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Research Article

Strong and Weak Convergence of Modified Mann Iteration for New Resolvents of Maximal Monotone Operators in Banach Spaces

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We prove strong and weak convergence theorems for a new resolvent of maximal monotone operators in a Banach space and give an estimate of the convergence rate of the algorithm. Finally, we apply our convergence theorem to the convex minimization problem. The result present in this paper extend and improve the corresponding result of Ibaraki and Takahashi (2007), and Kim and Xu (2005).

1. Introduction

Let E be a Banach space with norm $\|\cdot\|$, let E^* denote the dual of E, and let $\langle x, f\rangle$ denote the value of $f \in E^*$ at $x \in E$. Let $T : E \to E^*$ be an operator. The problem of finding $v \in E$ satisfying $0 \in Tv$ is connected with the convex minimization problems. When T is maximal monotone, a well-known method for solving the equation $0 \in Tv$ in Hilbert space H is the proximal point algorithm (see [1]): $x_1 = x \in H$ and

$$x_{n+1} = J_{r_n} x_n, \quad n = 1, 2, \ldots, \tag{1.1}$$

where $r_n \subset (0, \infty)$ and $J_r = (I + rT)^{-1}$ for all $r > 0$ is the resolvent operator for T. Rockafellar [1] proved the weak convergence of the algorithm (1.1).

The modifications of the proximal point algorithm for different operators have been investigated by many authors. Recently, Kohsaka and Takahashi [2] considered the algorithm (1.2) in a smooth and uniformly convex Banach space and Kamimura et al. [3] considered the

algorithm (1.3) in a uniformly smooth and uniformly convex Banach space E; $x_1 = x \in E, u \in E$ and

$$x_{n+1} = J^{-1}\left(\alpha_n Ju + (1-\alpha_n) J J_{r_n} x_n\right), \quad n = 1, 2, \ldots, \tag{1.2}$$

$$x_{n+1} = J^{-1}(\alpha_n Jx_n + (1-\alpha_n) J J_{r_n} x_n), \quad n = 1, 2, \ldots, \tag{1.3}$$

where $J_r = (J + rT)^{-1} J$, J is the duality mapping of E. They showed that the algorithm (1.2) converges strongly to some element of $T^{-1}0$ and the algorithm (1.3) converges weakly to some element of $T^{-1}0$ provided that the sequences $\{\alpha_n\}$ and $\{r_n\}$ of real numbers are chosen appropriately. These results extend the Kamimura and Takahashi [4] results in Hilbert spaces to those in Banach spaces.

In 2008, motivated by Kim and Xu [5], Li and Song [6] studied a combination of the schemes of (1.2) and (1.3); $x_1 = x \in E$ and

$$\begin{aligned} y_n &= J^{-1}(\alpha_n Jx_n + (1-\alpha_n) J J_{r_n} x_n), \\ x_{n+1} &= J^{-1}\left(\beta_n Jx + \left(1-\beta_n\right) Jy_n\right), \end{aligned} \tag{1.4}$$

for every $n = 1, 2, \ldots$, where $J_r = (J + rT)^{-1} J$, J is the duality mapping of E. They also proved strong and weak convergence theorems and give an estimate for the rate of convergence of the algorithm (1.4).

Very recently, Ibaraki and Takahashi [7] introduced the Mann iteration and Harpern iteration for new resovents of maximal monotone operator in a uniformly smooth and uniformly convex Banach space E; $x_1 = x \in E$ $u \in E$ and

$$x_{n+1} = \alpha_n u + (1-\alpha_n) J_{r_n} x_n, \quad n = 1, 2, \ldots, \tag{1.5}$$

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) J_{r_n} x_n, \quad n = 1, 2, \ldots, \tag{1.6}$$

where $J_r = (I + rBJ)^{-1}$, J is the duality mapping of E, and $B \subset E^* \times E$ is maximal monotone. They proved that Algorithm (1.5) converges strongly to some element of $(BJ)^{-1}0$ and Algorithm (1.6) converges weakly to some element of $(BJ)^{-1}0$ provided that the sequences $\{\alpha_n\}$ and $\{r_n\}$ of real numbers are chosen appropriately.

Inspired and motivated by Li and Song [6] and Ibaraki and Takahashi [7], we study a combination of the schemes of (1.5) and (1.6); $x_1 = x \in E$ and

$$\begin{aligned} y_n &= \alpha_n x_n + (1-\alpha_n) J_{r_n} x_n, \\ x_{n+1} &= \beta_n x + \left(1-\beta_n\right) y_n, \end{aligned} \tag{1.7}$$

for every $n = 1, 2, \ldots$, where $J_r = (I + rBJ)^{-1}$, J is the duality mapping of E, and $B \subset E^* \times E$ is maximal monotone. When $\alpha_n \equiv 0$, Algorithm (1.7) reduces to (1.5) and, when $\beta_n \equiv 0$, Algorithm (1.7) reduces to (1.6). Then, we prove strong and weak convergence theorems of the sequence and we also estimate the rate of the convergence of algorithm (1.7). Finally, by using our main result, we consider the problem of finding minimizes of convex functions defined on Banach spaces.

2. Preliminaries

Let E be a real Banach space with dual space E^*. When $\{x_n\}$ is a sequence in E, we denote strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$, respectively. As usual, we denote the duality pairing of E^* by $\langle x, x^* \rangle$, when $x^* \in E^*$ and $x \in E$, and the closed unit ball by U_E, and denote by $\mathbb{R}$ and $\mathbb{N}$ the set of all real numbers and the set of all positive integers, respectively. The set $\mathbb{R}_+$ stands for $[0, +\infty)$ and $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$. An operator $T \subset E \times E^*$ is said to be monotone if $\langle x - y, x^* - y^* \rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in T$. We denote the set $\{x \in E : 0 \in Tx\}$ by $T^{-1}0$. A monotone T is said to be *maximal* if its graph $G(T) = \{(x, y) : y \in Tx\}$ is not properly contained in the graph of any other monotone operator. If T is maximal monotone, then the solution set $T^{-1}0$ is closed and convex. If E is reflexive and strictly convex, then a monotone operator T is maximal if and only if $R(J + \lambda T) = E^*$ for each $\lambda > 0$ (see [8, 9] for more details).

The normalized duality mapping J from E into E^* is defined by

$$J(x) = \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \right\}, \quad \forall x \in E. \tag{2.1}$$

We recall [10] that E is reflexive if and only if J is surjective; E is smooth if and only if J is single-valued.

Let E be a smooth Banach space. Consider the following function: (see [11])

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \qquad \forall x, y \in E. \tag{2.2}$$

It is obvious from the definition of ϕ that $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$, for all $x, y \in E$. We also know that

$$\phi(x, y) + \phi(y, x) = 2\langle x - y, Jx - Jy \rangle, \quad \text{for each } x, y \in E. \tag{2.3}$$

We recall [12] that the functional $\|\cdot\|^2$ is called totally convex at x if the function $\nu(x, t) : [0, \infty) \to [0, \infty]$ defined by

$$\nu(x, t) = \inf\{\phi(y, x) : y \in E, \|y - x\| = t\}, \tag{2.4}$$

is positive whenever $t > 0$. The functional $\|\cdot\|^2$ is called totally convex on bounded sets if for each bounded nonempty subset A of E, the function $\nu(A, t) : [0, \infty) \to [0, \infty]$ defined by $\nu(A, t) = \inf\{\nu(x, t) : x \in A\}$ is positive on $(0, \infty)$.

It is well known that if a Banach space E is uniformly convex, then $\|\cdot\|^2$ is totally convex on any bounded nonempty set. It is known that (see [12]) if $\|\cdot\|^2$ is totally convex on a bounded set A, then $\nu(A, ct) \geq c\nu(A, t)$ for $c \geq 1$ and $t \geq 0$, and $\nu(A, \cdot)$ is strictly increasing on $[0, \infty)$.

Lemma 2.1 (see [13]). *Let E be a uniformly convex, smooth Banach space, and let $\{x_n\}$ and $\{y_n\}$ be sequences in E. If $\{x_n\}$ or $\{y_n\}$ is bounded and $\lim_{n\to\infty}\phi(x_n, y_n) = 0$, then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

Let E be a reflexive, strictly convex, smooth Banach space, and J the duality mapping from E into E^*. Then J^{-1} is also single-valued, one-to-one, surjective, and it is the duality mapping from E^* into E. We make use of the following mapping V studied in Alber [11]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2 \tag{2.5}$$

for all $x \in E$ and $x^* \in E^*$. In other words, $V(x, x^*) = \phi(x, J^{-1}(x))$ for all $x \in E$ and $x^* \in E^*$.

Lemma 2.2 (see [7]). *Let E be a reflexive, strictly convex, smooth Banach space, and let V be as in* (2.5). *Then*

$$V(x, x^*) + 2\langle y, Jx - x^*\rangle \leq V(x + y, x^*) \tag{2.6}$$

for all $x, y \in E$ and $x^* \in E^*$.

Let E be a smooth Banach space and let D be a nonempty closed convex subset of E. A mapping $R : D \to D$ is called generalized nonexpansive if $F(R) \neq \emptyset$ and $\phi(Rx, y) \leq \phi(x, y)$ for each $x \in D$ and $y \in F(R)$, where $F(R)$ is the set of fixed points of R. Let C be a nonempty closed subset of E. A mapping $R : E \to C$ is said to be sunny if

$$R(Rx + t(x - Rx)) = Rx, \quad \forall x \in E, \quad \forall t \geq 0. \tag{2.7}$$

A mapping $R : E \to C$ is said to be a retraction if $Rx = x$, for all $x \in C$. If E is smooth and strictly convex, then a sunny generalized nonexpansive retraction of E onto C is uniquely decided if it exists (see [14]). We also know that if E is reflexive, smooth, and strictly convex and C is a nonempty closed subset of E, then there exists a sunny generalized nonexpansive retraction R_C of E onto C if and only if $J(C)$ is closed and convex. In this case, R_C is given by $R_C = J^{-1}\Pi_{J(C)} J$ see [15]. Let C be a nonempty closed subset of a Banach space E. Then C is said to be a sunny generalized nonexpansive retract (resp., a generalized nonexpansive retract) of E if there exists a sunny generalized nonexpansive retraction (resp, a generalized nonexpansive retraction) of E onto C (see [14] for more detials). The set of fixed points of such a generalized nonexpansive retraction is C. The following lemma was obtained in [14].

Lemma 2.3 (see [14]). *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E. Let R_C be a retraction of E onto C. Then R_C is sunny and generalized nonexpansive if and only if*

$$\langle x - R_C x, JR_C x - Jy\rangle \geq 0, \tag{2.8}$$

for each $x \in E$ and $y \in C$, where J is the duality mapping of E.

Let E be a reflexive, strictly convex, and smooth Banach space with its dual E^*. If a monotone operator $B \subset E^* \times E$ is maximal, then $(BJ)^{-1}0$ is closed and $E = R(I + rBJ)$ for all $r > 0$ (see [14]). So, for each $r > 0$ and $x \in E$, we can consider the set $J_r(x) = \{z \in E : x \in z + rBJz\}$. From [14], $J_r x$ consists of one point. We denote such a J_r by $(I + rBJ)^{-1}$.

However J_r is called a generalized resolvent of B. We also know that $(BJ)^{-1}0 = F(J_r)$ for each $r > 0$, where $F(J_r)$ is the set of fixed points of J_r and J_r is generalized nonexpansive for each $r > 0$ (see [14]). The Yosida approximtion of B is defined by $A_r = (I - J_r)/r$. We know that $(J\,J_r x, A_r x) \in B$; (see [14] for more detials). The following result was obtained in [14].

Theorem 2.4 (see [14]). *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let $B \subset E^* \times E$ be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Then the following hold:*

(1) *for each $x \in E$, $\lim_{r\to\infty} J_r x$ exists and belongs to $(BJ)^{-1}0$,*

(2) *if $Rx := \lim_{r\to\infty} J_r x$ for each $x \in E$, then R is a sunny generalized nonexpansive retraction of E onto $(BJ)^{-1}0$.*

Lemma 2.5 (see [7]). *Let E be a reflexive, strictly convex, and smooth Banach space, let $B \subset E^* \times E$ be a maximal monotone operator with $B^{-1}0 \neq \emptyset$, and $J_r = (I + rBJ)^{-1}$ for all $r > 0$. Then*

$$\phi(x, J_r x) + \phi(J_r x, u) \leq \phi(x, u), \tag{2.9}$$

for all $r > 0$, $u \in (BJ)^{-1}0$, and $x \in E$.

Lemma 2.6 (see [16]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n t_n + r_n, \quad n \geq 1, \tag{2.10}$$

where $\{\alpha_n\}$, $\{t_n\}$, and $\{r_n\}$ satisfy the conditions: $\{\alpha_n\} \subset [0,1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\limsup_{n\to\infty} t_n \leq 0$, and $r_n \geq 0$, $\sum_{n=1}^{\infty} r_n < \infty$. Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.7 (see [17]). *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequence of nonnegative real numbers satisfying*

$$\alpha_{n+1} \leq \alpha_n + \beta_n, \tag{2.11}$$

for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} \beta_n < +\infty$. Then $\{\alpha_n\}$ has a limit in $[0, +\infty)$.

3. Convergence Theorems

In this section, we first prove a strong convergence theorem for the algorithm (1.7) which extends the previous result of Ibaraki and Takahashi [7] and we next prove a weak convergence theorem for algorithm (1.7) under different conditions on data, respectively.

Theorem 3.1. *Let E be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable. Let $B \subset E^* \times E$ be a maximal monotone operator with $B^{-1}0 \neq \emptyset$ and let $J_r = (I + rBJ)^{-1}$ for all $r > 0$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in E$ and*

$$\begin{aligned} y_n &= \alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n, \\ x_{n+1} &= \beta_n x + (1 - \beta_n) y_n, \end{aligned} \tag{3.1}$$

for every $n = 1, 2, \ldots$, *where* $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1]$, $\{r_n\} \subset (0,\infty)$ *satisfy* $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\beta_n = 0$, $\sum_{n=1}^{\infty}\beta_n = \infty$ *and* $\lim_{n\to\infty} r_n = \infty$. *Then the sequence* $\{x_n\}$ *converges strongly to* $R_{(BJ)^{-1}0}(x)$, *where* $R_{(BJ)^{-1}0}$ *is a sunny generalized nonexpansive retraction of* E *onto* $(BJ)^{-1}0$.

Proof. Note that $B^{-1}0 \neq \emptyset$ implies $(BJ)^{-1}0 \neq \emptyset$. In fact, if $u^* \in B^{-1}0$, we obtain $0 \in Bu^*$ and hence $0 \in BJ\,J^{-1}u^*$. So, we have $J^{-1}u^* \in (BJ)^{-1}0$. We denote a sunny generalized nonexpansive retraction $R_{(BJ)^{-1}0}$ of E onto $(BJ)^{-1}0$ by R. Let $z \in (BJ)^{-1}0$. We first prove that $\{x_n\}$ is bounded. From Lemma 2.5 and the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}
\phi(y_n, z) &= \phi(\alpha_n x_n + (1-\alpha_n)J_{r_n}x_n, z) \\
&\leq \alpha_n\phi(x_n, z) + (1-\alpha_n)\phi(J_{r_n}x_n, z) \\
&\leq \alpha_n\phi(x_n, z) + (1-\alpha_n)\{\phi(x_n, z) - \phi(x_n, J_{r_n}x_n)\} \\
&\leq \alpha_n\phi(x_n, z) + (1-\alpha_n)\phi(x_n, z) = \phi(x_n, z),
\end{aligned} \tag{3.2}$$

for all $n \in \mathbb{N}$. By (3.2), we have

$$\begin{aligned}
\phi(x_{n+1}, z) &= \phi(\beta_n x + (1-\beta_n)y_n, z) \\
&\leq \beta_n\phi(x, z) + (1-\beta_n)\phi(y_n, z) \\
&\leq \beta_n\phi(x, z) + (1-\beta_n)\phi(x_n, z),
\end{aligned} \tag{3.3}$$

for all $n \in \mathbb{N}$. Hence, by induction, we have $\phi(x_n, z) \leq \phi(x, z)$ for all $n \in \mathbb{N}$ and, therefore, $\{\phi(x_n, z)\}$ is bounded. This implies that $\{x_n\}$ is bounded. Since $\phi(y_n, z) \leq \phi(x_n, z)$ and $\phi(J_{r_n}x_n, z) \leq \phi(x_n, z)$ for all $n \in \mathbb{N}$, it follows that $\{y_n\}$ and $\{J_{r_n}x_n\}$ are also bounded. We next prove that

$$\lim_{n\to\infty}\sup\langle x - Rx, Jx_n - JRx\rangle \leq 0. \tag{3.4}$$

Put $u_n = x_{n+1}$ for all $n \in \mathbb{N}$. Since $\{Ju_n\}$ is bounded, without loss of generality, we have a subsequence $\{Ju_{n_i}\}$ of $\{Ju_n\}$ such that

$$\lim_{i\to\infty}\langle x - Rx, Ju_{n_i} - JRx\rangle = \lim_{n\to\infty}\sup\langle x - Rx, Ju_n - JRx\rangle, \tag{3.5}$$

and $\{Ju_{n_i}\}$ converges weakly to some $v^* \in E^*$. From the definition of $\{x_n\}$, we have

$$u_n - y_n = \beta_n(x - y_n), \qquad y_n - J_{r_n}x_n = \alpha_n(x_n - J_{r_n}x_n) \tag{3.6}$$

for all $n \in \mathbb{N}$. Since $\{y_n\}$ is bounded and $\beta_n \to 0$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty}\|u_n - y_n\| = \lim_{n\to\infty}\beta_n\|x - y_n\| = 0. \tag{3.7}$$

Moreover, we note that

$$\lim_{n\to\infty}\|y_n - J_{r_n}x_n\| = \lim_{n\to\infty}\alpha_n\|x_n - J_{r_n}x_n\| = 0. \tag{3.8}$$

By (3.7) and (3.8), we have

$$\lim_{n\to\infty}\|u_n - J_{r_n}x_n\| = 0. \tag{3.9}$$

Since E has a uniformly Gâteaux differentiable norm, the duality mapping J is norm to $weak^*$ uniformly continuous on each bounded subset of E. Therefore, we obtain from (3.9) that

$$Ju_{n_i} - JJ_{r_{n_i}}x_{n_i} \rightharpoonup 0, \quad \text{as } i \longrightarrow \infty. \tag{3.10}$$

This implies that $J\,J_{r_{n_i}}x_{n_i} \rightharpoonup v^*$ as $i \to \infty$. On the other hand, from $r_n \to \infty$ as $n \to \infty$, we have

$$\lim_{n\to\infty}\|A_{r_n}x_n\| = \lim_{n\to\infty}\frac{1}{r_n}\|x_n - J_{r_n}x_n\| = 0. \tag{3.11}$$

If $(y^*, y) \in B$, then it holds from the monotonicity of B that

$$\langle y - A_{r_{n_i}}x_{n_i}, y^* - JJ_{r_{n_i}}x_{n_i}\rangle \geq 0, \tag{3.12}$$

for all $i \in \mathbb{N}$. Letting $i \to \infty$, we get $\langle y, y^* - v^*\rangle \geq 0$. Then, the maximal of B implies $v^* \in B^{-1}0$. Put $v = J^{-1}v^*$. Applying Lemma 2.3, we obtain

$$\begin{aligned}\lim_{n\to\infty}\sup\langle x - Rx, Ju_n - JRx\rangle &= \lim_{i\to\infty}\langle x - Rx, Ju_{n_i} - JRx\rangle \\ &= \langle x - Rx, v^* - JRx\rangle \\ &= \langle x - Rx, Jv - JRx\rangle \leq 0.\end{aligned} \tag{3.13}$$

Finally, we prove that $x_n \to Rx$ as $n \to \infty$. From Lemma 2.2, the convexity of $\|\cdot\|^2$ and (3.2), we have

$$\begin{aligned}\phi(x_{n+1}, Rx) &= V(\beta_n x + (1-\beta_n)y_n, JRx) \\ &\leq V(\beta_n x + (1-\beta_n)y_n - \beta_n(x - Rx), JRx) \\ &\quad - 2\langle -\beta_n(x - Rx), Jx_{n+1} - JRx\rangle \\ &= V(\beta_n Rx + (1-\beta_n)y_n, JRx) + 2\beta_n\langle x - Rx, Jx_{n+1} - JRx\rangle \\ &= \phi(\beta_n Rx + (1-\beta_n)y_n, Rx) + 2\beta_n\langle x - Rx, Jx_{n+1} - JRx\rangle \\ &\leq \beta_n\phi(Rx, Rx) + (1-\beta_n)\phi(y_n, Rx) + 2\beta_n\langle x - Rx, Jx_{n+1} - JRx\rangle \\ &\leq (1-\beta_n)\phi(x_n, Rx) + \beta_n\sigma_n,\end{aligned} \tag{3.14}$$

for all $n \in \mathbb{N}$, where $\sigma_n = 2\langle x - Rx, Jx_{n+1} - JRx\rangle$. It easily verified from the assumption and (3.4) that $\sum_{n=1}^{\infty} \beta_n = \infty$ and $\limsup_{n\to\infty}\sigma_n \le 0$. Hence, by Lemma 2.6, $\lim_{n\to\infty}\phi(x_n, Rx) = 0$. Applying Lemma 2.1, we obtain $\lim_{n\to\infty}\|x_n - Rx\| = 0$. Therefore, $\{x_n\}$ converges strongly to $R_{(BJ)^{-1}0}(x)$. □

Put $\alpha_n \equiv 0$ in Theorem 3.1, then we obtain the following result.

Corollary 3.2 (see Ibaraki and Takahashi [7]). *Let E be a uniformly convex and uniformly smooth Banach space and let $B \subset E^* \times E$ be a maximal monotone operator with $B^{-1}0 \neq \emptyset$, let $J_r = (I + rBJ)^{-1}$ for all $r > 0$, and let $\{x_n\}$ be a sequence generated by $x_1 = x \in E$ and*

$$x_{n+1} = \beta_n x + (1 - \beta_n) J_{r_n} x_n, \tag{3.15}$$

for every $n = 1, 2, \ldots$, where $\{\beta_n\} \subset [0,1]$, $\{r_n\} \subset (0,\infty)$ satisfy $\lim_{n\to\infty}\beta_n = 0$, $\sum_{n=1}^{\infty}\beta_n = \infty$ and $\lim_{n\to\infty} r_n = \infty$. Then the sequence $\{x_n\}$ converges strongly to $R_{(BJ)^{-1}0}(x)$, where $R_{(BJ)^{-1}0}$ is the generalized projection of E onto $(BJ)^{-1}0$.

Theorem 3.3. *Let E be a uniformly convex and smooth Banach space whose duality mapping J is weakly sequentially continuous. Let $B \subset E^* \times E$ be a maximal monotone operator with $B^{-1}0 \neq \emptyset$ and let $J_r = (I + rBJ)^{-1}$ for all $r > 0$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in E$ and*

$$\begin{aligned} y_n &= \alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n, \\ x_{n+1} &= \beta_n x + (1 - \beta_n) y_n, \end{aligned} \tag{3.16}$$

for every $n = 1, 2, \ldots$, where $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1]$, $\{r_n\} \subset (0,\infty)$ satisfy $\sum_{n=1}^{\infty}\beta_n < \infty$, $\limsup_{n\to\infty}\alpha_n < 1$ and $\liminf_{n\to\infty} r_n > 0$. Then the sequence $\{x_n\}$ converges weakly to an element of $(BJ)^{-1}0$.

Proof. Let $v \in (BJ)^{-1}0$. Then, from (3.3), we have

$$\phi(x_{n+1}, v) \le (1 - \beta_n)\phi(x_n, v) + \beta_n\phi(x, v) \le \phi(x_n, v) + \beta_n\phi(x, v), \tag{3.17}$$

for all $n \in \mathbb{N}$. By Lemma 2.7, $\lim_{n\to\infty}\phi(x_n, v)$ exists. From $(\|x_n\| - \|v\|)^2 \le \phi(x_n, v)$ and $\phi(J_{r_n}x_n, v) \le \phi(x_n, v)$, we note that $\{x_n\}$ and $\{J_{r_n}x_n\}$ are bounded. From (3.3) and (3.2), we have

$$\begin{aligned} \phi(x_{n+1}, v) &\le \beta_n\phi(x, v) + (1 - \beta_n)\phi(y_n, v) \\ &\le \beta_n\phi(x, v) + (1 - \beta_n)\{\alpha_n\phi(x_n, v) + (1 - \alpha_n)(\phi(x_n, v) - \phi(x_n, J_{r_n}x_n))\} \\ &= \beta_n\phi(x, v) + (1 - \beta_n)\{\phi(x_n, v) - (1 - \alpha_n)\phi(x_n, J_{r_n}x_n)\} \\ &= \beta_n\phi(x, v) + (1 - \beta_n)\phi(x_n, v) - (1 - \beta_n)(1 - \alpha_n)\phi(x_n, J_{r_n}x_n), \end{aligned} \tag{3.18}$$

for all $n \in \mathbb{N}$ and hence,

$$\begin{aligned}(1-\beta_n)(1-\alpha_n)\phi(x_n, J_{r_n}x_n) &\le \beta_n\phi(x,v) + (1-\beta_n)\phi(x_n,v) - \phi(x_{n+1},v)\\ &= \beta_n(\phi(x,v) - \phi(x_n,v)) + \phi(x_n,v) - \phi(x_{n+1},v),\end{aligned} \tag{3.19}$$

for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty}\beta_n = 0$ and $\limsup_{n\to\infty}\alpha_n < 1$, $\lim_{n\to\infty}\phi(x_n, J_{r_n}x_n) = 0$. Applying Lemma 2.1, we obtain

$$\lim_{n\to\infty}\|x_n - J_{r_n}x_n\| = 0. \tag{3.20}$$

Since $\{x_n\}$ is bounded, we have a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup w \in E$ as $i \to \infty$. Then it follows from (3.20) that $J_{r_{n_i}}x_{n_i} \rightharpoonup w$ as $i \to \infty$. On the other hand, from (3.20) and $\liminf_{n\to\infty} r_n > 0$, we have

$$\lim_{n\to\infty}\|A_{r_n}x_n\| = \lim_{n\to\infty}\frac{1}{r_n}\|x_n - J_{r_n}x_n\| = 0. \tag{3.21}$$

Let $(z^*, z) \in B$. Then, it holds from monotonicity of B that

$$\langle z - A_{r_{n_i}}x_{n_i}, z^* - JJ_{r_{n_i}}x_{n_i}\rangle \ge 0, \tag{3.22}$$

for all $i \in \mathbb{N}$. Since J is weakly sequentially continuous, letting $i \to \infty$, we get $\langle z, z^* - Jw\rangle \ge 0$. Then, the maximality of B implies $Jw \in B^{-1}0$. Thus, $w \in (BJ)^{-1}0$.

Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup w_1$ and $x_{n_j} \rightharpoonup w_2$. By similar argument as above, we obtain $w_1, w_2 \in (BJ)^{-1}0$. Put $a := \lim_{n\to\infty}(\phi(x_n, w_1) - \phi(x_n, w_2))$.

Note that $\phi(x_n, w_1) - \phi(x_n, w_2) = 2\langle x_n, Jw_2 - Jw_1\rangle + \|w_1\|^2 - \|w_2\|^2, n = 1, 2, \ldots$. From $x_{n_i} \rightharpoonup w_1$ and $x_{n_j} \rightharpoonup w_2$, we have

$$a = 2\langle w_1, Jw_2 - Jw_1\rangle + \|w_1\|^2 - \|w_2\|^2, \tag{3.23}$$

$$a = 2\langle w_2, Jw_2 - Jw_1\rangle + \|w_1\|^2 - \|w_2\|^2, \tag{3.24}$$

respectively. Combining (3.23) and (3.24), we have

$$\langle w_1 - w_2, Jw_1 - Jw_2\rangle = 0. \tag{3.25}$$

Since J is strictly monotone, it follows that $w_1 = w_2$. Therefore, $\{x_n\}$ converges weakly to an element of $(BJ)^{-1}0$. □

Put $\beta_n \equiv 0$ in Theorem 3.3, then we obtain the following result.

Corollary 3.4 (see Ibaraki and Takahashi [7]). *Let E be a uniformly convex and smooth Banach space whose duality mapping J is weakly sequentially continuous. Let $B \subset E^* \times E$ be a maximal*

monotone operator with $B^{-1}0 \neq \emptyset$, *let* $J_r = (I + rBJ)^{-1}$ *for all* $r > 0$ *and let* $\{x_n\}$ *be a sequence generated by* $x_1 = x \in E$ *and*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) J_{r_n} x_n, \tag{3.26}$$

for every $n = 1, 2, \ldots,$ *where* $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$, $\{r_n\} \subset (0, \infty)$ *satisfy* $\sum_{n=1}^{\infty} \beta_n < \infty$, $\limsup_{n\to\infty} \alpha_n < 1$ *and* $\liminf_{n\to\infty} r_n > 0$. *Then the sequence* $\{x_n\}$ *converges weakly to an element of* $(BJ)^{-1}0$.

4. Rate of Convergence for the Algorithm

In this section, we study the rate of the convergence of the algorithm (1.7). We use the following notations in [6, 18]:

$$\begin{aligned}
N_0 &:= \left\{\varphi : \mathbb{R}_+ \to \overline{\mathbb{R}}_+ \mid t \mapsto \varphi(t) \text{ is nondecreasing for } t \geq 0,\ \varphi(0) = 0\right\},\\
\Omega_0 &:= \left\{\varphi : \mathbb{R}_+ \to \overline{\mathbb{R}}_+ \mid \varphi(0) = 0, \lim_{t\to 0^+} \varphi(t) = 0\right\},\\
\Gamma_0 &:= \left\{\varphi : \mathbb{R}_+ \to \overline{\mathbb{R}}_+ \mid \varphi \text{ is lsc and convex and } \varphi(t) = 0 \Longleftrightarrow t = 0\right\},\\
\Sigma_1 &:= \left\{\varphi : \mathbb{R}_+ \to \overline{\mathbb{R}}_+ \mid \varphi \text{ is lsc and convex}, \varphi(0) = 0, \lim_{t\to 0^+} t^{-1}\varphi(t) = 0\right\}.
\end{aligned} \tag{4.1}$$

We recall [18] that, for a function $\varphi : \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ satisfying $\varphi(0) = 0$, its pseudoconjugate $\varphi^{\#} : \mathbb{R}_+ \to \overline{\mathbb{R}}_+$, defined by

$$\varphi^{\#}(s) := \sup\{st - \varphi(t) \mid t \geq 0\} \in \overline{\mathbb{R}}, \tag{4.2}$$

is lower semicontinuous, convex and satisfies $\varphi^{\#}(0) = 0$, $\varphi^{\#}(s) \geq 0$ for all $s \geq 0$.

For a function $\varphi \in N_0$, its greatest quasi-inverse $\varphi^h : \mathbb{R}_+ \to \overline{\mathbb{R}}_+$, defined by

$$\varphi^h(s) := \sup\{t \geq 0 \mid \varphi(t) \leq s\}, \tag{4.3}$$

is nondecreasing. It is known [18] that $\varphi^h \in N_0 \cap \Omega_0$ if $\varphi(t) = 0 \Leftrightarrow t = 0$.

For a function $\varphi : \mathbb{R} \to \overline{\mathbb{R}}$, its lower semicontinuous convex hull, denoted by $\overline{\text{co}}\varphi$, is defined by $\text{epi}(\overline{\text{co}}\varphi) = \text{cl}(\text{co}(\text{epi}\varphi))$. It is obvious that $\overline{\text{co}}\varphi$ is lower semicontinuous convex and $\overline{\text{co}}\varphi \leq \varphi$.

Proposition 4.1. *Let E be uniformly convex and uniformly smooth. Then, for every* $r > 0$, *there exists* $\sigma_r \in \Sigma_1$ *such that, for all* $x, y \in rU_E$,

$$\langle y - x, Jy - Jx\rangle \leq \sigma_r(\|Jy - Jx\|). \tag{4.4}$$

Proof. Since E is uniformly convex, $f(x) = (1/2)\|x\|^2$ is uniform convex on rU_E for all $r > 0$. Since the norm of E is Fréchet differentiable, its Fréchet derivative $\nabla f(x) = Jx$. In [18, Proposition 3.6.5] for f and $B = rU_E$, where r is an arbitrary positive real number, we get the function $\vartheta_r(\cdot) : [0,+\infty) \to [0,+\infty)$, defined by

$$\vartheta_r(t) := \inf\left\{\frac{1}{2}\|y\|^2 - \frac{1}{2}\|x\|^2 - \langle y - x, Jx\rangle : x \in rU_E, y \in E, \|y - x\| = t\right\}, \tag{4.5}$$

satisfies that $\vartheta_r(t) = 0$ if and only if $t = 0$, and $t^{-1}\vartheta_r(t)$ is nondecreasing. Thus,

$$\frac{1}{2}\|y\|^2 - \frac{1}{2}\|x\|^2 - \langle y - x, Jx\rangle \geq \vartheta_r(\|y - x\|) \tag{4.6}$$

for all $x \in rU_E$, $y \in E$ and hence

$$\frac{1}{2}\|y + x\|^2 \geq \frac{1}{2}\|x\|^2 + \langle y, Jx\rangle + \overline{\text{co}}\ \vartheta_r(\|y\|) \tag{4.7}$$

for all $x \in rU_E$, $y \in E$. It follows that

$$\langle x, Jx\rangle - \frac{1}{2}\|x\|^2 + \langle x, Jy - Jx\rangle + \langle y, Jy - Jx\rangle - \overline{\text{co}}\varphi\vartheta_r(\|y\|) \geq \langle y + x, Jy\rangle - \frac{1}{2}\|y + x\|^2 \tag{4.8}$$

for all $x \in rU_E$, $y \in E$. Since $\langle x, Jx\rangle = (1/2)\|x\|^2 + (1/2)\|x\|^2$, we have

$$\frac{1}{2}\|x\|^2 + \langle y + x, Jy - Jx\rangle - \overline{\text{co}}\vartheta_r(\|y\|) \geq \langle y + x, Jy\rangle - \frac{1}{2}\|y + x\|^2 \tag{4.9}$$

for all $x \in rU_E$ and $y \in E$.

Taking the supremum on both sides of (4.9) over $y \in E$, by [18, Lemma 3.3.1(v)] (if $f(x) := \varphi(\|x\|)$, where $\varphi \in N_0$, then $f^*(x^*) = \varphi^{\#}(\|x^*\|)$), we get that

$$\frac{1}{2}\|x\|^2 + \langle x, Jy - Jx\rangle + (\overline{\text{co}}\vartheta_r)^{\#}(\|Jy - Jx\|) \geq \frac{1}{2}\|y\|^2 \tag{4.10}$$

for all $x \in rU_E$ and $y \in E$. Since $\vartheta_r(t)$ is nondecreasing and $\lim_{t\to\infty} t^{-1}\vartheta_r(t) \geq \vartheta_r(1) > 0$, we have $\overline{\text{co}}\vartheta_r \in \Gamma_0$. It follows from [20, Lemma 3.3.1(iii)] that $(\overline{\text{co}}\vartheta_r)^{\#} \in \Sigma_1$.

Interchanging x and y in (4.10) for $x, y \in rU_E$, it also holds that

$$\frac{1}{2}\|y\|^2 + \langle y, Jx - Jy\rangle + (\overline{\text{co}}\vartheta_r)^{\#}(\|Jy - Jx\|) \geq \frac{1}{2}\|x\|^2. \tag{4.11}$$

Thus, by taking $\sigma_r := 2(\overline{\text{co}}\vartheta_r)^{\#} \in \Sigma_1$, and adding side by side (4.10) and (4.11), we obtained

$$\langle y - x, Jy - Jx\rangle \leq \sigma_r(\|Jy - Jx\|), \quad \forall x, y \in rU_E. \tag{4.12}$$

□

Theorem 4.2. *Let E be a uniformly convex and uniformly smooth Banach space. Suppose that $B \subset E^* \times E$ is maximal monotone with $B^{-1}0 = \{v^*\}$ and B^{-1} is Lipschitz continuous at 0 with modulus $l \geq 0$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in E$ and*

$$\begin{aligned} y_n &= \alpha_n x_n + (1-\alpha_n) J_{r_n} x_n, \\ x_{n+1} &= \beta_n x + (1-\beta_n) y_n, \end{aligned} \tag{4.13}$$

for every $n = 1, 2, \ldots$, where $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1]$, $\{r_n\} \subset (0,\infty)$ satisfying $\lim_{n\to\infty} r_n = \infty$. If either $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\limsup_{n\to\infty} \alpha_n < 1$ or $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$, then $\{x_n\}$ converges strongly to $v := J^{-1}v^*$ and $\phi(x_n, v)$ converges to 0.

Moreover, there exists an integer $N > 0$ such that

$$\phi(x_{n+1}, v) \leq \tau_n \phi(x, v) + \theta_n + \delta_n, \quad \forall n \geq N, \tag{4.14}$$

where $\tau_n = \beta_n + \sum_{i=N}^{n-1} \beta_i \prod_{j=i+1}^{n} (1-\beta_j)\alpha_j$, $\theta_n = \prod_{i=N}^{n} (1-\beta_i)\alpha_i$, $\delta_n = l((1-\alpha_n)/r_n + \sum_{i=N}^{n-1}((1-\alpha_i)/r_i)\prod_{j=i+1}^{n}(1-\beta_{j-1})\alpha_j)$ and $\lim_{n\to\infty}\tau_n = \lim_{n\to\infty}\theta_n = \lim_{n\to\infty}\delta_n = 0$. Also, one obtains

$$\phi(x_{n+1}, v) \leq \beta_n \phi(x, v) + \left(\alpha_n + \circ\left(\frac{l}{r_n}\right)\right) k(\phi(x_n, v)), \tag{4.15}$$

for all $n \geq N$, where $k(t) = \max\{t, \nu_r^h(t)\} \in N_0 \cap \Omega_0$, and $\nu_r(t) := \nu(rU_E, t)$, ν_r^h is the greatest quasi-inverse of $\nu_r(t)$, and r is a positive number such that $\{v\} \cup \{x_n\} \cup \{J_{r_n} x_n\} \subset rU_E$.

Proof. Put $v = J^{-1}v^*$. Since $B^{-1}0 = \{v^*\}$, we have $(BJ)^{-1}0 = \{v\}$. We separate the proof into two cases.

Case 1. $\sum_{n=1}^{\infty} \beta_n < \infty$, and $\limsup_{n\to\infty} \alpha_n < 1$.

According to Theorem 3.3, we have $\lim_{n\to\infty}\phi(x_n, v)$ exists, $\{x_n\}$ and $\{J_{r_n}\}$ are bounded, and hence $\{v\} \cup \{x_n\} \cup \{J_{r_n} x_n\} \subset rU_E$ for some $r > 0$. Since B^{-1} is Lipschitz continuous at 0 with modulus $l \geq 0$, for some $\tau > 0$, we have $\|z^* - v^*\| \leq l\|w\|$ whenever $z^* \in B^{-1}(w)$ and $\|w\| \leq \tau$. Since $r_n \to \infty$, we may assume $l/r_n < 1$ for all $n \geq 1$. From Theorem 3.3, we have $\|x_n - J_{r_n} x_n\| \to 0$ and $\|A_{r_n} x_n\| \to 0$ as $n \to \infty$. Hence, there exists an integer $N > 0$ such that $\|A_{r_n} x_n\| \leq \tau$ for all $n \geq N$. Since $J J_{r_n} x_n \in B^{-1} A_{r_n} x_n$, we have

$$\|J J_{r_n} x_n - v^*\| \leq l\|A_{r_n} x_n\|, \quad \forall n \geq N. \tag{4.16}$$

By $\|Jx_n - v^*\| \leq \|Jx_n - JJ_{r_n}x_n\| + \|JJ_{r_n}x_n - v^*\|$ for all $n \in \mathbb{N}$. Since J is uniformly continuous on each bounded set, (3.20), (3.21) and (4.16), we obtain $\lim_{n\to\infty}\|Jx_n - v^*\| = 0$. By the uniform smoothness of E^*, we have $\lim_{n\to\infty}\|x_n - v\| = \lim_{n\to\infty}\|J^{-1}Jx_n - J^{-1}v^*\| = 0$. Since $\phi(x_n, v) + \phi(v, x_n) = 2\langle x_n - v, Jx_n - Jv\rangle = 2\langle x_n - v, Jx_n - v^*\rangle$ for all $n \in \mathbb{N}$, we get

$$\phi(x_n, v) \leq 2\langle x_n - v, Jx_n - v^*\rangle \leq 2\|x_n - v\|\|Jx_n - v^*\| \longrightarrow 0, \qquad \text{as } n \longrightarrow \infty. \tag{4.17}$$

Hence, $\phi(x_n, v) \to 0$ as $n \to \infty$. It follows from Proposition 4.1 and (4.16) that there exists $\sigma(t) \in \Sigma_1$, which implies $\sigma(lt) \le l\sigma(t)$ for all $t \ge 0$ and $l \in [0,1]$, such that

$$\begin{aligned}\phi(J_{r_n}x_n, v) &\le \phi(J_{r_n}x_n, v) + \phi(v, J_{r_n}x_n)\\ &= 2\langle J_{r_n}x_n - v, J\,J_{r_n}x_n - Jv\rangle \le \sigma(\|J\,J_{r_n}x_n - Jv\|)\\ &= 2\sigma(\|J\,J_{r_n}x_n - v^*\|) \le 2\sigma(l\|A_{r_n}x_n\|)\\ &= 2\sigma\left(l\left\|\frac{x_n - J_{r_n}x_n}{r_n}\right\|\right) \le \frac{2l}{r_n}\sigma(\|x_n - J_{r_n}x_n\|)\end{aligned} \tag{4.18}$$

for all $n \ge N$. It follows from $\sigma(t) \in \Sigma_1$ and (3.20) that

$$\lim_{n\to\infty}\sigma(\|x_n - J_{r_n}x_n\|) = 0. \tag{4.19}$$

From (3.3), (3.2), and (4.18), we have

$$\begin{aligned}\phi(x_{n+1}, v) &\le \beta_n\phi(x, v) + (1-\beta_n)\{\alpha_n\phi(x_n, v) + (1-\alpha_n)\phi(J_{r_n}x_n, v)\}\\ &\le \beta_n\phi(x, v) + (1-\beta_n)\alpha_n\phi(x_n, v) + (1-\beta_n)(1-\alpha_n)\frac{2l}{r_n}\sigma(\|x_n - J_{r_n}x_n\|)\end{aligned} \tag{4.20}$$

for all $n \ge N$. Since $\phi(x_n, v) \to 0$ and (4.19), we may assume $\phi(x_n, v) \le 1$ and $2\sigma(\|x_n - J_{r_n}x_n\|) \le 1$ for all $n \ge N$. By (4.20) and induction, we obtain

$$\begin{aligned}\phi(x_{n+1}, v) &\le \left(\beta_n + \sum_{i=N}^{n-1}\beta_i\prod_{j=i+1}^{n}(1-\beta_j)\alpha_j\right)\phi(x, v) + \left(\prod_{i=N}^{n}(1-\beta_i)\alpha_i\right)\phi(x_N, v)\\ &\quad + (1-\beta_n)l\left(\frac{(1-\alpha_n)}{r_n} + \sum_{i=N}^{n-1}\frac{(1-\alpha_i)}{r_i}2\sigma(\|x_n - J_{r_n}x_n\|)\prod_{j=i+1}^{n}(1-\beta_{j-1})\alpha_j\right)\\ &\le \tau_n\phi(x, v) + \theta_n + \delta_n,\end{aligned} \tag{4.21}$$

for all $n \ge N$, where $\tau_n = \beta_n + \sum_{i=N}^{n-1}\beta_i\prod_{j=i+1}^{n}(1-\beta_j)\alpha_j$, $\theta_n = \prod_{i=N}^{n}(1-\beta_i)\alpha_i$, $\delta_n = l((1-\alpha_n)/r_n + \sum_{i=N}^{n-1}((1-\alpha_i)/r_i)\prod_{j=i+1}^{n}(1-\beta_{j-1})\alpha_j)$.

Next, we prove τ_n, θ_n and δ_n tend to 0. By $\alpha_n,\ \beta_n \in [0,1]$ and $\sum_{i=1}^{\infty}\beta_i < \infty$, we get

$$0 \le \tau_n = \beta_n + \sum_{i=N}^{n-1}\beta_i\prod_{j=i+1}^{n}(1-\beta_j)\alpha_j \le \beta_n + \sum_{i=N}^{n-1}\beta_i = \sum_{i=N}^{n}\beta_i \le \sum_{i=1}^{\infty}\beta_i. \tag{4.22}$$

Thus $\{\tau_n\}$ is bounded. Since $\alpha_n \in [0,1]$ and $\limsup_{n\to\infty}\alpha_n < 1$, there exists some α $(0 < \alpha < 1)$ such that $0 \le \alpha_n \le \alpha$ whenever $n \ge N$. Then, we get

$$\begin{aligned}
0 &\le \limsup_{n\to\infty} \tau_n \le \limsup_{n\to\infty} \beta_n + \limsup_{n\to\infty} \sum_{i=N}^{n-1} \beta_i \prod_{j=i+1}^{n} (1-\beta_j)\alpha_j \\
&= \lim_{n\to\infty} \beta_n + \limsup_{n\to\infty} (1-\beta_n)\alpha_n \left(\beta_{n-1} + \sum_{i=N}^{n-2} \beta_i \prod_{j=i+1}^{n-1} (1-\beta_j)\alpha_j \right) \\
&= \limsup_{n\to\infty} (1-\beta_n)\alpha_n \tau_{n-1} \le \limsup_{n\to\infty} \alpha_n \tau_{n-1} \le \alpha \limsup_{n\to\infty} \tau_{n-1} \\
&= \alpha \limsup_{n\to\infty} \tau_n,
\end{aligned} \tag{4.23}$$

which implies

$$\lim_{n\to\infty} \tau_n = \limsup_{n\to\infty} \tau_n = 0. \tag{4.24}$$

Meanwhile, we also have

$$0 \le \theta_n = \prod_{i=N}^{n} (1-\beta_i)\alpha_i \le \prod_{i=N}^{n} \alpha_i \le \alpha^{n-N+1} \longrightarrow 0. \tag{4.25}$$

On the other hand,

$$\begin{aligned}
\delta_n &= l \left(\frac{(1-\alpha_n)}{r_n} + \sum_{i=N}^{n-1} \frac{(1-\alpha_i)}{r_i} \prod_{j=i+1}^{n} (1-\beta_{j-1})\alpha_j \right) \\
&\le (1-\alpha_n)\frac{l}{r_n} + \alpha_n l \left(\frac{(1-\alpha_{n-1})}{r_{n-1}} + \sum_{i=N}^{n-2} \frac{(1-\alpha_i)}{r_i} \prod_{j=i+1}^{n-1} (1-\beta_{j-1})\alpha_j \right) \\
&= (1-\alpha_n)\frac{l}{r_n} + \alpha_n \delta_{n-1}.
\end{aligned} \tag{4.26}$$

Since $\limsup_{n\to\infty}\alpha_n < 1$, $\sum_{n=1}^{\infty}(1-\alpha_n) = +\infty$, it follows from Lemma 2.6 that $\delta_n \to 0$.

From $\sigma(t) \in \Sigma_1$, which implies that $\lim_{t\to 0^+}(\sigma(t)/t) = 0$, and (3.20). It follows that

$$\lim_{n\to\infty} \frac{\sigma(\|x_n - J_{r_n}x_n\|)}{\|x_n - J_{r_n}x_n\|} = 0. \tag{4.27}$$

By (4.18), we have

$$\begin{aligned}\phi(J_{r_n}x_n, v) &\leq \frac{l}{r_n}2\sigma(\|x_n - J_{r_n}x_n\|)\\ &= \frac{l}{r_n}\frac{2\sigma(\|x_n - J_{r_n}x_n\|)}{\|x_n - J_{r_n}x_n\|}\|x_n - J_{r_n}x_n\|\\ &= \circ\left(\frac{l}{r_n}\right)\|x_n - J_{r_n}x_n\|,\end{aligned} \tag{4.28}$$

for all $n \geq N$, where $\circ(\frac{l}{r_n}) := \frac{l}{r_n}\frac{2\sigma(\|x_n - J_{r_n}x_n\|)}{\|x_n - J_{r_n}x_n\|}$.

Since E is uniformly convex, $\|\cdot\|^2$ is uniformly totally convex on each bounded set of E. Denote by $\nu_r(t) := \nu(rU_E, t)$ the modulus of uniformly total convexity on the bounded set rU_E. Then $\nu_r(t) \in N_0$ and satisfies $\nu(\|x_n - J_{r_n}x_n\|) \leq \phi(x_n, J_{r_n}x_n)$. From the definition of the greatest quasi-inverse of ν , we deduce that

$$\|x_n - J_{r_n}x_n\| \leq \nu_r^h(\phi(x_n, J_{r_n}x_n)). \tag{4.29}$$

From Lemma 2.5, we have

$$\phi(x_n, J_{r_n}x_n) \leq \phi(x_n, v) - \phi(J_{r_n}x_n, v) \leq \phi(x_n, v). \tag{4.30}$$

Since $\nu_r^h \in N_0$, it holds by (4.28), (4.29), and (4.30) that

$$\phi(J_{r_n}x_n, v) \leq \circ\left(\frac{l}{r_n}\right)\|x_n - J_{r_n}x_n\| \leq \circ\left(\frac{l}{r_n}\right)\nu_r^h(\phi(x_n, v)), \tag{4.31}$$

for all $n \in \mathbb{N}$. Let $k(t) = \max\{t, \nu_r^h(t)\}$. Since $\nu_r^h \in N_0$ and $\nu_r(t) \geq 0$ for $t > 0$, it follows in [18, Lemma 3.3.1(i)] that $\nu_r^h(t) \in \Omega_0$ and this implies that $k(t) \in N_0 \cap \Omega_0$. By the first inequality of (4.20), (4.31) and the definition of $k(t)$, we have

$$\phi(x_{n+1}, v) \leq \beta_n\phi(x, v) + \left(\alpha_n + \circ\left(\frac{l}{r_n}\right)\right)k(\phi(x_n, v)), \quad \forall n \geq N. \tag{4.32}$$

Case 2. $\lim_{n\to\infty}\beta_n = 0$, $\sum_{n=1}^{\infty}\beta_n = \infty$, and $\lim_{n\to\infty}\alpha_n = 0$.

From the proof of Theorem 3.1, we note that if $(BJ)^{-1}0 = \{v\}$, then $\{\phi(x_n, v)\}$ converges to 0, $\{x_n\}$ converges strongly to v, and $\lim_{n\to\infty}\|x_n - J_{r_n}x_n\| = 0$. By the same argument as in the proof of Case 1, we obtain $\phi(x_{n+1}, v) \leq \tau_n\phi(x, v) + \theta_n + \delta_n$ for all $n \geq N$, where τ_n, θ_n and δ_n are those of Case 1.

It remains to show that $\{\tau_n\}$, $\{\theta_n\}$, and $\{\delta_n\}$ converge to 0. Since $\alpha_n, \beta_n \in [0,1]$ and $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\beta_n = 0$, it follows that

$$\begin{aligned}
0 \le \tau_n &= \beta_n + \sum_{i=N}^{n-1}\beta_i \prod_{j=i+1}^{n}(1-\beta_j)\alpha_j \\
&= \beta_n(1-\alpha_n) + \beta_n\alpha_n + \sum_{i=N}^{n-1}\beta_i \prod_{j=i+1}^{n}(1-\beta_j)\alpha_j \\
&\le \beta_n + \alpha_n\left(\beta_n + \sum_{i=N}^{n-1}\beta_i \prod_{j=i+1}^{n}(1-\beta_j)\right) \\
&= \beta_n + \alpha_n\left(1 - \prod_{i=N}^{n}(1-\beta_i)\right) \le \beta_n + \alpha_n \longrightarrow 0, \\
0 \le \theta_n &= \prod_{i=N}^{n}(1-\beta_i)\alpha_i \le \alpha_n \prod_{i=N}^{n}(1-\beta_i) \longrightarrow 0
\end{aligned} \tag{4.33}$$

whenever $n \ge N$ large enough.

On the other hand,

$$\delta_n = l\left(\frac{(1-\alpha_n)}{r_n} + \sum_{i=N}^{n-1}\frac{(1-\alpha_i)}{r_i}\prod_{j=i+1}^{n}(1-\beta_{j-1})\alpha_j\right) \le (1-\alpha_n)\frac{l}{r_n} + \alpha_n\delta_{n-1}. \tag{4.34}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}(1-\alpha_n) = +\infty$. It follows from Lemma 2.6 that $\delta_n \to 0$. Moreover, according to the proof of Case 1, we also have that, for all $n \ge N$,

$$\phi(x_{n+1}, v) \le \beta_n\phi(x,v) + \left(\alpha_n + \circ\left(\frac{l}{r_n}\right)\right)k(\phi(x_n, v)) \tag{4.35}$$

where $k(t)$ is the same as that of Case 1. Hence, the conclusion follows. □

If $\alpha_n \equiv 0$ for all $n \in \mathbb{N}$, then the algorithm (1.7) reduces to (1.5). Also, letting $\alpha_n = 0$ in (4.15) we obtain

$$\phi(x_{n+1}, v) \le \beta_n\phi(x,v) + \circ\left(\frac{l}{r_n}\right)k(\phi(x_n, v)), \quad \forall n \ge N. \tag{4.36}$$

Corollary 4.3 (see Li and Song [6]). *Let H be a Hilbert space and let $B \subset H \times H$ be a maximal monotone with $B^{-1}0 = \{v^*\}$ and B^{-1} is Lipschitz continuous at 0 with modulus $l \ge 0$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in H$ and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)J_{r_n}x_n, \tag{4.37}$$

for every $n = 1, 2, \ldots$, *where* $\{\alpha_n\} \subset [0,1]$, $\{r_n\} \subset (0,\infty)$ *satisfy* $\limsup_{n\to\infty}\alpha_n < 1$, *and* $\lim_{n\to\infty} r_n = \infty$. *Then the sequence* $\{x_n\}$ *converges strongly to* v. *Moreover, there exists an integer* $N > 0$ *such that*

$$\|x_{n+1} - v\| \le \sqrt{\theta_n + \delta_n} \quad \forall n \ge N, \tag{4.38}$$

where $\theta_n = \prod_{i=N}^{n}\alpha_i$, $\delta_n = l((1-\alpha_n)/r_n + \sum_{i=N}^{n-1}((1-\alpha_i)/r_i)\prod_{j=i+1}^{n}\alpha_j)$, *and* $\lim_{n\to\infty}\theta_n = \lim_{n\to\infty}\delta_n = 0$. *Meanwhile, one obtains the estimate, for all* $n \ge N$,

$$\|x_{n+1} - v\| \le \sqrt{\alpha_n + \circ\left(\frac{l}{r_n}\right)^2}\,\|x_n - v\|. \tag{4.39}$$

Proof. Note that, for $x, y \in H$, we have $J = I$ and $\phi(x,y) = \|x-y\|^2$. Under our assumptions, the iterative sequence (4.37) reduces to a special case of the algorithm (1.7) where $\beta_n = 0$. From the proof of Case 1 in Theorem 4.2, we know that $\{x_n\}$ converges strongly to v and there exists some $N > 0$ such that (4.20) and (4.21) hold for $n \ge N$.

Let $\beta_n = 0$ in the inequality (4.21). Then, $\tau_n = 0$. It follows from (4.21) that

$$\|x_{n+1} - v\| \le \sqrt{\theta_n + \delta_n}, \quad \forall n \ge N. \tag{4.40}$$

Since $\alpha_n \in [0,1]$ and $\limsup_{n\to\infty}\alpha_n < 1$, there exists some $\alpha (0 < \alpha < 1)$ such that $0 \le \alpha_n \le \alpha$ whenever $n \ge N$. Then, we get $\theta_n = \prod_{i=N}^{n}\alpha_i \le \alpha^{n-N+1} \to 0$. Also,

$$\begin{aligned}
\delta_n &= l\left(\frac{(1-\alpha_n)}{r_n} + \sum_{i=N}^{n-1}\frac{(1-\alpha_i)}{r_i}\prod_{j=i+1}^{n}\alpha_j\right) \\
&= (1-\alpha_n)\frac{l}{r_n} + \alpha_n l\left(\frac{(1-\alpha_{n-1})}{r_{n-1}} + \sum_{i=N}^{n-2}\frac{(1-\alpha_i)}{r_i}\prod_{j=i+1}^{n-1}\alpha_j\right) \\
&= (1-\alpha_n)\frac{l}{r_n} + \alpha_n\delta_{n-1}.
\end{aligned} \tag{4.41}$$

Since $\limsup_{n\to\infty}\alpha_n < 1$, $\sum_{n=1}^{\infty}(1-\alpha_n) = +\infty$. It follows from Lemma 2.6 that $\delta_n \to 0$. Additionally, from (4.20), (4.16), and (4.30), we have

$$\begin{aligned}
\|x_{n+1}-v\|^2 &\le \alpha_n\|x_n-v\|^2 + (1-\alpha_n)\|J_{r_n}x_n - v\|^2 \\
&\le \alpha_n\|x_n-v\|^2 + (1-\alpha_n)(l\|A_{r_n}x_n\|)^2 \\
&= \alpha_n\|x_n-v\|^2 + (1-\alpha_n)l^2\left\|\frac{x_n - J_{r_n}x_n}{r_n}\right\|^2 \\
&\le \alpha_n\|x_n-v\|^2 + (1-\alpha_n)\left(\frac{l}{r_n}\right)^2\|x_n-v\|^2 \\
&\le \left(\alpha_n + \left(\frac{l}{r_n}\right)^2\right)\|x_n-v\|^2,
\end{aligned} \tag{4.42}$$

for all $n \ge N$, which implies the equality (4.39). □

5. Applications

In this section, we study the problem of finding a minimizer of a proper lower semicontinuous convex function in a Banach space.

Theorem 5.1. *Let E be a uniformly convex and uniformly smooth Banach space and let $f^* : E^* \to (-\infty,\infty]$ be a proper lower semicontinuous convex function such that $(\partial f^*)^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence defined as follows: $x_1 = x \in E$ and*

$$\begin{aligned}
z_n^* &= \arg\min_{y^*\in E^*}\left\{f^*(y^*) + \frac{1}{2r_n}\|y^*\|^2 - \frac{1}{r_n}\langle x_n, y^*\rangle\right\}; \\
y_n^* &= \alpha_n x_n + (1-\alpha_n)J^{-1}z_n^*; \\
x_{n+1} &= \beta_n x + (1-\beta_n)y_n^*,
\end{aligned} \tag{5.1}$$

for every $n = 1,2,\ldots$, where $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1]$, and $\{r_n\} \subset (0,\infty)$ satisfy $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\beta_n = 0$, $\sum_{n=1}^{\infty}\beta_n = \infty$, and $\lim_{n\to\infty}r_n = \infty$. Then the sequence $\{x_n\}$ converges strongly to $R_{(\partial f^* J)^{-1}0}(x)$, where $R_{(\partial f^* J)^{-1}0}$ is a sunny generalized nonexpansive retraction of E onto $(\partial f^* J)^{-1}0$.

Proof. By Rockafellar's theorem [19, 20], the subdifferential mapping $\partial f^* \subset E^* \times E$ is maximal monotone. Let $J_r = (I + r\partial f J)^{-1}$ for all $r > 0$. As in the proof of [7, Corollary 5.1], we have $J^{-1}z_n^* = J_{r_n}x_n$ for all $n \in \mathbb{N}$. Hence, by Theorem 3.1, $\{x_n\}$ converges strongly to $R_{(\partial f^* J)^{-1}0}(x)$. □

When $\alpha_n = 0$ in Theorem 5.1 we obtain the following corollary.

Corollary 5.2 (see Ibaraki and Takahashi [7]). *Let E be a uniformly convex and uniformly smooth Banach space and let $f^* : E^* \to (-\infty, \infty]$ be a proper lower semicontinuous convex function such that $(\partial f^*)^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence defined as follows: $x_1 = x \in E$ and*

$$\begin{aligned} y_n^* &= \arg\min_{y^* \in E^*} \left\{ f^*(y^*) + \frac{1}{2r_n}\|y^*\|^2 - \frac{1}{r_n}\langle x_n, y^* \rangle \right\}; \\ x_{n+1} &= \beta_n x + (1-\beta_n) J^{-1} y_n^*, \end{aligned} \tag{5.2}$$

for every $n = 1, 2, \ldots$, where $\{\beta_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy $\lim_{n\to\infty}\beta_n = 0$, $\sum_{n=1}^{\infty}\beta_n = \infty$, and $\lim_{n\to\infty} r_n = \infty$. Then the sequence $\{x_n\}$ converges strongly to $R_{(\partial f^* J)^{-1}0}(x)$, where $R_{(\partial f^* J)^{-1}0}$ is a sunny generalized nonexpansive retraction of E onto $(\partial f^* J)^{-1}0$.

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