Research Article

Some Identities of Symmetry for the Generalized Bernoulli Numbers and Polynomials

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By the properties of $p$-adic invariant integral on $\mathbb{Z}_p$, we establish various identities concerning the generalized Bernoulli numbers and polynomials. From the symmetric properties of $p$-adic invariant integral on $\mathbb{Z}_p$, we give some interesting relationship between the power sums and the generalized Bernoulli polynomials.

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1. Introduction

Let $p$ be a fixed prime number. Throughout this paper, the symbols $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of rational integers, the ring of $p$-adic integers, the field of $p$-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $v_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$. Let $\text{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in \text{UD}(\mathbb{Z}_p)$, the $p$-adic invariant integral on $\mathbb{Z}_p$ is defined as

$$I(f) = \int_{\mathbb{Z}_p} f(x)dx = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \quad (1.1)$$

(see [1]). From the definition (1.1), we have

$$I(f_1) = I(f) + f'(0), \quad \text{where } f'(0) = \frac{df(x)}{dx} \bigg|_{x=0}, \ f_1(x) = f(x + 1). \quad (1.2)$$
Let \( f_n(x) = f(x + n), \) \( (n \in \mathbb{N}) \). Then we can derive the following equation from (1.2):

\[
I(f_n) = I(f) + \sum_{i=0}^{n-1} f'(i),
\]

(1.3)

(see [1]). It is well known that the ordinary Bernoulli polynomials \( B_n(x) \) are defined as

\[
\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!},
\]

(1.4)

(see [1–25]), and the Bernoulli number \( B_n \) are defined as \( B_n = B_n(0) \).

Let \( d \) be a fixed positive integer. For \( n \in \mathbb{N} \), we set

\[
X = X_d = \lim_{N \to \infty} \left( \mathbb{Z}/dp^N \mathbb{Z} \right), \quad X_1 = \mathbb{Z}_p;
\]

\[
X^* = \bigcup_{(\alpha, p) = 1} \left( \alpha + dp\mathbb{Z}_p \right);
\]

\[
a + dp^N \mathbb{Z}_p = \left\{ x \in X \mid x \equiv a \pmod{dp^N} \right\},
\]

(1.5)

where \( a \in \mathbb{Z} \) lies in \( 0 \leq a < dp^N \). It is easy to see that

\[
\int_X f(x)dx = \int_{\mathbb{Z}_p} f(x)dx, \quad \text{for } f \in \text{UD}(\mathbb{Z}_p).
\]

(1.6)

In [14], the Witt’s formula for the Bernoulli numbers are given by

\[
\int_{\mathbb{Z}_p} x^n dx = B_n, \quad n \in \mathbb{Z}_+.
\]

(1.7)

Let \( \chi \) be the Dirichlet’s character with conductor \( d \in \mathbb{N} \). Then the generalized Bernoulli polynomials attached to \( \chi \) are defined as

\[
\sum_{a=1}^{d} \chi(a) \frac{t e^{at}}{e^{at} - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n, \chi}(x) \frac{t^n}{n!},
\]

(1.8)

(see [22]), and the generalized Bernoulli numbers attached to \( \chi \), \( B_{n, \chi} \) are defined as \( B_{n, \chi} = B_{n, \chi}(0) \).

In this paper, we investigate the interesting identities of symmetry for the generalized Bernoulli numbers and polynomials attached to \( \chi \) by using the properties of \( p \)-adic invariant integral on \( \mathbb{Z}_p \). Finally, we will give relationship between the power sum polynomials and the generalized Bernoulli numbers attached to \( \chi \).
2. Symmetry of Power Sum and the Generalized Bernoulli Polynomials

Let \( \chi \) be the Dirichlet character with conductor \( d \in \mathbb{N} \). From (1.3), we note that

\[
\int_{X} \chi(x)e^{xt}dx = \frac{t\sum_{i=0}^{d-1} \chi(i)e^{it}}{e^{dt} - 1} = \sum_{n=0}^{\infty} B_{n, \chi} \frac{t^n}{n!},
\]

(2.1)

where \( B_{n, \chi}(x) \) are the \( n \)th generalized Bernoulli numbers attached to \( \chi \). Now, we also see that the generalized Bernoulli polynomials attached to \( \chi \) are given by

\[
\int_{X} \chi(y)e^{(x+y)t}dy = \frac{t\sum_{i=0}^{d-1} \chi(i)e^{it}}{e^{dt} - 1}e^{xt} = \sum_{n=0}^{\infty} B_{n, \chi}(x) \frac{t^n}{n!}.
\]

(2.2)

By (2.1) and (2.2), we easily see that

\[
\int_{X} \chi(x)x^n dx = B_{n, \chi}, \quad \int_{X} \chi(y)(x+y)^n dy = B_{n, \chi}(x).
\]

(2.3)

From (2.2), we have

\[
B_{n, \chi}(x) = \sum_{\ell=0}^{n} \binom{n}{\ell} B_{\ell, \chi} x^{n-\ell}.
\]

(2.4)

From (2.2), we can also derive

\[
\int_{X} \chi(x)e^{xt}dx = \sum_{i=0}^{d-1} \chi(i) \frac{t}{e^{dt} - 1}e^{(i/d)t} = \sum_{n=0}^{\infty} \left( d^{n-1} \sum_{i=0}^{d-1} \chi(i)B_{n}\left(\frac{i}{d}\right)\right) \frac{t^n}{n!}.
\]

(2.5)

Therefore, we obtain the following lemma.

**Lemma 2.1.** For \( n \in \mathbb{Z}_+ \), one has

\[
\int_{X} \chi(x)x^n dx = B_{n, \chi} = d^{n-1} \sum_{i=0}^{d-1} \chi(i)B_{i}\left(\frac{i}{d}\right).
\]

(2.6)

We observe that

\[
\frac{1}{t}\left( \int_{X} \chi(x)e^{(nd+\ell)t}dx - \int_{X} e^{xt}\chi(x)dx \right) = \frac{nd\int_{X} \chi(x)e^{xt}dx}{\int_{X} e^{ndxt}dx} = \frac{e^{ntd} - 1}{e^{dt} - 1} \left( \sum_{i=0}^{d-1} \chi(i)e^{it} \right).
\]

(2.7)

Thus, we have

\[
\frac{1}{t}\left( \int_{X} \chi(x)e^{(nd+\ell)t}dx - \int_{X} \chi(x)dx \right) = \sum_{k=0}^{\infty} \left( \sum_{\ell=0}^{d-1} \chi(\ell)e^{\ell k} \right) \frac{t^n}{k!}.
\]

(2.8)
Let us define the $p$-adic functional $T_k(\chi, n)$ as follows:

$$T_k(\chi, n) = \sum_{\ell=0}^{n} \chi(\ell)e^{k}, \quad \text{for } k \in \mathbb{Z}_+.$$  (2.9)

By (2.8) and (2.9), we see that

$$\frac{1}{t} \left( \int_{\chi} x \chi(e^{(nd+x)t})dx - \int_{\chi} x \chi(e^{x}dx) \right) = \frac{\sum_{m=0}^{\infty} (T_k(\chi, nd - 1))^{t^k}}{k!}. \quad \text{(2.10)}$$

By using Taylor expansion in (2.10), we have

$$\int_{\chi} x \chi(dn + x)^kdx - \int_{\chi} x \chi(x)^kdx = kT_{k-1}(\chi, nd - 1), \quad \text{for } k, n, d \in \mathbb{N}.$$  (2.11)

That is,

$$B_{k,\chi}(nd) - B_{k,\chi} = kT_{k-1}(\chi, nd - 1).$$  (2.12)

Let $w_1, w_2, d \in \mathbb{N}$. Then we consider the following integral equation:

$$\frac{d\int_{\chi} x \chi(x_1) \chi(x_2)e^{(w_1x_1 + w_2x_2)t}dx_1dx_2}{\int_{\chi} e^{(w_1x_1 + w_2x_2)t}dx} = \frac{t(e^{w_1t} - 1)}{(e^{w_1t} - 1)(e^{w_2t} - 1)} \left( \sum_{a=0}^{d-1} \chi(a)e^{w_1at} \right) \left( \sum_{b=0}^{d-1} \chi(b)e^{w_2bt} \right). \quad \text{(2.13)}$$

From (2.7) and (2.10), we note that

$$\frac{d\int_{\chi} x \chi(x)e^{xt}dx}{\int_{\chi} e^{xt}dx} = \sum_{k=0}^{\infty} (T_k(\chi, dw_1 - 1))^{t^k}.$$  (2.14)

Let us consider the $p$-adic functional $T_{\chi}(w_1, w_2)$ as follows:

$$T_{\chi}(w_1, w_2) = \frac{d\int_{\chi} x \chi(x_1) \chi(x_2)e^{(w_1x_1 + w_2x_2)t}dx_1dx_2}{\int_{\chi} e^{(w_1x_1 + w_2x_2+t)dx_3}.}.$$  (2.15)

Then we see that $T_{\chi}(w_1, w_2)$ is symmetric in $w_1$ and $w_2$, and

$$T_{\chi}(w_1, w_2) = \frac{t(e^{w_1t} - 1)}{(e^{w_1t} - 1)(e^{w_2t} - 1)} \left( \sum_{a=0}^{d-1} \chi(a)e^{w_1at} \right) \left( \sum_{b=0}^{d-1} \chi(b)e^{w_2bt} \right). \quad \text{(2.16)}$$
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By (2.15) and (2.16), we have

\[
T_x(w_1, w_2) = \left( \frac{1}{w_1} \int_x x(x_1) e^{w_1(x_1 + w_2x)} dx_1 \right) \left( \frac{d w_2 \int_x x(x_2) e^{w_2 x_2} dx_2}{\int e^{d w_2 x_2} dx_2} \right)
\]

\[
= \left( \frac{1}{w_1} \sum_{i=0}^{\infty} B_{i,\chi}(w_2x) \frac{w_1^i}{i!} \right) \left( \sum_{k=0}^{\infty} T_k(\chi, dw_1 - 1) \frac{w_2^k}{k!} \right)
\]

\[
= \frac{1}{w_1} \left( \sum_{i=0}^{\infty} \left( \sum_{l=0}^{i} \frac{w_1^l}{l!} \right) T_{\ell - i}(\chi, dw_1 - 1) w_2^{\ell - i} \right) \frac{t^\ell}{\ell!}.
\]  \hspace{1cm} (2.17)

From the symmetric property of \( T_x(w_1, w_2) \) in \( w_1 \) and \( w_2 \), we note that

\[
T_x(w_1, w_2) = \left( \frac{1}{w_2} \int_x x(x_2) e^{w_2(x_2 + w_1x)} dx_2 \right) \left( \frac{d w_1 \int_x x(x_1) e^{w_1 x_1} dx_1}{\int e^{d w_1 x_1} dx_1} \right)
\]

\[
= \left( \frac{1}{w_2} \sum_{i=0}^{\infty} B_{i,\chi}(w_1x) \frac{w_2^i}{i!} \right) \left( \sum_{k=0}^{\infty} T_k(\chi, dw_2 - 1) \frac{w_1^k}{k!} \right)
\]

\[
= \frac{1}{w_2} \left( \sum_{i=0}^{\infty} \left( \sum_{l=0}^{i} \frac{w_2^l}{l!} \right) T_{\ell - i}(\chi, dw_2 - 1) w_1^{\ell - i} \right) \frac{t^\ell}{\ell!}.
\]  \hspace{1cm} (2.18)

By comparing the coefficients on the both sides of (2.17) and (2.18), we obtain the following theorem.

**Theorem 2.2.** For \( w_1, w_2, d \in \mathbb{N} \), one has

\[
\sum_{i=0}^{\ell} \binom{\ell}{i} B_{i,\chi}(w_2x) T_{\ell - i}(\chi, dw_1 - 1) w_1^{i-1} w_2^{\ell - i} = \sum_{i=0}^{\ell} \binom{\ell}{i} B_{i,\chi}(w_1x) T_{\ell - i}(\chi, dw_2 - 1) w_2^{i-1} w_1^{\ell - i}.
\]  \hspace{1cm} (2.19)

Let \( x = 0 \) in Theorem 2.2. Then we have

\[
\sum_{i=0}^{\ell} \binom{\ell}{i} B_{i,\chi} T_{\ell - i}(\chi, dw_1 - 1) w_1^{i-1} w_2^{\ell - i} = \sum_{i=0}^{\ell} \binom{\ell}{i} B_{i,\chi} T_{\ell - i}(\chi, dw_2 - 1) w_2^{i-1} w_1^{\ell - i}.
\]  \hspace{1cm} (2.20)
By (2.14) and (2.16), we also see that

\[
T_x(w_1, w_2) = \left( \frac{e^{iw_1 x_1 t}}{w_1} \int_x \chi(x_1) e^{iw_1 x_1 t} dx_1 \right) \left( \frac{dw_1}{w_1} \int_x \chi(x_2) e^{iw_2 x_2 t} dx_2 \right)
\]

\[
= \left( \frac{e^{iw_1 x_1 t}}{w_1} \int_x \chi(x_1) e^{iw_1 x_1 t} dx_1 \right) \left( \frac{dw_1}{w_1} \int_x \chi(x_2) e^{iw_2 x_2 t} dx_2 \right)
\]

\[
= \left( \frac{e^{iw_1 x_1 t}}{w_1} \int_x \chi(x_1) e^{iw_1 x_1 t} dx_1 \right) \left( \frac{e^{dw_2 x_2 t} - 1}{e^{w_2 x_2 t} - 1} \right) \left( \sum_{i=0}^{d-1} \chi(i) e^{iw_2 i} \right)
\]

\[
= \left( \frac{e^{iw_1 x_1 t}}{w_1} \int_x \chi(x_1) e^{iw_1 x_1 t} dx_1 \right) \left( \sum_{i=0}^{w_2-1} e^{iw_2 i} \chi(i) \right)
\]

\[
= \frac{1}{w_1} \sum_{i=0}^{dw_2-1} \chi(i) \int_x \chi(x_1) e^{iw_1 x_1 t} dx_1
\]

\[
= \frac{1}{w_1} \sum_{i=0}^{dw_2-1} \chi(i) \sum_{k=0}^{\infty} B_{k,i} \left( \frac{w_2 x + w_2 i}{w_1} \right) \frac{w_1^k k!}{k!}
\]

\[
= \sum_{k=0}^{\infty} \left\{ \sum_{i=0}^{dw_2-1} \chi(i) B_{k,i} \left( \frac{w_2 x + w_2 i}{w_1} \right) w_1^{k-1} \right\} \frac{t^k}{k!}
\]

From the symmetric property of \(T_x(w_1, w_2)\) in \(w_1\) and \(w_2\), we can also derive the following equation:

\[
T_x(w_1, w_2) = \left( \frac{e^{iw_2 x_2 t}}{w_2} \int_x \chi(x_2) e^{iw_2 x_2 t} dx_2 \right) \left( \frac{dw_1}{w_1} \int_x \chi(x_1) e^{iw_1 x_1 t} dx_1 \right)
\]

\[
= \left( \frac{e^{iw_2 x_2 t}}{w_2} \int_x \chi(x_2) e^{iw_2 x_2 t} dx_2 \right) \left( \frac{dw_1}{w_1} \int_x \chi(x_1) e^{iw_1 x_1 t} dx_1 \right)
\]

\[
= \left( \frac{e^{iw_2 x_2 t}}{w_2} \int_x \chi(x_2) e^{iw_2 x_2 t} dx_2 \right) \left( \frac{e^{dw_1 x_1 t} - 1}{e^{w_1 x_1 t} - 1} \right) \left( \sum_{i=0}^{d-1} \chi(i) e^{iw_1 i} \right)
\]

\[
= \left( \frac{e^{iw_2 x_2 t}}{w_2} \int_x \chi(x_2) e^{iw_2 x_2 t} dx_2 \right) \left( \sum_{i=0}^{w_1-1} e^{iw_1 i} \chi(i) \right)
\]

\[
= \frac{1}{w_2} \sum_{i=0}^{dw_1-1} \chi(i) \int_x \chi(x_2) e^{iw_2 x_2 t} dx_2
\]

\[
= \frac{1}{w_2} \sum_{i=0}^{dw_1-1} \chi(i) \sum_{k=0}^{\infty} B_{k,i} \left( \frac{w_1 x + w_1 i}{w_2} \right) \frac{w_2^k k!}{k!}
\]

\[
= \sum_{k=0}^{\infty} \left\{ \sum_{i=0}^{dw_1-1} \chi(i) B_{k,i} \left( \frac{w_1 x + w_1 i}{w_2} \right) w_2^{k-1} \right\} \frac{t^k}{k!}
\]

By comparing the coefficients on the both sides of (2.21) and (2.22), we obtain the following theorem.
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Theorem 2.3. For $w_1, w_2, d \in \mathbb{N}$, one has

$$
\sum_{i=0}^{d w_1 - 1} \chi(i) B_{k,x} \left( \frac{w_2 x + \frac{w_2}{w_1} i}{w_1} \right) w_1^{k-1} = \sum_{i=0}^{d w_2 - 1} \chi(i) B_{k,x} \left( \frac{w_1 x + \frac{w_1}{w_2} i}{w_2} \right) w_2^{k-1}.
$$

(2.23)

Remark 2.4. Let $x = 0$ in Theorem 2.3. Then we see that

$$
\sum_{i=0}^{d w_1 - 1} \chi(i) B_{k,x} \left( \frac{w_2}{w_1} i \right) w_1^{k-1} = \sum_{i=0}^{d w_2 - 1} \chi(i) B_{k,x} \left( \frac{w_1}{w_2} i \right) w_2^{k-1}.
$$

(2.24)

If we take $w_2 = 1$, then we have

$$
\sum_{i=0}^{d w_1 - 1} \chi(i) B_{k,x} \left( \frac{i}{w_1} \right) w_1^{k-1} = \sum_{i=0}^{d w_2 - 1} \chi(i) B_{k,x} \left( w_1 i \right).
$$

(2.25)

Remark 2.5. Let $\chi$ be trivial character. Then we can easily derive the “multiplication theorem for Bernoulli polynomials” from Theorems 2.2 and 2.3 (see [14]).

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References


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