Research Article

Optimal Inequalities among Various Means of Two Arguments

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We establish two optimal inequalities among power mean $M_p(a, b) = (a^p/2 + b^p/2)^{1/p}$, arithmetic mean $A(a, b) = (a + b)/2$, logarithmic mean $L(a, b) = (a - b)/(\log a - \log b)$, and geometric mean $G(a, b) = \sqrt{ab}$.

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1. Introduction

For $p \in \mathbb{R}$, the power mean $M_p(a, b)$ of order p of two positive numbers a and b is defined by

$$M_p(a, b) = \begin{cases} \left(\frac{a^p + b^p}{2} \right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0. \end{cases} \quad (1.1)$$

In the recent past, the power mean $M_p(a, b)$ has been the subject of intensive research. In particular, many remarkable inequalities for the mean can be found in literature [1–11]. It is well-known that $M_p(a, b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed a and b .

If we denote by

$$I(a, b) = \begin{cases} \frac{1}{e} \left(\frac{a^a}{b^b} \right)^{1/(a-b)}, & b \neq a, \\ a, & b = a, \end{cases} \quad L(a, b) = \begin{cases} \frac{b-a}{\log b - \log a}, & b \neq a, \\ a, & b = a, \end{cases} \quad (1.2)$$

$A(a, b) = (a + b)/2$, $G(a, b) = \sqrt{ab}$ and $H(a, b) = 2ab/(a + b)$ the identric mean, logarithmic mean, arithmetic mean, geometric mean and the harmonic mean, of two positive real numbers a and b , respectively, then

$$\begin{aligned} \min\{a, b\} \leq H(a, b) = M_{-1}(a, b) \leq G(a, b) = M_0(a, b) \leq L(a, b) \\ \leq I(a, b) \leq A(a, b) = M_1(a, b) \leq \max\{a, b\} \end{aligned} \quad (1.3)$$

with equality if and only if $b = a$ in each inequality.

In [12], Alzer and Janous established the following sharp double inequality (see also [13, page 350]):

$$M_{\log 2 / \log 3}(a, b) \leq \frac{2}{3}A(a, b) + \frac{1}{3}G(a, b) \leq M_{2/3}(a, b) \quad (1.4)$$

for all real numbers $a, b > 0$.

In [14], Mao proved

$$M_{1/3}(a, b) \leq \frac{1}{3}A(a, b) + \frac{2}{3}G(a, b) \leq M_{1/2}(a, b) \quad (1.5)$$

for all real numbers $a, b > 0$, and the constant $1/3$ in the left side inequality cannot be improved.

In [15–17], the authors presented the bounds for L and I in terms of A and G as follows:

$$\begin{aligned} G^{2/3}(a, b)A^{1/3}(a, b) < L(a, b) < \frac{1}{3}A(a, b) + \frac{2}{3}G(a, b), \\ \frac{1}{3}G(a, b) + \frac{2}{3}A(a, b) < I(a, b) \end{aligned} \quad (1.6)$$

for all $a, b > 0$ with $a \neq b$.

Alzer [18] proved

$$\sqrt{G(a, b)A(a, b)} < \sqrt{L(a, b)I(a, b)} < \frac{1}{2}(L(a, b) + I(a, b)) < \frac{1}{2}(G(a, b) + A(a, b)). \quad (1.7)$$

In [5], Alzer and Qiu established

$$\alpha A(a, b) + (1 - \alpha)G(a, b) < I(a, b) < \beta A(a, b) + (1 - \beta)G(a, b) \quad (1.8)$$

for $\alpha \leq 2/3$, $\beta \geq 2/e = 0.73575 \dots$ and $a, b > 0$ with $a \neq b$.

The main purpose of this paper is to present the optimal bounds for $A^\alpha(a, b)L^{1-\alpha}(a, b)$ and $G^\alpha(a, b)L^{1-\alpha}(a, b)$ for all $\alpha \in (0, 1)$ in terms of the power mean $M_p(a, b)$. Moreover, two optimal inequalities among $A(a, b)$, $G(a, b)$, $L(a, b)$ and $M_p(a, b)$ are proved.

2. Lemmas

In order to establish our main results we need two inequalities, which we present in this section.

Lemma 2.1. *If $r \in (0, 1)$, then*

$$\begin{aligned} g(t) = & -\left[t^2 + t^{(1+2r)/3} + (1-2r)\left(t^{(1+2r)/3+1} + t\right)\right] \log t \\ & + (1-r)\left(t^{(1+2r)/3+2} - t^{(1+2r)/3} + t^2 - 1\right) > 0 \end{aligned} \quad (2.1)$$

for $t \in (1, +\infty)$.

Proof. Let $p = (1+2r)/3$, $g_1(t) = t^{1-p}g'(t)$, $g_2(t) = t^p g'_1(t)$, $g_3(t) = t^{1-p}g'_2(t)$, $g_4(t) = t^p g'_3(t)$, $g_5(t) = t^{3-p}g'_4(t)$, $g_6(t) = t^p g'_5(t)$, $g_7(t) = t^{1-p}g'_6(t)$, and $g_8(t) = t^p g'_7(t)$, then simple computation yields

$$g(1) = 0, \quad (2.2)$$

$$\begin{aligned} g_1(t) = & -\left[2t^{2-p} + (1-2r)t^{1-p} + (1-2r)(p+1)t + p\right] \log t \\ & + (1-r)(p+2)t^2 - (1-2r)t + (1-2r)t^{2-p} \\ & - (1-2r)t^{1-p} - p(1-r) - 1, \end{aligned} \quad (2.3)$$

$$\begin{aligned} g_1(1) = & 0, \\ g_2(t) = & -\left[2(2-p)t + (1-2r)(p+1)t^p + (1-2r)(1-p)\right] \log t \\ & + 2(1-r)(2+p)t^{1+p} - (1-2r)(p+2)t^p - pt^{p-1} \\ & + (2rp - 4r - p)t - (1-2r)(2-p), \end{aligned} \quad (2.4)$$

$$\begin{aligned} g_2(1) = & 0, \\ g_3(t) = & -\left[2(2-p)t^{1-p} + p(p+1)(1-2r)\right] \log t + p(1-p)t^{-1} \\ & + (2rp - 4r + p - 4)t^{1-p} - (1-2r)(1-p)t^{-p} \\ & + 2(1-r)(2+p)(1+p)t - (1-2r)(p^2 + 3p + 1), \end{aligned} \quad (2.5)$$

$$\begin{aligned} g_3(1) = & 6p - 2 - 4r = 0, \\ g_4(t) = & -2(1-p)(2-p) \log t + 2(1-r)(2+p)(1+p)t^p \\ & - p(1+p)(1-2r)t^{p-1} + p(p-1)t^{p-2} + p(1-p)(1-2r)t^{-1} \\ & - p^2 - 2rp^2 + 6rp - 4r + 7p - 8, \end{aligned} \quad (2.6)$$

$$g_4(1) = 12p - 4 - 8r = 0,$$

$$\begin{aligned}
g_5(t) &= 2p(1-r)(p+1)(p+2)t^2 + p(1-p^2)(1-2r)t \\
&\quad - 2(2-p)(1-p)t^{2-p} - p(1-p)(1-2r)t^{1-p} + p(p-1)(p-2), \\
g_5(1) &= 2p^3 + 2(1-4r)p^2 + 4(3-r)p - 4 \\
&= \frac{8}{27}(-10r^3 - 15r^2 + 24r + 1) > 0,
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
g_6(t) &= 4p(p+1)(p+2)(1-r)t^{p+1} + p(1-p^2)(1-2r)t^p \\
&\quad - 2(2-p)^2(1-p)t - p(1-p)^2(1-2r), \\
g_6(1) &= 4p^3 + 4(1-4r)p^2 + 8(3-r)p - 8 \\
&= \frac{16}{27}(-10r^3 - 15r^2 + 24r + 1) > 0,
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
g_7(t) &= 4p(2+p)(1+p)^2(1-r)t - 2(1-p)(2-p)^2t^{1-p} \\
&\quad + p^2(1-p^2)(1-2r), \\
g_7(1) &= (3-2r)p^4 + 2(9-8r)p^3 + 11(1-2r)p^2 + 8(3-r)p - 8 \\
&= \frac{1}{81}(-32r^5 - 400r^4 - 888r^3 - 412r^2 + 1576r + 156) > 0,
\end{aligned} \tag{2.9}$$

$$g_8(t) = 4p(1+p)^2(2+p)(1-r)t^p - 2(1-p)^2(2-p)^2, \tag{2.10}$$

$$\begin{aligned}
g_8(1) &= 2(1-2r)p^4 + 4(7-4r)p^3 - 2(3+10r)p^2 + 8(4-r)p - 8 \\
&= \frac{8}{81}(-8r^5 - 60r^4 - 82r^3 - 79r^2 + 198r + 31) > 0.
\end{aligned} \tag{2.11}$$

From (2.10) we clearly see that $g_8(t)$ is strictly increasing in $(1, +\infty)$. Therefore, Lemma 2.1 follows from (2.2)–(2.9) and (2.11) together with the monotonicity of $g_8(t)$. $\square$

Lemma 2.2. *If $r \in (0, 1)$, then*

$$\begin{aligned}
g(t) &= \left[rt^{(1-r)/3+1} + (r-2)t^{(1-r)/3} + (r-2)t + r \right] \log t \\
&\quad + 2(1-r) \left(t^{(1-r)/3+1} - t^{(1-r)/3} + t - 1 \right) > 0
\end{aligned} \tag{2.12}$$

for $t \in (1, +\infty)$.

Proof. Let $p = (1 - r)/3$, $g_1(t) = t^{1-p}g'_1(t)$, $g_2(t) = t^p g'_1(t)$, $g_3(t) = t^{1-p}g'_2(t)$, $g_4(t) = t^{p+2}g'_3(t)$, and $g_5(t) = t^{1-p}g'_4(t)$, then simple computation leads to

$$g(1) = 0, \quad (2.13)$$

$$\begin{aligned} g_1(t) &= \left[r(p+1)t + (r-2)t^{1-p} + p(r-2) \right] \log t - rt^{1-p} + rt^{-p} \\ &\quad + (2+2p-2pr-r)t + 2pr-2p+r-2, \end{aligned} \quad (2.14)$$

$$\begin{aligned} g_1(1) &= 0, \\ g_2(t) &= \left[r(p+1)t^p + (1-p)(r-2) \right] \log t + (2p+2-rp)t^p \\ &\quad + p(r-2)t^{p-1} - rpt^{-1} + pr-2, \end{aligned} \quad (2.15)$$

$$\begin{aligned} g_2(1) &= 0, \\ g_3(t) &= pr(p+1) \log t + (r-2)(1-p)t^{-p} + rpt^{-p-1} \\ &\quad + p(p-1)(r-2)t^{-1} + 2p^2 - rp^2 + pr + 2p + r, \end{aligned} \quad (2.16)$$

$$\begin{aligned} g_3(1) &= 6p + 2r - 2 = 0, \\ g_4(t) &= p(p+1)rt^{p+1} + p(1-p)(r-2)t^p \\ &\quad - p(1-p)(r-2)t - p(p+1)r, \end{aligned} \quad (2.17)$$

$$\begin{aligned} g_4(1) &= 0, \\ g_5(t) &= rp(1+p)^2t + p(1-p)(2-r)t^{1-p} + p^2(1-p)(r-2), \end{aligned} \quad (2.18)$$

$$\begin{aligned} g_5(1) &= 2p(p^2 + 2rp - 2p + 1) \\ &= \frac{2(1-r)}{27}(-5r^2 + 10r + 4) > 0. \end{aligned} \quad (2.19)$$

From (2.18) we clearly see that $g_5(t)$ is strictly increasing in $(1, +\infty)$. Therefore, Lemma 2.2 follows from (2.13)–(2.17) and (2.19) together with the monotonicity of $g_5(t)$. $\square$

3. Main Results

Theorem 3.1. *If $\alpha \in (0, 1)$, then*

$$A^\alpha(a, b)L^{1-\alpha}(a, b) \leq M_{(1+2\alpha)/3}(a, b) \quad (3.1)$$

with equality if and only if $a = b$, and the parameter $(1 + 2\alpha)/3$ cannot be improved.

Proof. If $a = b$, then we clearly see that

$$a = A^\alpha(a, b)L^{1-\alpha}(a, b) = M_{(1+2\alpha)/3}(a, b) = b. \quad (3.2)$$

If $a \neq b$, then without loss of generality we assume that $a > b$ and let $t = a/b > 1$; hence elementary calculation yields

$$M_{(1+2\alpha)/3}(a, b) - A^\alpha(a, b)L^{1-\alpha}(a, b) = b \left[\left(\frac{t^{(1+2\alpha)/3} + 1}{2} \right)^{3/(1+2\alpha)} - \left(\frac{1+t}{2} \right)^\alpha \left(\frac{t-1}{\log t} \right)^{1-\alpha} \right]. \quad (3.3)$$

Let

$$\begin{aligned} f(t) = & \frac{3}{1+2\alpha} \log \left(t^{(1+2\alpha)/3} + 1 \right) - \frac{3}{1+2\alpha} \log 2 - \alpha \log \frac{t+1}{2} + \alpha \log 2 \\ & - (1-\alpha) \log(t-1) + (1-\alpha) \log(\log t), \end{aligned} \quad (3.4)$$

then

$$\lim_{t \rightarrow 1} f(t) = 0, \quad (3.5)$$

$$f'(t) = \frac{g(t)}{t(t+1)(t-1)(t^{(1+2\alpha)/3} + 1) \log t}, \quad (3.6)$$

where

$$\begin{aligned} g(t) = & - \left[t^2 + t^{(1+2\alpha)/3} + (1-2\alpha) \left(t^{(1+2\alpha)/3+1} + t \right) \right] \log t \\ & + (1-\alpha) \left(t^{(1+2\alpha)/3+2} - t^{(1+2\alpha)/3} + t^2 - 1 \right). \end{aligned} \quad (3.7)$$

From Lemma 2.1 and (3.6) we know that

$$f'(t) > 0 \quad (3.8)$$

for $t \in (1, +\infty)$.

Therefore, we get

$$M_{(1+2\alpha)/3}(a, b) > A^\alpha(a, b)L^{1-\alpha}(a, b) \quad (3.9)$$

for $a > b$ that follows from (3.3)–(3.5) and (3.8).

Next, we prove that the parameter $(1+2\alpha)/3$ cannot be improved.

For any $0 < \varepsilon < (1 + 2\alpha)/3$, let $0 < t < 1$, then (1.1) leads to

$$\begin{aligned} & A^\alpha(t+1, 1)L^{1-\alpha}(t+1, 1) - M_{(1+2\alpha)/3-\varepsilon}(t+1, 1) \\ &= \left(1 + \frac{t}{2}\right)^\alpha \cdot \left[\frac{t}{\log(t+1)}\right]^{1-\alpha} - \left[\frac{(t+1)^{(1+2\alpha)/3-\varepsilon} + 1}{2}\right]^{1/((1+2\alpha)/3-\varepsilon)} \\ &= \frac{f_1(t)}{[\log(1+t)]^{1-\alpha}}, \end{aligned} \quad (3.10)$$

where

$$f_1(t) = \left(1 + \frac{t}{2}\right)^\alpha \cdot t^{1-\alpha} - \left[\frac{(t+1)^{(1+2\alpha-3\varepsilon)/3} + 1}{2}\right]^{3/(1+2\alpha-3\varepsilon)} [\log(1+t)]^{1-\alpha}. \quad (3.11)$$

Making use of the Taylor expansion we get

$$f_1(t) = \left[\frac{\varepsilon}{8}t^2 + o(t^2)\right] \cdot t^{1-\alpha}. \quad (3.12)$$

Equations (3.10) and (3.12) imply that for any $\alpha \in (0, 1)$ and $0 < \varepsilon < (1 + 2\alpha)/3$, there exists $0 < \delta_1(\varepsilon, \alpha) < 1$, such that

$$A^\alpha(1, 1+t)L^{1-\alpha}(1, 1+t) > M_{(1+2\alpha)/3-\varepsilon}(1, 1+t) \quad (3.13)$$

for $t \in (0, \delta_1)$. □

Remark 3.2. For any $0 < \alpha < 1$ we have

$$M_0(a, b) \leq A^\alpha(a, b)L^{1-\alpha}(a, b) \quad (3.14)$$

for all $a, b > 0$, with equality if and only if $a = b$, and the parameter 0 in the lower bound cannot be improved.

In fact, if $a = b$, then we clearly see that $M_0(a, b) = A^\alpha(a, b)L^{1-\alpha}(a, b) = a$. If $a \neq b$, then $M_0(a, b) < A^\alpha(a, b)L^{1-\alpha}(a, b)$ follows from $M_0(a, b) < L(a, b) < A(a, b)$.

Next, we prove that the parameter 0 in the lower bound cannot be improved.

For any $\varepsilon > 0$, we have

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{M_\varepsilon(t, 1)}{A^\alpha(t, 1)L^{1-\alpha}(t, 1)} &= \lim_{t \rightarrow \infty} \frac{((t^\varepsilon + 1)/2)^{1/\varepsilon}}{((t+1)/2)^\alpha ((t-1)/\log t)^{1-\alpha}} \\ &= \lim_{t \rightarrow \infty} \frac{((1+t^{-\varepsilon})/2)^{1/\varepsilon} (\log t)^{1-\alpha}}{((1+t^{-1})/2)^\alpha (1-t^{-1})^{1-\alpha}} \\ &= +\infty. \end{aligned} \quad (3.15)$$

Equation (3.15) implies that for any $\varepsilon > 0$, there exists $T_1 = T_1(\varepsilon) > 1$, such that

$$A^\alpha(t, 1)L^{1-\alpha}(t, 1) < M_\varepsilon(t, 1) \quad (3.16)$$

for $t \in (T_1, +\infty)$.

Theorem 3.3. *If $\alpha \in (0, 1)$, then*

$$G^\alpha(a, b)L^{1-\alpha}(a, b) \leq M_{(1-\alpha)/3}(a, b) \quad (3.17)$$

for all $a, b > 0$, with equality if and only if $a = b$, and the parameter $(1 - \alpha)/3$ cannot be improved.

Proof. If $a = b$, then we clearly see that

$$a = G^\alpha(a, b)L^{1-\alpha}(a, b) = M_{(1-\alpha)/3}(a, b) = b. \quad (3.18)$$

If $a \neq b$, then without loss of generality, we assume that $a > b$, let $t = a/b > 1$; hence simple computation leads to

$$M_{(1-\alpha)/3}(a, b) - G^\alpha(a, b)L^{1-\alpha}(a, b) = b \left[\left(\frac{t^{(1-\alpha)/3} + 1}{2} \right)^{3/(1-\alpha)} - t^{\alpha/2} \left(\frac{t-1}{\log t} \right)^{1-\alpha} \right]. \quad (3.19)$$

Let

$$\begin{aligned} h(t) &= \frac{3}{1-\alpha} \log \left(t^{(1-\alpha)/3} + 1 \right) - \frac{3}{1-\alpha} \log 2 - \alpha \log \frac{t+1}{2} + \alpha \log 2 \\ &\quad - (1-\alpha) \log(t-1) + (1-\alpha) \log(\log t), \end{aligned} \quad (3.20)$$

then

$$\lim_{t \rightarrow 1} h(t) = 0, \quad (3.21)$$

$$h'(t) = \frac{g(t)}{2t(t-1)(t^{(1-\alpha)/3} + 1) \log t}, \quad (3.22)$$

where

$$\begin{aligned} g(t) &= \left[\alpha t^{(1-\alpha)/3+1} + (\alpha-2)t^{(1-\alpha)/3} + (\alpha-2)t + \alpha \right] \log t \\ &\quad + 2(1-\alpha) \left(t^{(1-\alpha)/3+1} - t^{(1-\alpha)/3} + t - 1 \right). \end{aligned} \quad (3.23)$$

From Lemma 2.2 and (3.22) we know that

$$h'(t) > 0 \quad (3.24)$$

for $t \in (1, +\infty)$.

Therefore, we get

$$M_{(1-\alpha)/3}(a, b) > G^\alpha(a, b)L^{1-\alpha}(a, b) \quad (3.25)$$

for $a > b$ that follows from (3.19)–(3.21) and (3.24).

Next, we prove that the constants $(1 - \alpha)/3$ cannot be improved.

For any $0 < \varepsilon < (1 - \alpha)/3$, let $0 < t < 1$, then (1.1) leads to

$$\begin{aligned} & G^\alpha(t+1, 1)L^{1-\alpha}(t+1, 1) - M_{(1-\alpha)/3-\varepsilon}(t+1, 1) \\ &= (1+t)^{\alpha/2} \cdot \left[\frac{t}{\log(t+1)} \right]^{1-\alpha} - \left[\frac{(t+1)^{(1-\alpha)/3-\varepsilon} + 1}{2} \right]^{1/((1-\alpha)/3-\varepsilon)} \\ &= \frac{h_1(t)}{[\log(1+t)]^{1-\alpha}}, \end{aligned} \quad (3.26)$$

where

$$h_1(t) = (1+t)^{\alpha/2} \cdot t^{1-\alpha} - \left[\frac{(t+1)^{(1-\alpha-3\varepsilon)/3} + 1}{2} \right]^{3/(1-\alpha-3\varepsilon)} [\log(1+t)]^{1-\alpha}. \quad (3.27)$$

Making use of the Taylor expansion we get

$$h_1(t) = \left[\frac{\varepsilon}{8} t^2 + o(t^2) \right] \cdot t^{1-\alpha}. \quad (3.28)$$

Equations (3.26) and (3.28) imply that for any $\alpha \in (0, 1)$ and $0 < \varepsilon < (1 - \alpha)/3$, there exists $0 < \delta_2(\varepsilon, \alpha) < 1$, such that

$$A^\alpha(1, 1+t)L^{1-\alpha}(1, 1+t) > M_{(1-\alpha)/3-\varepsilon}(1, 1+t) \quad (3.29)$$

for $t \in (0, \delta_2)$. □

Remark 3.4. For any $\varepsilon > 0$, we have

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{M_\varepsilon(t, 1)}{G^\alpha(t, 1)L^{1-\alpha}(t, 1)} &= \lim_{t \rightarrow \infty} \frac{((t^\varepsilon + 1)/2)^{1/\varepsilon}}{t^{\alpha/2}((t-1)/\log t)^{1-\alpha}} \\ &= \lim_{t \rightarrow \infty} \frac{((1+t^{-\varepsilon})/2)^{1/\varepsilon} t^{\alpha/2} (\log t)^{1-\alpha}}{(1-t^{-1})^{1-\alpha}} = +\infty. \end{aligned} \quad (3.30)$$

Therefore, (3.30) implies that inequality

$$M_0(a, b) \leq G^\alpha(a, b)L^{1-\alpha}(a, b) \quad (3.31)$$

holds for all $\alpha \in (0, 1)$ and $a, b > 0$, with equality if and only if $a = b$ and the parameter 0 in the lower bound cannot be improved.

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