Research Article

The Stability of a Quadratic Functional Equation with the Fixed Point Alternative

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Lee, An and Park introduced the quadratic functional equation \( f(2x+y) + f(2x-y) = 8f(x) + 2f(y) \) and proved the stability of the quadratic functional equation in the spirit of Hyers, Ulam and Th. M. Rassias. Using the fixed point method, we prove the generalized Hyers-Ulam stability of the quadratic functional equation in Banach spaces.

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1. Introduction


**Theorem 1.1** (Th. M. Rassias). Let \( f : E \to E' \) be a mapping from a normed vector space \( E \) into a Banach space \( E' \) subject to the inequality

\[
\| f(x + y) - f(x) - f(y) \| \leq \epsilon (\| x \|^p + \| y \|^p) \tag{1.1}
\]

for all \( x, y \in E \), where \( \epsilon \) and \( p \) are constants with \( \epsilon > 0 \) and \( p < 1 \). Then the limit

\[
L(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n} \tag{1.2}
\]
exists for all \( x \in E \), and \( L : E \to E' \) is the unique additive mapping which satisfies

\[
\|f(x) - L(x)\| \leq \frac{2\varepsilon}{2 - 2p}\|x\|^p
\]  
(1.3)

for all \( x \in E \). Also, if for each \( x \in E \) the function \( f(tx) \) is continuous in \( t \in \mathbb{R} \), then \( L \) is \( \mathbb{R} \)-linear.

The above inequality (1.1) has provided a lot of influence in the development of what is now known as a generalized Hyers-Ulam stability of functional equations. Beginning around the year 1980, the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was studied by a number of mathematicians. Gavruta [5] generalized the Rassias’ result.

**Theorem 1.2** (see [6–8]). Let \( X \) be a real normed linear space and \( Y \) a real complete normed linear space. Assume that \( f : X \to Y \) is an approximately additive mapping for which there exist constants \( \theta \geq 0 \) and \( p \in \mathbb{R} - \{1\} \) such that \( f \) satisfies inequality

\[
\|f(x + y) - f(x) - f(y)\| \leq \theta \cdot \|x\|^{p/2} \cdot \|y\|^{p/2}
\]  
(1.4)

for all \( x, y \in X \). Then there exists a unique additive mapping \( L : X \to Y \) satisfying

\[
\|f(x) - L(x)\| \leq \frac{\theta}{2p - 2}\|x\|^p
\]  
(1.5)

for all \( x \in X \). If, in addition, \( f : X \to Y \) is a mapping such that the transformation \( t \to f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in X \), then \( L \) is an \( \mathbb{R} \)-linear mapping.

The functional equation

\[
f(x + y) + f(x - y) = 2f(x) + 2f(y)
\]  
(1.6)

is called a quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic function. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [9] for mappings \( f : X \to Y \), where \( X \) is a normed space and \( Y \) is a Banach space. Cholewa [10] noticed that the theorem of Skof is still true if the relevant domain \( X \) is replaced by an Abelian group. Czerwik [11] proved the generalized Hyers-Ulam stability of the quadratic functional equation. Several functional equations have been investigated in [12–25].

Let \( X \) be a set. A function \( d : X \times X \to [0, \infty) \) is called a generalized metric on \( X \) if \( d \) satisfies

1. \( d(x, y) = 0 \) if and only if \( x = y \);
2. \( d(x, y) = d(y, x) \) for all \( x, y \in X \);
3. \( d(x, z) \leq d(x, y) + d(y, z) \) for all \( x, y, z \in X \).

We recall a fundamental result in fixed point theory.
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Theorem 1.3 (see [26–28]). Let \((X, d)\) be a complete generalized metric space and let \(J : X \to X\) be a strictly contractive mapping with Lipschitz constant \(L < 1\). Then for each given element \(x \in X\), either

\[
d(J^n x, J^{n+1} x) = \infty
\]  

for all nonnegative integers \(n\) or there exists a positive integer \(n_0\) such that

(1) \(d(J^n x, J^{n+1} x) < \infty\), for all \(n \geq n_0\);

(2) the sequence \(\{J^n x\}\) converges to a fixed point \(y^*\) of \(J\);

(3) \(y^*\) is the unique fixed point of \(J\) in the set \(Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}\);

(4) \(d(y, y^*) \leq (1/(1 - L))d(y, Jy)\) for all \(y \in Y\).

Lee et al. [29] proved that a mapping \(f : X \to Y\) satisfies

\[
f(2x + y) + f(2x - y) = 8f(x) + 2f(y)
\]  

for all \(x, y \in X\) if and only if the mapping \(f : X \to Y\) satisfies

\[
f(x + y) + f(x - y) = 2f(x) + 2f(y)
\]  

for all \(x, y \in X\).

Using the fixed point method, Park [14] proved the generalized Hyers-Ulam stability of the quadratic functional equation

\[
f(2x + y) = 4f(x) + f(y) + f(x + y) - f(x - y)
\]  

in Banach spaces.

In this paper, using the fixed point method, we prove the generalized Hyers-Ulam stability of the quadratic functional equation (1.8) in Banach spaces.

Throughout this paper, assume that \(X\) is a normed vector space with norm \(|| \cdot ||\) and that \(Y\) is a Banach space with norm \(|| \cdot ||\).

### 2. Fixed Points and Generalized Hyers-Ulam Stability of a Quadratic Functional Equation

For a given mapping \(f : X \to Y\), we define

\[
Cf(x, y) := f(2x + y) + f(2x - y) - 8f(x) - 2f(y)
\]  

for all \(x, y \in X\).

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the quadratic functional equation \(Cf(x, y) = 0\).
Theorem 2.1. Let $f : X \to Y$ be a mapping for which there exists a function $\varphi : X^2 \to [0, \infty)$ with $f(0) = 0$ such that

$$\|Df(x, y)\| \leq \varphi(x, y)$$

for all $x, y \in X$. If there exists an $L < 1$ such that $\varphi(x, y) \leq 4L\varphi(x/2, y/2)$ for all $x, y \in X$, then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (1.8) and

$$\|f(x) - Q(x)\| \leq \frac{1}{8 - 8L}\varphi(x, 0)$$

for all $x \in X$.

Proof. Consider the set

$$S := \{ g : X \to Y \},$$

and introduce the generalized metric on $S$:

$$d(g, h) = \inf\{ K \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq K\varphi(x, 0), \ \forall x \in X \}.$$

It is easy to show that $(S, d)$ is complete.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g(2x)$$

for all $x \in X$.

By [30, Theorem 3.1],

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

Letting $y = 0$ in (2.2), we get

$$\|2f(2x) - 8f(x)\| \leq \varphi(x, 0)$$

for all $x \in X$. So

$$\left\| f(x) - \frac{1}{4}f(2x) \right\| \leq \frac{1}{8}\varphi(x, 0)$$

for all $x \in X$. Hence $d(f, Jf) \leq 1/8$. 
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By Theorem 1.3, there exists a mapping $Q : X \rightarrow Y$ such that

(1) $Q$ is a fixed point of $J$, that is,

$$Q(2x) = 4Q(x)$$  \hspace{1cm} (2.10)

for all $x \in X$. The mapping $Q$ is a unique fixed point of $J$ in the set

$$M = \{ g \in S : d(f, g) < \infty \}. $$  \hspace{1cm} (2.11)

This implies that $Q$ is a unique mapping satisfying (2.10) such that there exists $K \in (0, \infty)$ satisfying

$$\|f(x) - Q(x)\| \leq K \varphi(x, 0)$$  \hspace{1cm} (2.12)

for all $x \in X$.

(2) $d(J^n f, Q) \rightarrow 0$ as $n \rightarrow \infty$. This implies the equality

$$\lim_{n \rightarrow \infty} \frac{f(2^n x)}{4^n} = Q(x)$$  \hspace{1cm} (2.13)

for all $x \in X$.

(3) $d(f, Q) \leq (1/(1 - L))d(f, Jf)$, which implies the inequality

$$d(f, Q) \leq \frac{1}{8 - 8L}. $$  \hspace{1cm} (2.14)

This implies that the inequality (2.3) holds.

It follows from (2.2) and (2.13) that

$$\|CQ(x, y)\| = \lim_{n \rightarrow \infty} \frac{1}{4^n} \|Cf(2^n x, 2^n y)\| \leq \lim_{n \rightarrow \infty} \frac{1}{4^n} \varphi(2^n x, 2^n y) \leq \lim_{n \rightarrow \infty} L^n \varphi(x, y) = 0 $$  \hspace{1cm} (2.15)

for all $x, y \in X$. So $CQ(x, y) = 0$ for all $x, y \in X$.

By [29, Proposition 2.1], the mapping $Q : X \rightarrow Y$ is quadratic, as desired. \hfill $\Box$

**Corollary 2.2.** Let $0 < p < 2$ and $\theta$ be positive real numbers, and let $f : X \rightarrow Y$ be a mapping such that

$$\|Cf (x, y)\| \leq \theta(\|x\|^p + \|y\|^p) $$  \hspace{1cm} (2.16)

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ satisfying (1.8) and

$$\|f(x) - Q(x)\| \leq \frac{\theta}{8 - 2^{p+1}} \|x\|^p $$  \hspace{1cm} (2.17)

for all $x \in X$. 

Proof. The proof follows from Theorem 2.1 by taking
\[
\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)
\] (2.18)
for all \(x, y \in X\). Then \(L = 2^{p-2}\), and we get the desired result. \(\square\)

**Theorem 2.3.** Let \(f : X \to Y\) be a mapping for which there exists a function \(\varphi : X^2 \to [0, \infty)\) satisfying (2.2) and \(f(0) = 0\). If there exists an \(L < 1\) such that \(\varphi(x, y) \leq (L/4)\varphi(2x, 2y)\) for all \(x, y \in X\), then there exists a unique quadratic mapping \(Q : X \to Y\) satisfying (1.8) and
\[
\|f(x) - Q(x)\| \leq \frac{L}{8 - 8L}\varphi(x, 0)
\] (2.19)
for all \(x \in X\).

**Proof.** We consider the linear mapping \(J : S \to S\) such that
\[
Jg(x) := 4g\left(\frac{x}{2}\right)
\] (2.20)
for all \(x \in X\).

It follows from (2.8) that
\[
\|f(x) - 4f\left(\frac{x}{2}\right)\| \leq \frac{1}{2}\varphi\left(\frac{x}{2}, 0\right) \leq \frac{L}{8}\varphi(x, 0)
\] (2.21)
for all \(x \in X\). Hence \(d(f, Jf) \leq L/8\).

By Theorem 1.3, there exists a mapping \(Q : X \to Y\) such that
(1) \(Q\) is a fixed point of \(J\), that is,
\[
Q(2x) = 4Q(x)
\] (2.22)
for all \(x \in X\). The mapping \(Q\) is a unique fixed point of \(J\) in the set
\[
M = \{g \in S : d(f, g) < \infty\}.
\] (2.23)
This implies that \(Q\) is a unique mapping satisfying (2.22) such that there exists \(K \in (0, \infty)\) satisfying
\[
\|f(x) - Q(x)\| \leq K\varphi(x, 0)
\] (2.24)
for all \(x \in X\).

(2) \(d(J^nf, Q) \to 0\) as \(n \to \infty\). This implies the equality
\[
\lim_{n \to \infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)
\] (2.25)
for all \(x \in X\).
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(3) \(d(f, Q) \leq \frac{1}{1 - L} d(f, Jf)\), which implies the inequality

\[ d(f, Q) \leq \frac{L}{8 - 8L}, \]  

which implies that the inequality (2.19) holds.

The rest of the proof is similar to the proof of Theorem 2.1. \(\square\)

**Corollary 2.4.** Let \(p > 2\) and \(\theta\) be positive real numbers, and let \(f : X \to Y\) be a mapping satisfying (2.16). Then there exists a unique quadratic mapping \(Q : X \to Y\) satisfying (1.8) and

\[ \| f(x) - Q(x) \| \leq \frac{\theta}{2^{p+1} - 8} \| x \|^p \]  

for all \(x \in X\).

**Proof.** The proof follows from Theorem 2.3 by taking

\[ \varphi(x, y) := \theta(\|x\|^p + \|y\|^p) \]  

for all \(x, y \in X\). Then \(L = 2^{-p}\) and, we get the desired result. \(\square\)

**Theorem 2.5.** Let \(f : X \to Y\) be a mapping for which there exists a function \(\varphi : X^2 \to [0, \infty)\) satisfying (2.2). If there exists an \(L < 1\) such that \(\varphi(x, y) \leq 9L \varphi(x/3, y/3)\) for all \(x, y \in X\), then there exists a unique quadratic mapping \(Q : X \to Y\) satisfying (1.8) and

\[ \| f(x) - Q(x) \| \leq \frac{1}{9 - 9L} \varphi(x, x) \]  

for all \(x \in X\).

**Proof.** Consider the set

\[ S := \{ g : X \to Y \}, \]  

and introduce the **generalized metric** on \(S\):

\[ d(g, h) = \inf \{ K \in \mathbb{R}_+ : \| g(x) - h(x) \| \leq K \varphi(x, x), \ \forall x \in X \}. \]  

It is easy to show that \((S, d)\) is complete.

Now we consider the linear mapping \(J : S \to S\) such that

\[ Jg(x) := \frac{1}{9} g(3x) \]  

for all \(x \in X\).
By [30, Theorem 3.1],
\[ d(fg, fh) \leq Ld(g, h) \] (2.33)
for all \( g, h \in S \).

Letting \( y = x \) in (2.2), we get
\[ \| f(3x) - 9f(x) \| \leq \varphi(x, x) \] (2.34)
for all \( x \in X \). So
\[ \left\| f(x) - \frac{1}{9} f(3x) \right\| \leq \frac{1}{9} \varphi(x, x) \] (2.35)
for all \( x \in X \). Hence \( d(f, Jf) \leq 1/9 \).

By Theorem 1.3, there exists a mapping \( Q : X \rightarrow Y \) such that
(1) \( Q \) is a fixed point of \( J \), that is,
\[ Q(3x) = 9Q(x) \] (2.36)
for all \( x \in X \). The mapping \( Q \) is a unique fixed point of \( f \) in the set
\[ M = \{ g \in S : d(f, g) < \infty \}. \] (2.37)
This implies that \( Q \) is a unique mapping satisfying (2.36) such that there exists \( K \in (0, \infty) \) satisfying
\[ \| f(x) - Q(x) \| \leq K\varphi(x, x) \] (2.38)
for all \( x \in X \).
(2) \( d(J^n f, Q) \rightarrow 0 \) as \( n \rightarrow \infty \). This implies the equality
\[ \lim_{n \rightarrow \infty} \frac{f(3^n x)}{9^n} = Q(x) \] (2.39)
for all \( x \in X \).
(3) \( d(f, Q) \leq (1/(1 - L))d(f, Jf) \), which implies the inequality
\[ d(f, Q) \leq \frac{1}{9 - 9L}. \] (2.40)
This implies that the inequality (2.29) holds.

It follows from (2.2) and (2.39) that
\[ \| CQ(x, y) \| = \lim_{n \rightarrow \infty} \frac{1}{y^n} \| Cf(3^n x, 3^n y) \| \leq \lim_{n \rightarrow \infty} \frac{1}{y^n} \varphi(3^n x, 3^n y) \leq \lim_{n \rightarrow \infty} L^n \varphi(x, y) = 0 \] (2.41)
for all \( x, y \in X \). So \( CQ(x, y) = 0 \) for all \( x, y \in X \).

By [29, Proposition 2.1], the mapping \( Q : X \rightarrow Y \) is quadratic, as desired. \( \square \)
Corollary 2.6. Let $0 < p < 2$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying (2.16). Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (1.8) and

$$
\|f(x) - Q(x)\| \leq \frac{2\theta}{9 - 3^p} \|x\|^p
$$

(2.42)

for all $x \in X$.

Proof. The proof follows from Theorem 2.5 by taking

$$
\varphi(x, y) := \theta (\|x\|^p + \|y\|^p)
$$

(2.43)

for all $x, y \in X$. Then $L = 3^{p-2}$ and, we get the desired result. \qed

Corollary 2.7. Let $0 < p < 1$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be a mapping such that

$$
\|Df(x, y)\| \leq \theta \cdot \|x\|^p \cdot \|y\|^p
$$

(2.44)

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (1.8) and

$$
\|f(x) - Q(x)\| \leq \frac{\theta}{9 - 9^p} \|x\|^{2p}
$$

(2.45)

for all $x \in X$.

Proof. The proof follows from Theorem 2.5 by taking

$$
\varphi(x, y) := \theta \cdot \|x\|^p \cdot \|y\|^p
$$

(2.46)

for all $x, y \in X$. Then $L = 9^{p-1}$ and, we get the desired result. \qed

Theorem 2.8. Let $f : X \to Y$ be a mapping for which there exists a function $\varphi : X^2 \to [0, \infty)$ satisfying (2.2). If there exists an $L < 1$ such that $\varphi(x, y) \leq (L/9)\varphi(3x, 3y)$ for all $x, y \in X$, then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (1.8) and

$$
\|f(x) - Q(x)\| \leq \frac{L}{9 - 9L} \varphi(x, x)
$$

(2.47)

for all $x \in X$.

Proof. We consider the linear mapping $J : S \to S$ such that

$$
Jg(x) := 9g\left(\frac{x}{3}\right)
$$

(2.48)

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. \qed
Corollary 2.9. Let \( p > 2 \) and \( \theta \) be positive real numbers, and let \( f : X \to Y \) be a mapping satisfying (2.16). Then there exists a unique quadratic mapping \( Q : X \to Y \) satisfying (1.8) and
\[
\| f(x) - Q(x) \| \leq \frac{2\theta}{3^p - 9} \| x \|^p
\]
for all \( x \in X \).

Proof. The proof follows from Theorem 2.8 by taking
\[
\varphi(x, y) := \theta \left( \| x \|^p + \| y \|^p \right)
\]
for all \( x, y \in X \). Then \( L = 3^{2-p} \), and we get the desired result. \( \square \)

Corollary 2.10. Let \( p > 1 \) and \( \theta \) be positive real numbers, and let \( f : X \to Y \) be a mapping satisfying (2.44). Then there exists a unique quadratic mapping \( Q : X \to Y \) satisfying (1.8) and
\[
\| f(x) - Q(x) \| \leq \frac{\theta}{9^{p-9}} \| x \|^{2p}
\]
for all \( x \in X \).

Proof. The proof follows from Theorem 2.8 by taking
\[
\varphi(x, y) := \theta \cdot \| x \|^p \cdot \| y \|^p
\]
for all \( x, y \in X \). Then \( L = 9^{1-p} \), and we get the desired result. \( \square \)

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References


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