Research Article

On the Generalized Hyers-Ulam-Rassias Stability of Quadratic Functional Equations

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We achieve the general solution and the generalized Hyers-Ulam-Rassias and Ulam-Gavruta-Rassias stabilities for quadratic functional equations $f(ax + by) + f(ax - by) = (b(a + b)/2)f(x + y) + (b(a + b)/2)f(x - y) + (2a^2 - ab - b^2)f(x) + (b^2 - ab)f(y)$ where $a, b$ are nonzero fixed integers with $b \neq \pm a, -3a$, and $f(ax + by) + f(ax - by) = 2a^2 f(x) + 2b^2 f(y)$ for fixed integers $a, b$ with $a, b \neq 0$ and $a \pm b \neq 0$.

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1. Introduction

In 1940, Ulam [1] proposed the stability problem for functional equations in the following question regarding to the stability of group homomorphism.

Let $(G_1, \cdot)$ be a group and let $(G_2, \ast)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \rightarrow G_2$ satisfies the inequality $d(h(x \cdot y), h(x) \ast h(y)) < \delta$, for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \rightarrow G_2$ with $d(h(x), H(x)) < \varepsilon$, for all $x \in G_1$? In other words, under what conditions does a homomorphism exist near an approximately homomorphism? Generally, the concept of stability for a functional equation comes up when we the functional equation is replaced by an inequality which acts as a perturbation of that equation. Hyers [2] answered to the question affirmatively in 1941 so if $f : E \rightarrow E'$ such that

$$\|f(x + y) - f(x) - f(y)\| \leq \delta,$$  \hspace{1cm} (1.1)

for all $x, y \in E$, and for some $\delta > 0$ where $E, E'$ are Banach spaces; then there exists a unique additive mapping $T : E \rightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \delta,$$ \hspace{1cm} (1.2)
for all $x \in E$. However, if $f(tx)$ is a continuous mapping at $t \in \mathbb{R}$ for each fixed $x \in E$ then $T$ is linear. In 1950, Hyers’s theorem was generalized by Aoki [3] for additive mappings and independently, in 1978, by Rassias [4] for linear mappings considering the Cauchy difference controlled by sum of powers of norms. This stability phenomenon is called the Hyers-Ulam-Rassias stability.

On the other hand, Rassias [5–10] considered the Cauchy difference controlled by a product of different powers of norm. However, there was a singular case; for this singularity a counterexample was given by Gavruta [11]. This stability phenomenon is called the Ulam-Gavruta-Rassias stability (see also [12, 13]). In addition, J. M. Rassias considered the mixed product-sum of powers of norms control function [14]. This stability is called JMRassias mixed product-sum stability (see also [15–22]).

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y), \tag{1.3}$$

is related to symmetric biadditive function and is called a quadratic functional equation naturally, and every solution of the quadratic equation (1.3) is said to be a quadratic function. 

It is well known that a function $f$ between two real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function $B$ such that $f(x) = B(x, x)$ for all $x$ where

$$B(x, y) = \frac{1}{4}(f(x + y) - f(x - y)) \tag{1.4}$$

(see [23, 24]). Skof proved Hyers-Ulam-Rassias stability problem for quadratic functional equation (1.3) for a class of functions $f : A \to B$, where $A$ is normed space and $B$ is a Banach space, (see [25]). Cholewa [26] noticed that Skof’s theorem is still true if relevant domain $A$ alters to an abelian group. In 1992, Czerwik proved the Hyers-Ulam-Rassias stability of (1.3) (see [27]) and four years later,Grabiec [28] generalized the result mentioned above.

Throughout this paper, assume that $a, b$ are fixed integers with $a, b \neq 0$, we introduce the following functional equations, which are different from (1.3):

$$f(ax + by) + f(ax - by) = \frac{b(a + b)}{2}f(x + y) + \frac{b(a + b)}{2}f(x - y)$$

$$+ \left(2a^2 - ab - b^2\right)f(x) + \left(b^2 - ab\right)f(y), \tag{1.5}$$

where $b \neq \pm a, -3a$, and

$$f(ax + by) + f(ax - by) = 2a^2f(x) + 2b^2f(y), \tag{1.6}$$

where $b \neq \pm a$.

In this paper, we establish the general solution and the generalized Hyers-Ulam-Rassias and Ulam-Gavruta-Rassias stabilities problem for (1.5), (1.6) which are equivalent to (1.3).
2. Solution of (1.5), (1.6)

Let $X$ and $Y$ be real vector spaces. We here present the general solution of (1.5), (1.6).

**Theorem 2.1.** A function $f : X \to Y$ satisfies the functional equation (1.3) if and only if $f : X \to Y$ satisfies the functional equation (1.5). Therefore, every solution of functional equation (1.5) is also a quadratic function.

**Proof.** Let $f$ satisfy the functional equation (1.3). Putting $x = y = 0$ in (1.3), we get $f(0) = 0$. Set $x = 0$ in (1.3) to get $f(-y) = f(y)$. Letting $y = x$ and $y = 2x$ in (1.3), respectively, we obtain that $f(2x) = 4f(x)$ and $f(3x) = 9f(x)$ for all $x \in X$. By induction, we lead to $f(kx) = k^2 f(x)$ for all positive integers $k$. Replacing $x$ and $y$ by $2x + y$ and $2x - y$ in (1.3), respectively, gives

$$f(2x + y) + f(2x - y) = 8f(x) + 2f(y)$$

(2.1)

for all $x, y \in X$. Using (1.3) and (2.1), we lead to

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 4f(x) - 2f(y)$$

(2.2)

for all $x, y \in X$. Suppose that $k \neq 0$ is a fixed integer by using (1.3), we get

$$kf(x + y) + kf(x - y) - 2kf(x) - 2kf(y) = 0$$

(2.3)

for all $x, y \in X$. Using (2.2) and (2.3), we obtain

$$f(2x + y) + f(2x - y) = (2 + k)f(x + y) + (2 + k)f(x - y) + 2(2 - k)f(x) - 2(1 + k)f(y)$$

(2.4)

for all $x, y \in X$. Replacing $x$ and $y$ by $3x + y$ and $3x - y$ in (1.3), respectively, then using (1.3) and (2.3), we have

$$f(3x + y) + f(3x - y) = (3 + k)f(x + y) + (3 + k)f(x - y) + 2(6 - k)f(x) - 2(2 + k)f(y)$$

(2.5)

for all $x, y \in X$. By using the above method, by induction, we infer that

$$f(ax + y) + f(ax - y) = (a + k)f(x + y) + (a + k)f(x - y) + 2(a^2 - a - k)f(x) - 2(a + k - 1)f(y)$$

(2.6)
for all \(x, y \in X\) and each positive integer \(a \geq 1\). For a negative integer \(a \leq -1\), replacing \(a\) by \(-a\) one can easily prove the validity of (2.6). Therefore (1.3) implies (2.6) for any integer \(a \neq 0\). First, it is noted that (2.6) also implies the following equation

\[
f(bx + y) + f(bx - y) = (b + k)f(x + y) + (b + k)f(x - y) \\
+ 2\left(b^2 - b - k\right)f(x) - 2(b + k - 1)f(y)
\]  

(2.7)

for all integers \(b \neq 0\). Setting \(y = 0\) in (2.7) gives \(f(bx) = b^2 f(x)\). Substituting \(y\) with \(by\) into (2.7), one gets

\[
(b + k)f(x + by) + (b + k)f(x - by) = b^2 f(x + y) + b^2 f(x - y) \\
- 2\left(b^2 - b - k\right)f(x) + 2b^2(b + k - 1)f(by)
\]

(2.8)

for all \(x, y \in X\). Replacing \(y\) by \(by\) in (2.6), we observe that

\[
f(ax + by) + f(ax - by) = (a + k)f(x + by) + (a + k)f(x - by) \\
+ 2\left(a^2 - a - k\right)f(x) - 2(a + k - 1)f(by)
\]

(2.9)

for all \(x, y \in X\). Hence, according to (2.8) and (2.9), we get

\[
(b + k)f(ax + by) + (b + k)f(ax - by) = b^2(a + k)f(x + y) + b^2(a + k)f(x - y) \\
+ 2\left(a^2(b + k) - b^2(a + k)\right)f(x) - 2b^2(a - b)f(y)
\]

(2.10)

for all \(x, y \in X\). In particular, if we substitute \(k := b\) in (2.10) and dividing it by \(2b\), we conclude that \(f\) satisfies (1.5).

Let \(f\) satisfy the functional equation (1.5), for nonzero fixed integers \(a, b\) with \(b \neq \pm a, -3a\). Putting \(x = y = 0\) in (1.5), we get

\[
\left(2a^2 - ba + b^2 - 2\right)f(0) = 0,
\]

(2.11)

so

\[
\left(2a - \frac{b + \sqrt{16 - 7b^2}}{2}\right)\left(a - \frac{b - \sqrt{16 - 7b^2}}{4}\right)f(0) = 0,
\]

(2.12)
but since \( a, b \neq 0 \) and \( b \neq \pm a, -3a \), therefore \( f(0) = 0 \). Setting \( y = 0 \) in (1.5) gives \( f(ax) = a^2 f(x) \) for all \( x \in X \). Letting \( y = -y \) in (1.5), we get

\[
f(ax - by) + f(ax + by) = \frac{b(a+b)}{2} f(x - y) + \frac{b(a+b)}{2} f(x + y)
\]

\[
+ \left( 2a^2 - ab - b^2 \right) f(x) + \left( b^2 - ab \right) f(-y)
\]

(2.13) for all \( x, y \in X \). If we compare (1.5) with (2.13), then since \( a, b \neq 0 \) and \( b \neq \pm a, -3a \), we conclude that \( f(-y) = f(y) \) for all \( y \in X \). Letting \( x = 0 \) in (1.5) and using the evenness of \( f \) give \( f(by) = b^2 f(y) \) for all \( y \in X \). Therefore for all \( x \in X \), we get \( f(abx) = a^2 b^2 f(x) \). Replacing \( x \) and \( y \) by \( bx \) and \( ay \) in (1.5), respectively, we have

\[
a^2 b^2 f(x + y) + a^2 b^2 f(x - y) = \frac{b(a+b)}{2} f(bx + ay) + \frac{b(a+b)}{2} f(bx - ay)
\]

\[
+ b^2 \left( 2a^2 - ab - b^2 \right) f(x) + a^2 \left( b^2 - ab \right) f(y)
\]

(2.14) for all \( x, y \in X \). On the other hand, if we interchange \( x \) with \( y \) in (1.5), we obtain

\[
f(ay + bx) + f(ay - bx) = \frac{b(a+b)}{2} f(y + x) + \frac{b(a+b)}{2} f(y - x)
\]

\[
+ \left( 2a^2 - ab - b^2 \right) f(y) + \left( b^2 - ab \right) f(x)
\]

(2.15) for all \( x, y \in X \). But since \( f \) is even, it follows from (2.15) that

\[
f(bx + ay) + f(bx - ay) = \frac{b(a+b)}{2} f(x + y) + \frac{b(a+b)}{2} f(x - y)
\]

\[
+ \left( b^2 - ab \right) f(x) + \left( 2a^2 - ab - b^2 \right) f(y)
\]

(2.16) for all \( x, y \in X \). Hence, according to (2.14) and (2.16), we obtain that

\[
a^2 b^2 f(x + y) + a^2 b^2 f(x - y) = \frac{b(a+b)}{2} \left[ \frac{b(a+b)}{2} (f(x + y) + f(x - y)) \right]
\]

\[
+ \left( b^2 - ab \right) f(x) + \left( 2a^2 - ab - b^2 \right) f(y)
\]

\[
+ b^2 \left( 2a^2 - ab - b^2 \right) f(x) + a^2 \left( b^2 - ab \right) f(y)
\]

(2.17)
for all \(x, y \in X\). Therefore, \(f\) satisfies (1.3).

\[\]
Corollary 2.3 ([29, Proposition 2.1]). A function \( f : X \to Y \) satisfies the following functional equation:

\[
f(ax + y) + f(ax - y) = 2a^2 f(x) + 2f(y)
\]

(2.24)

for all \( x, y \in X \) if and only if \( f : X \to Y \) satisfies the functional equation (1.3) for all \( x, y \in X \).

Proof. Assume that \( b = 1 \) in functional equation (1.6) and apply Theorem 2.2. \( \square \)

3. Stability

We now investigate the generalized Hyers-Ulam-Rassias and Ulam-Gavruta-Rassias stabilities problem for functional equations (1.5), (1.6). From this point on, let \( X \) be a real vector space and let \( Y \) be a Banach space. Before taking up the main subject, we define the difference operator \( \Delta_f : X \times X \to Y \) by

\[
\Delta_f(x, y) = f(ax + by) + f(ax - by) - \frac{b(a + b)}{2} f(x + y) - \frac{b(a + b)}{2} f(x - y)
\]

(3.1)

for all \( x, y \in X \) and \( a, b \) fixed integers such that \( a, b \neq 0 \) and \( a \pm b \neq 0 \) where \( f : X \to Y \) is a given function.

Theorem 3.1. Let \( j \in \{-1, 1\} \) be fixed, and let \( \varphi : X \times X \to [0, \infty) \) be a function such that

\[
\tilde{\varphi}(x) := \sum_{i=(1-j)/2}^{\infty} \frac{1}{a^{2i}} \varphi(a^{i}x, 0) < \infty
\]

(3.2)

\[
\lim_{n \to \infty} \frac{1}{a^{2nj}} \varphi(a^{nj}x, a^{nj}y) = 0
\]

(3.3)

for all \( x, y \in X \). Suppose that \( f : X \to Y \) be a function satisfies

\[
\|\Delta_f(x, y)\| \leq \varphi(x, y)
\]

(3.4)

for all \( x, y \in X \). Furthermore, assume that \( f(0) = 0 \) in (3.4) for the case \( j = 1 \). Then there exists a unique quadratic function \( Q : X \to Y \) such that

\[
\|f(x) - Q(x)\| \leq \frac{1}{2a^{1+j}} \tilde{\varphi}\left(\frac{x}{a^{(1-j)/2}}\right),
\]

(3.5)

for all \( x \in X \).
Proof. For \( j = 1 \), putting \( y = 0 \) in (3.4), we have

\[
\|2f(ax) - 2ax^2 f(x)\| \leq \varphi(x, 0) \tag{3.6}
\]

for all \( x \in X \). So

\[
\left\| f(x) - \frac{1}{a^2} f(ax) \right\| \leq \frac{1}{2a^2} \varphi(x, 0) \tag{3.7}
\]

for all \( x \in X \). Replacing \( x \) by \( ax \) in (3.7) and dividing by \( a^2 \) and summing the resulting inequality with (3.7), we get

\[
\left\| f(x) - \frac{1}{a^4} f\left(a^2 x\right)\right\| \leq \frac{1}{2a^2} \left( \varphi(x, 0) + \frac{\varphi(ax, o)}{a^2} \right) \tag{3.8}
\]

for all \( x \in X \). Hence

\[
\left\| \frac{1}{a^2k} f\left(a^k x\right) - \frac{1}{a^{2m}} f\left(a^m x\right)\right\| \leq \frac{1}{2a^2} \sum_{i=k}^{m-1} \frac{1}{a^{2i}} \varphi\left(a^i x, 0\right) \tag{3.9}
\]

for all nonnegative integers \( m \) and \( k \) with \( m > k \) and for all \( x \in X \). It follows from (3.2) and (3.9) that the sequence \( \{(1/a^{2n}) f(a^n x)\} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{(1/a^{2n}) f(a^n x)\} \) converges. So one can define the function \( Q : X \rightarrow Y \) by

\[
Q(x) := \lim_{n \to \infty} \frac{1}{a^{2n}} f(a^n x) \tag{3.10}
\]

for all \( x \in X \). By (3.3) for \( j = 1 \) and (3.4),

\[
\| \Delta_Q(x, y) \| = \lim_{n \to \infty} \frac{1}{a^{2n}} \| \Delta_j(a^n x, a^n y) \| \leq \lim_{n \to \infty} \frac{1}{a^{2n}} \varphi(a^n x, a^n y) = 0 \tag{3.11}
\]

for all \( x, y \in X \). So \( \Delta_Q(x, y) = 0 \). By Theorem 2.1, the function \( Q : X \rightarrow Y \) is quadratic. Moreover, letting \( k = 0 \) and passing the limit \( m \to \infty \) in (3.9), we get the inequality (3.5) for \( j = 1 \).

Now, let \( Q' : X \rightarrow Y \) be another quadratic function satisfying (1.5) and (3.5). Then we have

\[
\left\| Q(x) - Q'(x) \right\| = \frac{1}{a^{2n}} \left\| Q(a^n x) - Q'(a^n x) \right\|
\]

\[
\leq \frac{1}{a^{2n}} \left( \left\| Q(a^n x) - f(a^n x) \right\| + \left\| Q'(a^n x) - f(a^n x) \right\| \right)
\]

\[
\leq \frac{1}{a^2a^{2n}} \tilde{\varphi}(a^n x, 0),
\]

where \( \tilde{\varphi}(x, y) \) is defined in (3.4).
which tends to zero as \( n \to \infty \) for all \( x \in X \). So we can conclude that \( Q(x) = Q'(x) \) for all \( x \in X \). This proves the uniqueness of \( Q \).

Also, for \( j = -1 \), it follows from (3.6) that

\[
\left\| f(x) - a^2 f\left(\frac{x}{a}\right) \right\| \leq \frac{1}{2} \phi\left(\frac{x}{a}, 0\right)
\]

for all \( x \in X \). Hence

\[
\left\| a^{2k} f\left(\frac{x}{a^k}\right) - a^{2m} f\left(\frac{x}{a^m}\right) \right\| \leq \frac{1}{2} \sum_{i=k}^{m-1} a^2 \phi\left(\frac{x}{a^{i+1}}, 0\right)
\]

for all nonnegative integers \( m \) and \( k \) with \( m > k \) and for all \( x \in X \). It follows from (3.14) that the sequence \( \{a^{2n} f(x/a^n)\} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{a^{2n} f(x/a^n)\} \) converges. So one can define the function \( Q : X \to Y \) by

\[
Q(x) := \lim_{n \to \infty} a^{2n} f\left(\frac{x}{a^n}\right)
\]

for all \( x \in X \). By (3.3) for \( j = -1 \) and (3.4),

\[
\left\| \Delta_Q(x, y) \right\| = \lim_{n \to \infty} a^{2n} \left\| \Delta_f\left(\frac{x}{a^n}, \frac{y}{a^n}\right) \right\| \leq \lim_{n \to \infty} a^{2n} \phi\left(\frac{x}{a^n}, \frac{y}{a^n}\right) = 0,
\]

for all \( x, y \in X \). So \( \Delta_Q(x, y) = 0 \). By Theorem 2.1, the function \( Q : X \to Y \) is quadratic. Moreover, letting \( k = 0 \) and passing the limit \( m \to \infty \) in (3.14), we get the inequality (3.5) for \( j = -1 \). The rest of the proof is similar to the proof of previous section.

From Theorem 3.1, we obtain the following corollaries concerning the JMRassias mixed product-sum stability of the functional equation (1.5).

**Corollary 3.2.** Let \( \varepsilon, p, q \geq 0 \) and \( r, s > 0 \) be real numbers such that \( p, q < 2 \) and \( r + s \neq 2 \). Suppose that a function \( f : X \to Y \) satisfies

\[
\left\| \Delta_j(x, y) \right\| \leq \varepsilon (\|x\|^p + \|y\|^q + \|x\|^r \|y\|^s)
\]

for all \( x, y \in X \). Then there exists a unique quadratic function \( Q : X \to Y \) such that

\[
\left\| f(x) - Q(x) \right\| \leq \frac{\varepsilon}{2(\alpha^2 - \alpha^p)} \|x\|^p
\]

for all \( x \in X \).

**Proof.** In Theorem 3.1, put \( j := 1 \) and \( \varphi(x, y) := \varepsilon (\|x\|^p + \|y\|^q + \|x\|^r \|y\|^s) \).
Corollary 3.3. Let $\varepsilon, p, q \geq 0$ and $r, s > 0$ be real numbers such that $p + q > 2$ and $r + s \neq 2$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies (3.17) for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ such that

$$
\|f(x) - Q(x)\| \leq \frac{\varepsilon}{2(a^p - a^2)} \|x\|^p
$$

(3.19)

for all $x \in X$.

Proof. In Theorem 3.1, put $j := -1$ and $\varphi(x, y) := \varepsilon(\|x\|^p + \|y\|^q + \|\alpha x\|^r \|\beta y\|^s)$.

Theorem 3.4. Let $j \in \{-1, 1\}$ be fixed, and let $\varphi : X \times X \to [0, \infty)$ be a function such that

$$
\tilde{\varphi}(x) := \sum_{i = (1-j)/2}^{\infty} \frac{1}{a^{2ij}} \varphi(a^{ij}x, 0) < \infty,
$$

(3.20)

$$
\lim_{n \to \infty} \frac{1}{a^{2nj}} \varphi(a^{nj}x, a^{nj}y) = 0
$$

for all $x, y \in X$. Suppose that $f : X \to Y$ be a function satisfies

$$
\left\|f(ax + by) + f(ax - by) - 2a^2 f(x) - 2b^2 f(y)\right\| \leq \varphi(x, y)
$$

(3.21)

for all $x, y \in X$. Furthermore, assume that $f(0) = 0$ in (3.21) for the case $j = 1$. Then there exists a unique quadratic function $Q : X \to Y$ such that

$$
\|f(x) - Q(x)\| \leq \frac{1}{2a^{1+j}} \tilde{\varphi}\left(\frac{x}{a^{1+j/2}}\right),
$$

(3.22)

for all $x \in X$.

Proof. The proof is similar to the proof of Theorem 3.1.

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