_\lambda$ (see Lemma 2.2), we get that $\{u_n\}$ is bounded in W . Going if necessary to a subsequence, we can assume that there exists $u_\lambda \in W$ such that

$$\begin{aligned} u_n &\rightharpoonup u_\lambda \quad \text{weakly in } W, \\ u_n &\longrightarrow u_\lambda \quad \text{almost every where in } \Omega, \\ u_n &\longrightarrow u_\lambda \quad \text{strongly in } L^s(\Omega) \quad \forall 1 \leq s < p^*. \end{aligned} \quad (3.28)$$

First, we claim that u_λ is a nontrivial solution of $(E_{\lambda f, g})$. By (3.27) and (3.28), it is easy to see that u_λ is a solution of $(E_{\lambda f, g})$. From $u_n \in \mathcal{N}_\lambda$ and (2.6), we deduce that

$$\lambda \int_{\Omega} f |u_n|^q dx = \frac{q(p^*-p)}{p(p^*-q)} \|u_n\|^p - \frac{p^*q}{p^*-q} J_\lambda(u_n). \quad (3.29)$$

Let $n \rightarrow \infty$ in (3.29), by (3.27), (3.28), and $\alpha_\lambda < 0$, we get

$$\int_{\Omega} f |u_\lambda|^q dx \geq -\frac{p^*q}{p^*-q} \alpha_\lambda > 0. \quad (3.30)$$

Thus, $u_\lambda \in \mathcal{N}_\lambda$ is a nontrivial solution of $(E_{\lambda f, g})$. Now we prove that $u_n \rightarrow u_\lambda$ strongly in W and $J_\lambda(u_\lambda) = \alpha_\lambda$. By (3.29), if $u \in \mathcal{N}_\lambda$, then

$$J_\lambda(u) = \frac{p^* - p}{p^* p} \|u\|^p - \frac{p^* - q}{p^* q} \lambda \int_{\Omega} f |u|^q dx. \quad (3.31)$$

In order to prove that $J_\lambda(u_\lambda) = \alpha_\lambda$, it suffices to recall that $u_\lambda \in \mathcal{N}_\lambda$, by (3.31), and applying Fatou's lemma to get

$$\begin{aligned} \alpha_\lambda &\leq J_\lambda(u_\lambda) = \frac{p^* - p}{p^* p} \|u_\lambda\|^p - \frac{p^* - q}{p^* q} \lambda \int_{\Omega} f |u_\lambda|^q dx \\ &\leq \liminf_{n \rightarrow \infty} \left(\frac{p^* - p}{p^* p} \|u_n\|^p - \frac{p^* - q}{p^* q} \lambda \int_{\Omega} f |u_n|^q dx \right) \\ &\leq \liminf_{n \rightarrow \infty} J_\lambda(u_n) = \alpha_\lambda. \end{aligned} \quad (3.32)$$

This implies that $J_\lambda(u_\lambda) = \alpha_\lambda$ and $\lim_{n \rightarrow \infty} \|u_n\|^p = \|u_\lambda\|^p$. Let $v_n = u_n - u_\lambda$, then Brézis and Lieb lemma [15] implies that

$$\|v_n\|^p = \|u_n\|^p - \|u_\lambda\|^p + o_n(1). \quad (3.33)$$

Therefore, $u_n \rightarrow u_\lambda$ strongly in W . Moreover, we have $u_\lambda \in \mathcal{N}_\lambda^+$. On the contrary, if $u_\lambda \in \mathcal{N}_\lambda^-$, then by Lemma 2.7, there are unique t_0^+ and t_0^- such that $t_0^+ u_\lambda \in \mathcal{N}_\lambda^+$ and $t_0^- u_\lambda \in \mathcal{N}_\lambda^-$. In particular, we have $t_0^+ < t_0^- = 1$. Since

$$\frac{d}{dt} J_\lambda(t_0^+ u_\lambda) = 0, \quad \frac{d^2}{dt^2} J_\lambda(t_0^+ u_\lambda) > 0, \quad (3.34)$$

there exists $t_0^+ < \bar{t} \leq t_0^-$ such that $J_\lambda(t_0^+ u_\lambda) < J_\lambda(\bar{t} u_\lambda)$. By Lemma 2.7,

$$J_\lambda(t_0^+ u_\lambda) < J_\lambda(\bar{t} u_\lambda) \leq J_\lambda(t_0^- u_\lambda) = J_\lambda(u_\lambda), \quad (3.35)$$

which is a contradiction. Since $J_\lambda(u_\lambda) = J_\lambda(|u_\lambda|)$ and $|u_\lambda| \in \mathcal{N}_\lambda^+$, by Lemma 2.3 we may assume that u_λ is a nontrivial nonnegative solution of $(E_{\lambda f, g})$. Moreover, from $f, g \in L^\infty(\Omega)$, then using the standard bootstrap argument (see, e.g., [16]) we obtain $u_\lambda \in L^\infty(\Omega)$; hence by applying regularity results [17, 18] we derive that $u_\lambda \in C^{1, \alpha}(\Omega)$ for some $\alpha \in (0, 1)$ and finally, by the Harnack inequality [19] we deduce that $u_\lambda > 0$. This completes the proof. $\square$

Now, we begin the proof of Theorem 1.4. By Theorem 3.4, we obtain $(E_{\lambda f, g})$ that has a positive solution u_λ in $C^{1, \alpha}(\Omega)$ for some $\alpha \in (0, 1)$.

4. Proof of Theorem 1.5

Next, we will establish the existence of the second positive solution of $(E_{\lambda f, g})$ by proving that J_λ satisfies the $(PS)_{\alpha_\lambda^-}$ condition.

Lemma 4.1. Assume that (f1) and (g1) hold. If $\{u_n\} \subset W$ is a $(PS)_c$ -sequence for J_λ , then $\{u_n\}$ is bounded in W .

Proof. We argue by contradiction. Assume that $\|u_n\| \rightarrow \infty$. Let $\hat{u}_n = u_n / \|u_n\|$. We may assume that $\hat{u}_n \rightharpoonup \hat{u}$ in W . This implies that $\hat{u}_n \rightarrow \hat{u}$ strongly in $L^s(\Omega)$ for all $1 \leq s < p^*$ and

$$\frac{\lambda}{q} \int_{\Omega} f|\hat{u}_n|^q dx = \frac{\lambda}{q} \int_{\Omega} f|\hat{u}|^q dx + o_n(1). \quad (4.1)$$

Since $\{u_n\}$ is a $(PS)_c$ -sequence for J_λ and $\|u_n\| \rightarrow \infty$, there hold

$$\begin{aligned} \frac{1}{p} \int_{\Omega} |\nabla \hat{u}_n|^p dx - \frac{\lambda \|u_n\|^{q-p}}{q} \int_{\Omega} f|\hat{u}_n|^q dx - \frac{\|u_n\|^{p^*-p}}{p^*} \int_{\Omega} g|\hat{u}_n|^{p^*} dx &= o_n(1), \\ \int_{\Omega} |\nabla \hat{u}_n|^p dx - \lambda \|u_n\|^{q-p} \int_{\Omega} f|\hat{u}_n|^q dx - \|u_n\|^{p^*-p} \int_{\Omega} g|\hat{u}_n|^{p^*} dx &= o_n(1). \end{aligned} \quad (4.2)$$

From (4.1)-(4.2), we can deduce that

$$\int_{\Omega} |\nabla \hat{u}_n|^p dx = \frac{p(p^* - q)}{q(p^* - p)} \|u_n\|^{q-p} \lambda \int_{\Omega} f|\hat{u}|^q dx + o_n(1). \quad (4.3)$$

Since $1 \leq q < 2$ and $\|u_n\| \rightarrow \infty$, (4.3) implies

$$\int_{\Omega} |\nabla \hat{u}_n|^p dx \rightarrow 0, \quad \text{as } n \rightarrow \infty, \quad (4.4)$$

which is contrary to the fact $\|\hat{u}_n\| = 1$ for all n . $\square$

Lemma 4.2. Assume that (f1) and (g1) hold. If $\{u_n\} \subset W$ is a $(PS)_c$ -sequence for J_λ with $c \in (0, (1/N)|g^+|_{\infty}^{-(N-p)/p} S^{N/p})$, then there exists a subsequence of $\{u_n\}$ converging weakly to a nontrivial solution of $(E_{\lambda f, g})$.

Proof. Let $\{u_n\} \subset W$ be a $(PS)_c$ -sequence for J_λ with $c \in (0, (1/N)|g^+|_{\infty}^{-(N-p)/p} S^{N/p})$. We know from Lemma 4.1 that $\{u_n\}$ is bounded in W , and then there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) and $u_0 \in W$ such that

$$\begin{aligned} u_n &\rightharpoonup u_0 \quad \text{weakly in } W, \\ u_n &\rightarrow u_0 \quad \text{almost every where in } \Omega, \\ u_n &\rightarrow u_0 \quad \text{strongly in } L^s(\Omega) \quad \forall 1 \leq s < p^*. \end{aligned} \quad (4.5)$$

It is easy to see that $J'_\lambda(u_0) = 0$ and

$$\lambda \int_{\Omega} f(x)|u_n|^q dx = \lambda \int_{\Omega} f(x)|u_0|^q dx + o_n(1). \quad (4.6)$$

Next we verify that $u_0 \neq 0$. Arguing by contradiction, we assume $u_0 \equiv 0$. Setting

$$l = \lim_{n \rightarrow \infty} \int_{\Omega} g|u_n|^{p^*} dx. \quad (4.7)$$

Since $J'_\lambda(u_n) = o_n(1)$ and $\{u_n\}$ is bounded, then by (4.6), we can deduce that

$$0 = \left\langle \lim_{n \rightarrow \infty} J'_\lambda(u_n), u_n \right\rangle = \lim_{n \rightarrow \infty} \left(\|u_n\|^p - \int_{\Omega} g|u_n|^{p^*} \right) = \lim_{n \rightarrow \infty} \|u_n\|^p - l, \quad (4.8)$$

that is,

$$\lim_{n \rightarrow \infty} \|u_n\|^p = l. \quad (4.9)$$

If $l = 0$, then we get $c = \lim_{n \rightarrow \infty} J_\lambda(u_n) = 0$, which contradicts with $c > 0$. Thus we conclude that $l > 0$. Furthermore, the Sobolev inequality implies that

$$\|u_n\|^p \geq S \left(\int_{\Omega} |u_n|^{p^*} \right)^{p/p^*} \geq S \left(\int_{\Omega} \frac{g}{|g^+|_{\infty}} |u_n|^{p^*} \right)^{p/p^*} = S |g^+|_{\infty}^{-(N-p)/N} \left(\int_{\Omega} g|u_n|^{p^*} \right)^{p/p^*}. \quad (4.10)$$

Then as $n \rightarrow \infty$ we have

$$l = \lim_{n \rightarrow \infty} \|u_n\|^p \geq S |g^+|_{\infty}^{-(N-p)/N} \lim_{n \rightarrow \infty} \left(\int_{\Omega} g|u_n|^{p^*} \right)^{p/p^*} = S |g^+|_{\infty}^{-(N-p)/N} l^{p/p^*}, \quad (4.11)$$

which implies that

$$l \geq |g^+|_{\infty}^{-(N-p)/p} S^{N/p}. \quad (4.12)$$

Hence, from (4.6) to (4.12) we get

$$\begin{aligned} c &= \lim_{n \rightarrow \infty} J_\lambda(u_n) \\ &= \frac{1}{p} \lim_{n \rightarrow \infty} \|u_n\|^p - \frac{\lambda}{q} \lim_{n \rightarrow \infty} \int_{\Omega} f|u_n|^q dx - \frac{1}{p^*} \lim_{n \rightarrow \infty} \int_{\Omega} g|u_n|^{p^*} dx \\ &= \left(\frac{1}{p} - \frac{1}{p^*} \right) l \\ &\geq \frac{1}{N} |g^+|_{\infty}^{-(N-p)/p} S^{N/p}. \end{aligned} \quad (4.13)$$

This is a contradiction to $c < (1/N)|g^+|_{\infty}^{-(N-p)/p} S^{N/p}$. Therefore u_0 is a nontrivial solution of $(E_{\lambda f, g})$. $\square$

Lemma 4.3. Assume that (f1)-(f2) and (g1)-(g4) hold. Then for any $\lambda > 0$, there exists $v_\lambda \in W$ such that

$$\sup_{t \geq 0} J_\lambda(tv_\lambda) < \frac{1}{N} |g^+|_\infty^{-(N-p)/p} S^{N/p}. \quad (4.14)$$

In particular, $\alpha_\lambda^- < (1/N) |g^+|_\infty^{-(N-p)/p} S^{N/p}$ for all $\lambda \in (0, \Lambda_1)$ where Λ_1 is as in (1.5).

Proof. For convenience, we introduce the following notations:

$$\begin{aligned} I(u) &= \int_{\Omega} \left\{ \frac{1}{p} |\nabla u|^p - \frac{1}{p^*} g |u|^{p^*} \right\} dx, \\ \chi_{B(0, 2\rho_0)} &= \begin{cases} 1 & \text{if } x \in B(0, 2\rho_0), \\ 0 & \text{if } x \notin B(0, 2\rho_0), \end{cases} \\ Q(u) &= \frac{|\nabla u|_p^p}{\left| (g \chi_{B(0, 2\rho_0)})^{1/p^*} u \right|_{p^*}^p}. \end{aligned} \quad (4.15)$$

From (g3) to (g4), we know that there exists $\delta_0 \in (0, \rho_0)$ such that for all $x \in B(0, 2\delta_0)$,

$$g(x) = g(0) + o(|x|^\beta) \quad \text{for some } \beta > \frac{N}{p-1}. \quad (4.16)$$

Motivated by some ideas of selecting cut-off functions in [20, Lemma 4.1], we take such cut-off function $\eta(x)$ that satisfies $\eta(x) \in C_0^\infty(B(0, 2\delta_0))$, $\eta(x) = 1$ for $|x| < \delta_0$, $\eta(x) = 0$ for $|x| > 2\delta_0$, $0 \leq \eta \leq 1$, and $|\nabla \eta| \leq C$. Define, for $\varepsilon > 0$,

$$u_\varepsilon(x) = \frac{\varepsilon^{(N-p)/p^2} \eta(x)}{\left(\varepsilon + |x|^{p/(p-1)} \right)^{(N-p)/p}}. \quad (4.17)$$

Step 1. Show that $\sup_{t \geq 0} I(tu_\varepsilon) \leq (1/N) |g^+|_\infty^{-(N-p)/p} S^{N/p} + O(\varepsilon^{(N-p)/p})$.

On that purpose, we need to establish the following estimates (as $\varepsilon \rightarrow 0$):

$$\left| (g \chi_{B(0, 2\rho_0)})^{1/p^*} u_\varepsilon \right|_{p^*}^p = |g^+|_\infty^{-(N-p)/N} |u|_{L^{p^*}(\mathbb{R}^N)}^p + O(\varepsilon^{N/p}), \quad (4.18)$$

$$|\nabla u_\varepsilon|_p^p = |\nabla u|_{L^p(\mathbb{R}^N)}^p + O(\varepsilon^{(N-p)/p}), \quad (4.19)$$

where $U(x) = (1 + (x)^{p/(p-1)})^{-(N-p)/p} \in W^{1,p}(\mathbb{R}^N)$ is a minimizer of $\{|\nabla u|_p^p / |u|_p^p\}_{u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}}$, that is,

$$\frac{|\nabla U|_{L^p(\mathbb{R}^N)}^p}{|U|_{L^{p^*}(\mathbb{R}^N)}^p} = S = \inf_{u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}} \frac{|\nabla u|_{L^p(\mathbb{R}^N)}^p}{|u|_{L^{p^*}(\mathbb{R}^N)}^p}, \quad (4.20)$$

and $\omega_N = 2\pi^{N/2}/N\Gamma(N/2)$ which is the volume of the unit ball $B(0,1)$ in $\mathbb{R}^N$. We only show that equality (4.18) is valid; proofs of (4.19) are very similar to [20]. In view of (4.17), we get that

$$\left| \left(g\chi_{B(0,2\rho_0)} \right)^{1/p^*} u_\varepsilon \right|_{p^*}^{p^*} = \int_{B(0,2\rho_0)} g(x) |u_\varepsilon|^{p^*} dx = \int_{\mathbb{R}^N} \frac{\varepsilon^{N/p} \eta^{p^*}(x) g(x)}{(\varepsilon + |x|^{p/(p-1)})^N} dx. \quad (4.21)$$

On the other hand, let $x = \varepsilon^{(p-1)/p} y$, we can deduce that

$$\int_{\mathbb{R}^N} \frac{1}{(\varepsilon + |x|^{p/(p-1)})^N} dx = \varepsilon^{-N/p} \int_{\mathbb{R}^N} \frac{1}{(1 + |y|^{p/(p-1)})^N} dy = \varepsilon^{-N/p} |U|_{L^{p^*}(\mathbb{R}^N)}^{p^*}. \quad (4.22)$$

Combining with $g(0) = g^+|_\infty$ and the equalities above, we have

$$\begin{aligned} & \varepsilon^{-N/p} |g^+|_\infty |U|_{L^{p^*}(\mathbb{R}^N)}^{p^*} - \varepsilon^{-N/p} \left| \left(g\chi_{B(0,2\rho_0)} \right)^{1/p^*} u_\varepsilon \right|_{p^*}^{p^*} \\ &= \int_{\mathbb{R}^N \setminus B(0,\delta_0)} \frac{g(0) - \eta^{p^*}(x) g(x)}{(\varepsilon + |x|^{p/(p-1)})^N} dx + \int_{B(0,\delta_0)} \frac{g(0) - g(x)}{(\varepsilon + |x|^{p/(p-1)})^N} dx, \end{aligned} \quad (4.23)$$

hence

$$\begin{aligned} 0 &\leq \varepsilon^{-N/p} |g^+|_\infty |U|_{L^{p^*}(\mathbb{R}^N)}^{p^*} - \varepsilon^{-N/p} \left| \left(g\chi_{B(0,2\rho_0)} \right)^{1/p^*} u_\varepsilon \right|_{p^*}^{p^*} \\ &\leq \int_{\mathbb{R}^N \setminus B(0,\delta_0)} \frac{g(0)}{(\varepsilon + |x|^{p/(p-1)})^N} dx + \int_{B(0,\delta_0)} \frac{o(|x|^\beta)}{(\varepsilon + |x|^{p/(p-1)})^N} dx \\ &\leq \int_{\mathbb{R}^N \setminus B(0,\delta_0)} \frac{g(0)}{|x|^{Np/(p-1)}} dx + \int_{B(0,\delta_0)} \frac{o(|x|^\beta)}{|x|^{Np/(p-1)}} dx \\ &= N\omega_N \int_{\delta_0}^\infty \frac{r^{N-1} g(0)}{r^{pN/(p-1)}} dr + \int_0^{\delta_0} \frac{o(r^\beta) r^{N-1}}{r^{pN/(p-1)}} dr \\ &= (p-1)\omega_N \delta_0^{-N/(p-1)} g(0) + \frac{o(1)\delta_0^{\beta-(N/(p-1))}}{\beta - (N/(p-1))} \leq C_1 = \text{Const.}, \end{aligned} \quad (4.24)$$

which leads to

$$0 \leq 1 - |g^+|_\infty^{-1} \left| (g\chi_{B(0,2\rho_0)})^{1/p^*} u_\varepsilon \right|_{p^*}^{p^*} |U|_{L^{p^*}(\mathbb{R}^N)}^{-p^*} \leq C_1 |g^+|_\infty^{-1} |U|_{L^{p^*}(\mathbb{R}^N)}^{-p^*} \varepsilon^{N/p}, \quad (4.25)$$

that is,

$$1 - C_1 |g^+|_\infty^{-1} |U|_{L^{p^*}(\mathbb{R}^N)}^{-p^*} \varepsilon^{N/p} \leq |g^+|_\infty^{-1} \left| (g\chi_{B(0,2\rho_0)})^{1/p^*} u_\varepsilon \right|_{p^*}^{p^*} |U|_{L^{p^*}(\mathbb{R}^N)}^{-p^*} \leq 1. \quad (4.26)$$

Now, let ε be small enough such that $C_1 |g^+|_\infty^{-1} |U|_{L^{p^*}(\mathbb{R}^N)}^{-p^*} \varepsilon^{N/p} < 1$, then from (4.26) we can deduce that

$$\begin{aligned} 1 - C_1 |g^+|_\infty^{-1} |U|_{L^{p^*}(\mathbb{R}^N)}^{-p^*} \varepsilon^{N/p} &\leq \left(1 - C_1 |g^+|_\infty^{-1} |U|_{L^{p^*}(\mathbb{R}^N)}^{-p^*} \varepsilon^{N/p} \right)^{p/p^*} \\ &\leq |g^+|_\infty^{-(N-p)/N} \left| (g\chi_{B(0,2\rho_0)})^{1/p^*} u_\varepsilon \right|_{p^*}^p |U|_{L^{p^*}(\mathbb{R}^N)}^{-p} \leq 1, \end{aligned} \quad (4.27)$$

which yields that

$$|g^+|_\infty^{(N-p)/N} |U|_{L^{p^*}(\mathbb{R}^N)}^p - C_1 |g^+|_\infty^{-p/N} |U|_{L^{p^*}(\mathbb{R}^N)}^{p-p^*} \varepsilon^{N/p} \leq \left| (g\chi_{B(0,2\rho_0)})^{1/p^*} u_\varepsilon \right|_{p^*}^p \leq |g^+|_\infty^{(N-p)/N} |U|_{L^{p^*}(\mathbb{R}^N)}^p, \quad (4.28)$$

equivalently, equality (4.18) is valid.

Combining (4.18) and (4.19), we obtain that

$$\begin{aligned} Q(u_\varepsilon) &= \frac{|\nabla U|_{L^p(\mathbb{R}^N)}^p + O(\varepsilon^{(N-p)/p})}{|g^+|_\infty^{(N-p)/N} |U|_{L^{p^*}(\mathbb{R}^N)}^p + O(\varepsilon^{N/p})} \\ &= |g^+|_\infty^{-(N-p)/N} \frac{|\nabla U|_{L^p(\mathbb{R}^N)}^p + O(\varepsilon^{(N-p)/p})}{|U|_{L^{p^*}(\mathbb{R}^N)}^p + O(\varepsilon^{N/p})}. \end{aligned} \quad (4.29)$$

Hence

$$\begin{aligned} Q(u_\varepsilon) - |g^+|_\infty^{-(N-p)/N} S &= |g^+|_\infty^{-(N-p)/N} \left[\frac{|\nabla U|_{L^p(\mathbb{R}^N)}^p + O(\varepsilon^{(N-p)/p})}{|U|_{L^{p^*}(\mathbb{R}^N)}^p + O(\varepsilon^{N/p})} - \frac{|\nabla U|_{L^p(\mathbb{R}^N)}^p}{|U|_{L^{p^*}(\mathbb{R}^N)}^p} \right] \\ &= |g^+|_\infty^{-(N-p)/N} \left[\frac{|U|_{L^{p^*}(\mathbb{R}^N)}^p O(\varepsilon^{(N-p)/p}) - |\nabla U|_{L^p(\mathbb{R}^N)}^p O(\varepsilon^{N/p})}{(|U|_{L^{p^*}(\mathbb{R}^N)}^p + O(\varepsilon^{N/p})) |U|_{L^{p^*}(\mathbb{R}^N)}^p} \right] \\ &= O(\varepsilon^{(N-p)/p}). \end{aligned} \quad (4.30)$$

Using the fact that

$$\max_{t \geq 0} \left(\frac{t^p}{p} a - \frac{t^{p^*}}{p^*} b \right) = \frac{1}{N} \left(\frac{a}{b^{p/p^*}} \right)^{N/p} \quad \text{for any } a, b > 0, \quad (4.31)$$

we can deduce that

$$\sup_{t \geq 0} I(tu_\varepsilon) = \frac{1}{N} (Q(u_\varepsilon))^{N/p}. \quad (4.32)$$

From (4.30), we conclude that $\sup_{t \geq 0} I(tu_\varepsilon) \leq (1/N) |g^+|_\infty^{-(N-p)/p} S^{N/p} + O(\varepsilon^{(N-p)/p})$.

Step 2. We claim that for any $\lambda > 0$ there exists a constant $\varepsilon_\lambda > 0$ such that $\sup_{t \geq 0} J_\lambda(tu_{\varepsilon_\lambda}) < (1/N) |g^+|_\infty^{-(N-p)/p} S^{N/p}$.

Using the definitions of J_λ, u_ε and by (f2), (g3), we get

$$J_\lambda(tu_\varepsilon) \leq \frac{t^p}{p} |\nabla u_\varepsilon|_p^p, \quad \forall t \geq 0, \quad \forall \lambda > 0. \quad (4.33)$$

Combining this with (4.19), let $\varepsilon \in (0, 1)$, then there exists $t_0 \in (0, 1)$ independent of ε such that

$$\sup_{0 \leq t \leq t_0} J_\lambda(tu_\varepsilon) < \frac{1}{N} |g^+|_\infty^{-(N-p)/p} S^{N/p}, \quad \forall \lambda > 0, \quad \forall \varepsilon \in (0, 1). \quad (4.34)$$

Using the definitions of J_λ, u_ε , and by the results in Step 1 and (f2), we have

$$\begin{aligned} \sup_{t \geq t_0} J_\lambda(tu_\varepsilon) &= \sup_{t \geq t_0} \left(I(tu_\varepsilon) - \frac{t^q}{q} \lambda \int f(x) |u_\varepsilon|^q dx \right) \\ &\leq \frac{1}{N} |g^+|_\infty^{-(N-p)/p} S^{N/p} + O(\varepsilon^{(N-p)/p}) - \frac{t_0^q}{q} \beta_0 \lambda \int_{B(0, \delta_0)} |u_\varepsilon|^q dx. \end{aligned} \quad (4.35)$$

Let $0 < \varepsilon \leq \delta_0^{p/(p-1)}$, we have

$$\begin{aligned} \int_{B(0, \delta_0)} |u_\varepsilon|^q dx &= \int_{B(0, \delta_0)} \frac{\varepsilon^{q(N-p)/p^2}}{(\varepsilon + |x|^{p/(p-1)})^{((N-p)/p)q}} dx \\ &\geq \int_{B(0, \delta_0)} \frac{\varepsilon^{q(N-p)/p^2}}{(2\delta_0^{p/(p-1)})^{((N-p)/p)q}} dx \\ &= C_2(N, p, q, \delta_0) \varepsilon^{(q(N-p))/p^2}. \end{aligned} \quad (4.36)$$

Combining (4.35) and (4.36), for all $\varepsilon \in (0, \delta_0^{p/(p-1)})$, we get

$$\sup_{t \geq t_0} J_\lambda(tu_\varepsilon) \leq \frac{1}{N} |g^+|_\infty^{-(N-p)/p} S^{N/p} + O(\varepsilon^{(N-p)/p}) - \frac{t_0^q}{q} \beta_0 C_2 \lambda \varepsilon^{q(N-p)/p^2}. \quad (4.37)$$

Hence, for any $\lambda > 0$, we can choose small positive constant $\varepsilon_\lambda < \min\{1, \delta_0^{p/(p-1)}\}$ such that

$$O(\varepsilon_\lambda^{(N-p)/p}) - \frac{t_0^q}{q} \beta_0 C_2 \lambda \varepsilon_\lambda^{q(N-p)/p^2} < 0. \quad (4.38)$$

From (4.34), (4.37), (4.38), we can deduce that for any $\lambda > 0$, there exists $\varepsilon_\lambda > 0$ such that

$$\sup_{t \geq 0} J_\lambda(tu_{\varepsilon_\lambda}) < \frac{1}{N} |g^+|_\infty^{-(N-p)/p} S^{N/p}. \quad (4.39)$$

Step 3. Prove that $\alpha_\lambda^- < (1/N) S^{N/p}$ for all $\lambda \in (0, \Lambda_1)$.

By (f2), (g2), and the definition of u_ε , we have

$$\int_{\Omega} f(x) |u_\varepsilon|^q dx > 0, \quad \int_{\Omega} g(x) |u_\varepsilon|^{p^*} dx > 0. \quad (4.40)$$

Combining this with Lemma 2.7(ii), from the definition of α_λ^- and the results in Step 2, for any $\lambda \in (0, \Lambda_1)$, we obtain that there exists $t_{\varepsilon_\lambda} > 0$ such that $t_{\varepsilon_\lambda} u_{\varepsilon_\lambda} \in \mathcal{N}_\lambda^-$ and

$$\alpha_\lambda^- \leq J_\lambda(t_{\varepsilon_\lambda} u_{\varepsilon_\lambda}) \leq \sup_{t \geq 0} J_\lambda(tu_{\varepsilon_\lambda}) < \frac{1}{N} |g^+|_\infty^{-(N-p)/p} S^{N/p}. \quad (4.41)$$

This completes the proof. $\square$

Now, we establish the existence of a local minimum of J_λ on $\mathcal{N}_\lambda^-$.

Theorem 4.4. *If $\lambda \in (0, (q/p)\Lambda_1)$, then J_λ satisfies the $(PS)_{\alpha_\lambda^-}$ condition. Moreover, J_λ has a minimizer U_λ in $\mathcal{N}_\lambda^-$ and satisfies that*

(i) $J_\lambda(U_\lambda) = \alpha_\lambda^-$;

(ii) U_λ is a positive solution of $(E_{\lambda f, g})$ in $C^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$,

where Λ_1 is as in (1.5).

Proof. If $\lambda \in (0, (q/p)\Lambda_1)$, then by Theorem 2.6(ii), Proposition 3.3(ii), and Lemma 4.3, there exists a $(PS)_{\alpha_\lambda^-}$ -sequence $\{u_n\} \subset \mathcal{N}_\lambda^-$ in W for J_λ with $\alpha_\lambda^- \in (0, (1/N) |g^+|_\infty^{-(N-p)/p} S^{N/p})$. From Lemma 4.2, there exists a subsequence still denoted by $\{u_n\}$ and nontrivial solution $U_\lambda \in W$ of $(E_{\lambda f, g})$ such that $u_n \rightharpoonup U_\lambda$ weakly in W . Now we prove that $u_n \rightarrow U_\lambda$ strongly in W and $J_\lambda(U_\lambda) = \alpha_\lambda^-$. By (3.29), if $u \in \mathcal{N}_\lambda$, then

$$J_\lambda(u) = \frac{p^* - p}{p^* p} \|u\|^p - \frac{p^* - q}{p^* q} \lambda \int_{\Omega} f |u|^q dx. \quad (4.42)$$

First, we prove that $U_\lambda \in \mathcal{N}_\lambda^-$. On the contrary, if $U_\lambda \in \mathcal{N}_\lambda^+$, then by $\mathcal{N}_\lambda^-$ closed in W , we have $\|U_\lambda\| < \liminf_{n \rightarrow \infty} \|u_n\|$. By Lemma 2.7, there exists a unique t_λ^- such that $t_\lambda^- U_\lambda \in \mathcal{N}_\lambda^-$. Since $u_n \in \mathcal{N}_\lambda^-$, $J_\lambda(u_n) \geq J_\lambda(tu_n)$ for all $t \geq 0$ and by (4.42), we have

$$\alpha_\lambda^- \leq J_\lambda(t_\lambda^- U_\lambda) < \lim_{n \rightarrow \infty} J_\lambda(t_\lambda^- u_n) \leq \lim_{n \rightarrow \infty} J_\lambda(u_n) = \alpha_\lambda^-, \quad (4.43)$$

and this is contradiction.

In order to prove that $J_\lambda(U_\lambda) = \alpha_\lambda^-$, it suffices to recall that $u_n, U_\lambda \in \mathcal{N}_\lambda^-$ for all n , by (4.42), and applying Fatou's lemma to get

$$\begin{aligned} \alpha_\lambda^- &\leq J_\lambda(U_\lambda) = \frac{p^* - p}{p^* p} \|U_\lambda\|^p - \frac{p^* - q}{p^* q} \lambda \int_{\Omega} f |U_\lambda|^q dx \\ &\leq \liminf_{n \rightarrow \infty} \left(\frac{p^* - p}{p^* p} \|u_n\|^p - \frac{p^* - q}{p^* q} \lambda \int_{\Omega} f |u_n|^q dx \right) \\ &\leq \liminf_{n \rightarrow \infty} J_\lambda(u_n) = \alpha_\lambda^-. \end{aligned} \quad (4.44)$$

This implies that $J_\lambda(U_\lambda) = \alpha_\lambda^-$ and $\lim_{n \rightarrow \infty} \|u_n\|^p = \|U_\lambda\|^p$. Let $v_n = u_n - U_\lambda$, then Brézis and Lieb lemma [15] implies that

$$\|v_n\|^p = \|u_n\|^p - \|U_\lambda\|^p + o_n(1). \quad (4.45)$$

Therefore, $u_n \rightarrow U_\lambda$ strongly in W .

Since $J_\lambda(U_\lambda) = J_\lambda(|U_\lambda|)$ and $|U_\lambda| \in \mathcal{N}_\lambda^-$, by Lemma 2.3 we may assume that U_λ is a nontrivial nonnegative solution of $(E_{\lambda f, g})$. Finally, by using the same arguments as in the proof of Theorem 3.4, for all $\lambda \in (0, (q/p)\Lambda_1)$, we have that U_λ is a positive solution of $(E_{\lambda f, g})$ in $C^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$. $\square$

Now, we complete the proof of Theorem 1.5. By Theorems 3.4 and 4.4, if $\lambda \in (0, (q/p)\Lambda_1)$, then we obtain $(E_{\lambda f, g})$ that has two positive solutions u_λ and U_λ such that $u_\lambda \in \mathcal{N}_\lambda^+$, $U_\lambda \in \mathcal{N}_\lambda^-$, and $u_\lambda, U_\lambda \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$. Since $\mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- = \emptyset$, this implies that u_λ and U_λ are distinct.

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