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Research Article

Necessary Conditions for a Class of Optimal Control Problems on Time Scales

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Based on the Gateaux differential on time scales, we investigate and establish necessary conditions for Lagrange optimal control problems on time scales. Moreover, we present an economic model to demonstrate the effectiveness of our results.

1. Introduction

In this paper, we consider optimal control problem (P). Find $u^0 \in U_{\mathrm{ad}}$ such that

$$J\left(u^0(\cdot)\right) \leq J(u(\cdot)), \quad \forall u \in U_{\mathrm{ad}}, \tag{1.1}$$

where J is the cost functional given by

$$\text{minimize } J(u(\cdot)) := \int_{[0,\sigma(T))_{\mathbb{T}}} l(x(t;u), u(t))\Delta t, \tag{1.2}$$

and $x(\cdot;u) \in \mathrm{AC}([0,T]_{\mathbb{T}}, \mathbb{R})$ is a solution corresponding to the control $u \in \mathcal{U}_{\mathrm{ad}}$ of the following equation:

$$\begin{gathered} x^{\Delta}(t) = p(t)x(t) + f(t) + u(t), \quad \text{for } \Delta\text{-a.e., } t \in \left[0, \rho(T)\right]_{\mathbb{T}}, \\ x(0) = x_0, \end{gathered} \tag{1.3}$$

where $\mathbb{T} \subset \mathbb{R}$ is a bounded time scale, $[0,T]_{\mathbb{T}} = [0,T] \cap \mathbb{T}$ and $\sigma(T) := \max \mathbb{T}$. The admissible control set is

$$\mathcal{U}_{\text{ad}} := \{u(t), t \in [0,T]_{\mathbb{T}}, u \text{ is } \Delta\text{-measurable and } u(t) \in \mathcal{U}\}. \tag{1.4}$$

Here, the control set $\mathcal{U}$ is a bounded, closed, and convex subset of $\mathbb{R}$.

Time scale calculus was initiated by Hilger in his Ph.D. thesis in 1988 [1] in order to unite two existing approaches of dynamic models-difference and differential equations into a general framework, which can be used to model dynamic processes whose time domains are more complex than the set of integers (difference equations) or real numbers (differential equation). There are many potential applications for this relatively new theory. The optimal control problems on time scales are also an interesting topic, and many researchers are working in this area. Existing results on the literature of time scales are restricted to problems of the calculus of variations, which were introduced by Bohner [2] and by Hilscher and Zeidan [3]. There are many opportunities for applications in economics [4, 5]. More general optimal control problems on time scales were studied in [6, 7].

To the best of our knowledge, it seems that there is not too much work about the necessary conditions of optimal control problems on time scales by adapting the method of calculus of variations. That motivates us to investigate new necessary conditions of optimal control problem on time scales. In this paper, based on the Gateaux differential on time scales, we establish necessary conditions for Lagrange optimal control problems on time scales. Moreover, we present an economic model to demonstrate our results.

The paper is organized as follows. We present some necessary preliminary definitions and results about the time scales $\mathbb{T}$ in Section 2. In Section 3, based on the existence and uniqueness of solutions of a linear dynamic equation on time scales, we derive existence and uniqueness of system solutions for the controlled system. Then, we prove the minimum principle on time scales for the optimal control problem (P) in Section 4. Finally, in Section 5, an example is given to demonstrate our results.

2. Preliminaries

A time scale $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$. The two most popular examples are $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}. \tag{2.1}$$

We put $\inf \varnothing = \sup \mathbb{T}$ and $\sup \varnothing = \inf \mathbb{T}$, where $\varnothing$ denotes the empty set. If there is the finite $\max \mathbb{T}$, then $\sigma(\max \mathbb{T}) = \max \mathbb{T}$, and if there exists the finite $\min \mathbb{T}$, then $\rho(\min \mathbb{T}) = \min \mathbb{T}$. The graininess function $\mu : \mathbb{T} \to [0, +\infty)$ is $\mu(t) := \sigma(t) - t$. A point $t \in \mathbb{T}$ is called left-dense (left-scattered, right-dense, and right-scattered) if $\rho(t) = t$ ($\rho(t) < t$, $\sigma(t) = t$, and $\sigma(t) > t$) holds. If $\mathbb{T}$ has a left-scattered maximum value M, then we denote $\mathbb{T}^k := \mathbb{T} - \{M\}$. Otherwise, $\mathbb{T}^k := \mathbb{T}$.

Definitions and propositions of Lebesgue Δ-measure μ_Δ and Lebesgue integral can be seen in [8–10].

Definition 2.1. Let P denote a proposition with respect to $t \in \mathbb{T}$ and A a subset of $\mathbb{T}$. If there exists $E_1 \subset A$ with $\mu_\Delta(E_1) = 0$ such that P holds on $A \setminus E_1$, then P is said to hold Δ-a.e., on A.

Remark 2.2. For each $t_0 \in \mathbb{T} \setminus \{\max \mathbb{T}\}$, the single-point set $\{t_0\}$ is Δ-measurable, and its Δ-measure is given by

$$\mu_\Delta(\{t_0\}) = \sigma(t_0) - t_0 = \mu(t_0). \tag{2.2}$$

Obviously, $E_1 \subset A$ does not have any right-scattered points. For a set $E \subset \mathbb{T}$, define the Lebesgue Δ-integral of f over E by $\int_E f(t)\Delta t$ and let $f \in L^1_{\mathbb{T}}(E, \mathbb{R})$ (see [8]).

Lemma 2.3 (see [8]). *Let $f : [a,b)_{\mathbb{T}} \to \mathbb{R}$. $\tilde{f} : [a,b) \to \mathbb{R}$ is the extension of f to real interval $[a,b]$, defined by*

$$\tilde{f}(t) := \begin{cases} f(t) & \text{if } t \in [a,b)_{\mathbb{T}}, \\ f(t_i) & \text{if } t \in (t_i, \sigma(t_i)), \text{ for some } i \in I, \end{cases} \tag{2.3}$$

where $\{t_i\}_{i\in I}, I \subsetneq \mathbb{N}$ is the index of the set of all right-scattered points of $[a,b]_{\mathbb{T}}$. Then, $f \in L^1_{\mathbb{T}}([a,b)_{\mathbb{T}}, \mathbb{R})$ if and only if $\tilde{f} \in L^1([a,b], \mathbb{R})$. In this case,

$$\int_{[a,b)_{\mathbb{T}}} f(t)\Delta t = \int_{[a,b]} \tilde{f}(t)dt. \tag{2.4}$$

Definition 2.4. Suppose that $f : [a,b)_{\mathbb{T}} \to \mathbb{R}$. $f \in L^\infty_{\mathbb{T}}([a,b)_{\mathbb{T}}, \mathbb{R})$, if there exists a constant $C \in \mathbb{R}$ such that

$$|f(t)| \le C \quad \Delta\text{-a.e. } t \in [a,b)_{\mathbb{T}}. \tag{2.5}$$

Definition 2.5 (see [10]). A function $f : \mathbb{T} \to \mathbb{R}$ is said to be absolutely continuous on $\mathbb{T}$ if for every given constant $\varepsilon > 0$, there is a constant $\delta > 0$ such that if $\{[a_k, b_k)_{\mathbb{T}}\}_{k=1}^n$, with $a_k, b_k \in \mathbb{T}$, is a finite pairwise disjoint family of subintervals of $\mathbb{T}$ satisfying

$$\sum_{k=1}^n (b_k - a_k) < \delta, \tag{2.6}$$

then

$$\sum_{k=1}^n |f(b_k) - f(a_k)| < \varepsilon. \tag{2.7}$$

If $\mathbb{T} = [a,b]_{\mathbb{T}}$, then we denote all absolutely continuous functions on $[a,b]_{\mathbb{T}}$ as $\mathrm{AC}([a,b]_{\mathbb{T}}, \mathbb{R})$.

Lemma 2.6. *If f is Lebesgue Δ-integrable on $[a,b)_{\mathbb{T}}$, then the integral $F(t) = \int_{[a,t)_{\mathbb{T}}} f(l)\Delta l$, $t \in [a,b)_{\mathbb{T}}$ is absolutely continuous on $[a,b]_{\mathbb{T}}$. Moreover,*

$$F^\Delta(t) = f(t), \quad \text{for } \Delta\text{-a.e. } t \in [a,b)_{\mathbb{T}}. \tag{2.8}$$

Proof. $F(t) = \int_{[a,t)_{\mathbb{T}}} f(l)\Delta l = \int_{[a,t]} \widetilde{f}(l)dl$ where $\widetilde{f}(l)$ is introduced in (2.3) and $\widetilde{f} \in L^1([a,t],\mathbb{R})$ from Lemma 2.3. Now, by the standard Lebesgue integration theory, $F(\cdot)$ is an absolutely continuous function on the real interval $[a,b]$ and

$$F'(t) = \widetilde{f}(t), \quad \text{a.e. } t \in [a,b]. \tag{2.9}$$

Using Definition 2.5, $F(t) = \int_{[a,t)_{\mathbb{T}}} f(l)\Delta l$ is also absolutely continuous on $[a,b]_{\mathbb{T}}$.

Let F be differentiable at t for $t \in [a,b)_{\mathbb{T}}$. If t is right-scattered, that is, $t = t_i$ for some $\{t_i\}_{i\in I}$, it follows from the continuity of F at t that

$$F^{\Delta}(t) = \frac{F(\sigma(t_i)) - F(t_i)}{\mu(t_i)} = \frac{\int_{[t_i,\sigma(t_i))} \widetilde{f}(s)ds}{\sigma(t_i) - t_i} = \frac{\int_{[t_i,\sigma(t_i))} f(s)ds}{\sigma(t_i) - t_i} = f(t_i) = F'(t_i). \tag{2.10}$$

If t is right dense,

$$\lim_{s\to t, s\in\mathbb{T}} \frac{F(t) - F(s)}{t - s} = \lim_{s\to t} \frac{F(t) - F(s)}{t - s} = F'(t). \tag{2.11}$$

Hence, F is Δ-differentiable at t and $F^{\Delta}(t) = F'(t)$. That is,

$$E_1 := \left\{ t \in [a,b)_{\mathbb{T}} : \nexists F^{\Delta}(t) \right\} \subset \left\{ t \in [a,b) : \nexists F'(t) \right\} =: E_2. \tag{2.12}$$

The continuity of F guarantees that F is Δ-differentiable at every right-scattered point t_i. Moreover, (2.9) implies $\lambda(E_2) = 0$. We deduce that E_1 does not contain any right-scattered points and

$$\mu_{\Delta}(E_1) = \lambda(E_1) = 0. \tag{2.13}$$

Hence, F is Δ-differentiable Δ-a.e., on $[a,b)_{\mathbb{T}}$ and

$$F^{\Delta}(t) = F'(t) = \widetilde{f}(t) = f(t) \quad \text{for } \Delta\text{-a.e. } t \in [a,b)_{\mathbb{T}}. \tag{2.14}$$

The proof is complete. □

It follows from Definition 2.5 and Lemma 2.6 that one can easy to prove the following integration by parts formula on time scales.

Lemma 2.7. *If $f, g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ are absolutely continuous functions on $[a,b]_{\mathbb{T}}$, then $f \cdot g$ is absolutely continuous on $[a,b]_{\mathbb{T}}$ and the following equality is valid:*

$$\int_{[a,b)_{\mathbb{T}}} \left(f^{\Delta} g + f^{\sigma} g^{\Delta} \right)(s)\Delta s = f(b)g(b) - f(a)g(a) + \int_{[a,b)_{\mathbb{T}}} \left(f g^{\Delta} + f^{\Delta} g^{\sigma} \right)(s)\Delta s. \tag{2.15}$$

Let $C([a,b]_{\mathbb{T}},\mathbb{R})$ denote the linear space of all continuous functions $f:[a,b]_{\mathbb{T}}\to\mathbb{R}$ on time scale $\mathbb{T}$ with the maximum norm $\|f\|_C=\max_{t\in[a,b]_{\mathbb{T}}}|f(t)|$. The following statement can be understood as a time scale version of the Arzela-Ascoli theorem.

Lemma 2.8 (see [11] (Arzela-Ascoli theorem)). *Let X be a subset of $C([a,b]_{\mathbb{T}},\mathbb{R})$ satisfying the following conditions:*

(i) *X is bounded;*

(ii) *for any given $\varepsilon>0$, there exists $\delta>0$ such that $t_1,t_2\in[a,b]_{\mathbb{T}},|t_1-t_2|<\delta$ implies $|f(t_1)-f(t_2)|<\varepsilon$ for all $f\in X$ (i.e., the functions in X are equicontinuous).*

Then, X is relatively compact.

3. Existence and Uniqueness of Solutions for a Controlled System Equation

In order to derive necessary conditions, we prove the existence and uniqueness of solutions for controlled system equation (1.3).

Definition 3.1. A function $x\in \mathrm{AC}([0,T]_{\mathbb{T}},\mathbb{R})$ is said to be a solution of problem (1.3) if

(i) x is Δ-differentiable Δ-a.e. on $[0,T)_{\mathbb{T}}$ and $x^{\Delta}\in L^1_{\mathbb{T}}([0,T)_{\mathbb{T}},\mathbb{R})$;

(ii) $x(0)=x_0$ and $x^{\Delta}(t)=p(t)x(t)+f(t)+u(t)$, Δ-a.e. on $[0,T)_{\mathbb{T}}$.

We assume the following.

$[HF]$ p is regressive rd-continuous function and $f\in L^1_{\mathbb{T}}([0,T)_{\mathbb{T}},\mathbb{R})$.

$[HL]$ The scalar functions $l(x,u)$ along with their partial derivation $\{l_x,l_u\}$ are continuous and uniformly bounded on $\mathbb{R}\times\mathcal{U}$ for almost all $t\in[0,T]_{\mathbb{T}}$.

Theorem 3.2 (existence and uniqueness of solutions for the controlled system equation). *If assumption $[HF]$ holds, for any $u\in\mathcal{U}_{ad}$, problem* (1.3) *has a unique solution in $[0,T]_{\mathbb{T}}$ which given by*

$$x(t)=e_p(t,0)x_0+\int_{[0,t)_{\mathbb{T}}}e_p(t,\sigma(\tau))\big(f(\tau)+u(\tau)\big)\Delta\tau,\quad t\in[0,T]_{\mathbb{T}}. \tag{3.1}$$

Proof. For conciseness, we just give a brief proof. Define a function F as

$$F(t)=f(t)+u(t),\quad t\in[0,T)_{\mathbb{T}}. \tag{3.2}$$

Then, problem (1.3) is equivalent to

$$\begin{gathered} x^{\Delta}(t)=p(t)x(t)+F(t),\quad \text{for } \Delta\text{-a.e., } t\in[0,\rho(T)]_{\mathbb{T}},\\ x(0)=x_0. \end{gathered} \tag{3.3}$$

Since $F \in L^1_{\mathbb{T}}([0,T)_{\mathbb{T}},\mathbb{R})$, there exists a sequence $\{F_n\}$ in $C([0,T]_{\mathbb{T}},\mathbb{R})$ such that $\|F-F_n\|_{L^1_{\mathbb{T}}} \to 0$. Therefore, the Cauchy problem

$$\begin{aligned} x_n^{\Delta} &= p(t)x_n + F_n(t), \\ x_n(0) &= x_0, \quad t \in [0,\rho(T)]_{\mathbb{T}}, \end{aligned} \tag{3.4}$$

has an unique classical solution given by

$$x_n(t) = e_p(t,0)x_0 + \int_{[0,t)_{\mathbb{T}}} e_p(t,\sigma(\tau))F_n(\tau)\Delta\tau, \quad t \in [0,T]_{\mathbb{T}}. \tag{3.5}$$

Now, we define

$$x(t) = e_p(t,0)x_0 + \int_{[0,t)_{\mathbb{T}}} e_p(t,\sigma(\tau))F(\tau)\Delta\tau, \quad t \in [0,T]_{\mathbb{T}}. \tag{3.6}$$

Then,

$$\begin{aligned} \|x_n - x\|_C &= \max_{t\in[0,T]_{\mathbb{T}}} |x_n(t) - x(t)| \\ &\le \int_{[0,T)_{\mathbb{T}}} \left|e_p(t,\sigma(\tau))\right| \cdot |F_n(\tau) - F(\tau)|\Delta\tau \\ &\le \sup_{t,\tau\in[0,T]_{\mathbb{T}}} \left|e_p(t,\tau)\right| \cdot \|F-F_n\|_{L^1_{\mathbb{T}}} \longrightarrow 0, \end{aligned} \tag{3.7}$$

and Lemma 2.6 can be applied to testify that x tailors to Definition 3.1. □

Let

$$M_1 = \sup_{t\in[0,T]_{\mathbb{T}}} \left|e_p(t,0)\right|, \qquad M_2 = \sup_{t,\tau\in[0,T]_{\mathbb{T}}} \left|e_p(t,\tau)\right|. \tag{3.8}$$

Define the Hamiltonian $H(x,\psi^{\sigma},u)$ as

$$H(x,\psi^{\sigma},u) = l(x,u) + \psi^{\sigma}(px + f + u). \tag{3.9}$$

4. Necessary Conditions for Optimal Control Problem (P)

In this section, we will present the minimum principle on time scales for the optimal control problem (P).

Theorem 4.1 (minimum principle on time scales). *Suppose that* $[HF]$ *and* $[HL]$ *hold. If* u^0 *is an optimal solution for problem (P) and* $x^0(\cdot;u^0)$ *is an optimal trajectory corresponding to* u^0*, then it is necessary that there exists a function* $\psi \in \mathrm{AC}([\sigma(0),\sigma(T)]_{\mathbb{T}},\mathbb{R})$ *satisfying the following conditions:*

$$\int_{[0,\sigma(T))_{\mathbb{T}}} \left\langle H_u\left(x^0(t),\psi^\sigma(t),u^0(t)\right), u(t)-u^0(t)\right\rangle \Delta t \geq 0, \quad \forall u \in \mathcal{U}_{\mathrm{ad}}, \tag{4.1}$$

$$\psi^\Delta(t) = -H_x\left(x^0(t),\psi^\sigma(t),u^0(t)\right) = -p(t)\psi^\sigma(t) - l_x\left(x^0(t),u^0(t)\right), \quad t \in [\sigma(0),T]_{\mathbb{T}}, \tag{4.2}$$

$$\psi(\sigma(T)) = 0. \tag{4.3}$$

Proof. This theorem can be proved in the following several steps.

(i) For all $\varepsilon \in [0,1]$ and for all $u \in \mathcal{U}_{\mathrm{ad}}$, define $u^\varepsilon = u^0 + \varepsilon(u-u^0)$. Since $\mathcal{U}$ is a bounded closed convex set, then $\mathcal{U}_{<\mathrm{ad}}$ is also a closed convex subset of $L^\infty_{\mathbb{T}}([0,T]_{\mathbb{T}},\mathbb{R})$ and $u^\varepsilon \in \mathcal{U}_{\mathrm{ad}}$. Because $u^0 \in \mathcal{U}_{\mathrm{ad}}$ is optimal,

$$J\left(u^0(\cdot)\right) \leq J(u^\varepsilon(\cdot)), \quad \forall \varepsilon \in [0,1],\ \forall u \in \mathcal{U}_{\mathrm{ad}}. \tag{4.4}$$

$$\lim_{\varepsilon\to 0} u^\varepsilon(t) = u^0(t), \quad \text{on } [0,T]_{\mathbb{T}}. \tag{4.5}$$

(ii) Now, we verify that $\{x^\varepsilon(\cdot;u^\varepsilon)\}$ converges to $x^0(\cdot;u^0)$ in $C([0,T]_{\mathbb{T}},\mathbb{R})$ as $\varepsilon \to 0$ by using Arzela-Ascoli theorem (Lemma 2.8). By boundedness of $\mathcal{U}_{\mathrm{ad}}$, we have

$$\begin{aligned}
|x^\varepsilon(t;u^\varepsilon)| &= \left| e_p(t,0)x_0 + \int_{[0,t)_{\mathbb{T}}} e_p(t,\sigma(\tau))\left[f(\tau)+u^\varepsilon(\tau)\right]\Delta\tau \right| \\
&\leq \left|e_p(t,0)x_0\right| + \int_{[0,t)_{\mathbb{T}}} \left|e_p(t,\sigma(\tau))\left[f(\tau)+u^\varepsilon(\tau)\right]\right|\Delta\tau \\
&\leq M_1|x_0| + M_2\int_{[0,T)_{\mathbb{T}}} \left|\left[f(\tau)+u^\varepsilon(\tau)\right]\right|\Delta\tau \\
&\leq M_1|x_0| + M_2\left[\int_{[0,T)_{\mathbb{T}}} \left|f(\tau)\right|\Delta\tau + \int_{[0,T)_{\mathbb{T}}} |u^\varepsilon(\tau)|\Delta\tau\right] \\
&\leq M.
\end{aligned} \tag{4.6}$$

$\{x^\varepsilon(\cdot;u^\varepsilon)\}$ is uniformly bounded on $[0,T]_{\mathbb{T}}$.

Taking arbitrary points t_1 and t_2 of the segment $[0,T]_{\mathbb{T}}$ and using the absolutely continuity of integral and the boundedness of $\mathcal{U}_{\mathrm{ad}}$, we obtain

$$\begin{aligned}
&|(x^\varepsilon(t_1;u^\varepsilon) - x^\varepsilon(t_2;u^\varepsilon))| \\
&\quad\leq \left|e_p(t_1,t_2)-1\right| \cdot \left|\left[e_p(t_2,0)x_0 + \int_{[0,t_2)_{\mathbb{T}}} e_p(t_2,\sigma(\tau))\left[f(\tau)+u^\varepsilon(\tau)\right]\Delta\tau\right]\right| \\
&\qquad + \left|e_p(t_1,t_2)\right| \cdot \left|\int_{[t_2,t_1)_{\mathbb{T}}} e_p(t_2,\sigma(\tau))\left[f(\tau)+u^\varepsilon(\tau)\right]\Delta\tau\right|.
\end{aligned} \tag{4.7}$$

Since

$$\begin{gathered} e_p(t_1,t_2) \longrightarrow 1, \\ |(x^\varepsilon(t_1;u^\varepsilon) - x^\varepsilon(t_2;u^\varepsilon))| \longrightarrow 0, \quad \text{as } |t_1 - t_2| \longrightarrow 0. \end{gathered} \tag{4.8}$$

Hence, $\{x^\varepsilon(\cdot;u^\varepsilon)\}$ is equicontinuous in $[0,T]_{\mathbb{T}}$.

It follows from (4.5) that

$$\begin{aligned} \left|x^\varepsilon(t;u^\varepsilon) - x^0\left(t;u^0\right)\right| &= \left|\int_{[0,t)_{\mathbb{T}}} e_p(t,\sigma(\tau))\left(u^\varepsilon(\tau) - u^0(\tau)\right)\Delta\tau\right| \\ &\le \int_{[0,t)_{\mathbb{T}}} \left|e_p(t,\sigma(\tau))\right| \cdot \left|\left(u^\varepsilon(\tau) - u^0(\tau)\right)\right|\Delta\tau \\ &\le M_2 \int_{[0,T)_{\mathbb{T}}} \left|\left(u^\varepsilon(\tau) - u^0(\tau)\right)\right|\Delta\tau \\ &\longrightarrow 0, \quad \text{as } \varepsilon \longrightarrow 0. \end{aligned} \tag{4.9}$$

By Arzela-Ascoli theorem (Lemma 2.8), we obtain

$$x^\varepsilon \longrightarrow x^0 \quad \text{in } C([0,T]_{\mathbb{T}},\mathbb{R}). \tag{4.10}$$

(iii) Denote

$$y(t) := \lim_{\varepsilon\downarrow 0} \frac{x^\varepsilon(t) - x^0(t)}{\varepsilon}. \tag{4.11}$$

Then, y satisfies the following initial value problem:

$$y^\Delta(t) = p(t)y(t) + \left(u(t) - u^0(t)\right), \quad \text{for } \Delta\text{-a.e., } t \in [0,\rho(T)]_{\mathbb{T}} \tag{4.12}$$

with

$$y(0) = 0. \tag{4.13}$$

We call (4.12) and (4.13) the variational equations.

(iv) We calculate the Gateaux differential of J at $u^0 \in \mathcal{U}_{\mathrm{ad}}$ in the direction $u - u^0$. It follows from hypotheses $[HL]$, Lemma 2.3, and (4.4) that

$$\begin{aligned}
0 &\le \lim_{\varepsilon\downarrow 0}\frac{J(u^\varepsilon(\cdot)) - J(u^0(\cdot))}{\varepsilon} \\
&= \lim_{\varepsilon\downarrow 0}\int_{[0,\sigma(T))_{\mathbb{T}}} \frac{l(x^\varepsilon(t;u^\varepsilon),u^\varepsilon(t)) - l(x^0(t;u^0),u^0(t))}{\varepsilon}\Delta t \\
&= \lim_{\varepsilon\downarrow 0}\int_{[0,\sigma(T)]} \frac{l\left(\widetilde{x^\varepsilon(t;u^\varepsilon)},\widetilde{u^\varepsilon(t)}\right) - l\left(\widetilde{x^0(t;u^0)},\widetilde{u^0(t)}\right)}{\varepsilon}dt \\
&= \lim_{\varepsilon\downarrow 0}\int_{[0,\sigma(T)]} \left\{\int_0^1\left[\left\langle l_x\left(\widetilde{x^0}+\theta\left(\widetilde{x^\varepsilon}-\widetilde{x^0}\right),\widetilde{u^0}+\theta\varepsilon\left(\widetilde{u}-\widetilde{u^0}\right)\right),\frac{\widetilde{x^\varepsilon}-\widetilde{x^0}}{\varepsilon}\right\rangle\right.\right. \\
&\qquad\left.\left.+\left\langle l_u\left(\widetilde{x^0}+\theta\left(\widetilde{x^\varepsilon}-\widetilde{x^0}\right),\widetilde{u^0}+\theta\varepsilon\left(\widetilde{u}-\widetilde{u^0}\right)\right),\widetilde{u}-\widetilde{u^0}\right\rangle\right]d\theta\right\}dt \\
&= \int_{[0,\sigma(T)]}\left[\left\langle l_x\left(\widetilde{x^0(t)},\widetilde{u^0}\right),\widetilde{y(t)}\right\rangle + \left\langle l_u\left(\widetilde{x^0(t)},\widetilde{u^0}\right),\widetilde{u(t)}-\widetilde{u^0(t)}\right\rangle\right]dt \\
&= \int_{[0,\sigma(T))_{\mathbb{T}}}\left[\left\langle l_x\left(x^0(t),u^0\right),y(t)\right\rangle + \left\langle l_u\left(x^0(t),u^0\right),u(t)-u^0(t)\right\rangle\right]\Delta t.
\end{aligned} \tag{4.14}$$

Here, the "title" is the corresponding extension function in Lemma 2.3. That is,

$$\widetilde{u^\varepsilon(t)} := \begin{cases} u^\varepsilon(t) = u^0(t) + \varepsilon\left(u(t) - u^0(t)\right) & \text{if } t \in [0,T]_{\mathbb{T}}, \\ u^\varepsilon(t_i) = u^0(t_i) + \varepsilon\left(u(t_i) - u^0(t_i)\right) & \text{if } t \in (t_i,\sigma(t_i)), \text{ for some } i \in I, \end{cases} \tag{4.15}$$

where $\{t_i\}_{i\in I}, I \subsetneq \mathbb{N}$, is the set of all right-scattered points of $[0,T]_{\mathbb{T}}$. Obviously

$$\widetilde{u^\varepsilon(t)} = \widetilde{u^0(t)} + \varepsilon\left(\widetilde{u(t)} - \widetilde{u^0(t)}\right), \quad t \in [0,T]. \tag{4.16}$$

By the variational equations (4.12) and (4.13), we define an operator $T_1 : L_T^1([0,T)_{\mathbb{T}},\mathbb{R}) \to C([0,T]_{\mathbb{T}},\mathbb{R})$ as

$$y(t) := T_1\left(u - u^0\right)(t) = \int_{[0,t)_{\mathbb{T}}} e_p(t,\sigma(\tau))\left[u(\tau) - u^0(\tau)\right]\Delta\tau, \quad t \in [0,T]_{\mathbb{T}}. \tag{4.17}$$

Then, T_1 is a continuous linear operator. Furthermore, due to the uniform bound of l_x, $T_2 : C([0,T]_{\mathbb{T}},\mathbb{R}) \to \mathbb{R}$, given by

$$T_2 y := \int_{[0,\sigma(T))_{\mathbb{T}}}\left\langle l_x\left(x^0(t),u^0(t)\right),y(t)\right\rangle\Delta t, \tag{4.18}$$

is also a linear continuous functional. Hence, $T_2 \circ T_1 : L_T^1([0,T)_{\mathbb{T}}, \mathbb{R}) \to \mathbb{R}$ defined by

$$T_2 \circ T_1\left(u - u^0\right) = \int_{[0,\sigma(T))_{\mathbb{T}}} \left\langle l_x\left(x^0(t), u^0(t)\right), y(t)\right\rangle \Delta t \tag{4.19}$$

is a bounded linear functional.

By the Riesz representation theorem (see [12, Theorem 2.34]), there is a $\psi^\sigma \in L_T^\infty([0,T]_{\mathbb{T}}, \mathbb{R})$ such that

$$\int_{[0,\sigma(T))_{\mathbb{T}}} \left\langle l_x\left(x^0(t), u^0(t)\right), y(t)\right\rangle \Delta t = \int_{[0,\sigma(T))_{\mathbb{T}}} \left\langle u(t) - u^0(t), \psi^\sigma(t)\right\rangle \Delta t. \tag{4.20}$$

Using (4.20), (4.14), and (3.9), we obtain

$$\begin{aligned} 0 &\le \int_{[0,\sigma(T))_{\mathbb{T}}} \left[\left\langle l_x\left(x^0(t), u^0(t)\right), y(t)\right\rangle + \left\langle l_u\left(x^0(t), u^0(t)\right), u(t) - u^0(t)\right\rangle\right] \Delta t \\ &= \int_{[0,\sigma(T))_{\mathbb{T}}} \left[\left\langle l_u\left(x^0(t), u^0(t)\right) + \psi^\sigma(t), u(t) - u^0(t)\right\rangle\right] \Delta t \\ &= \int_{[0,\sigma(T))_{\mathbb{T}}} \left\langle H_u\left(x^0(t), \psi^\sigma(t), u^0(t)\right), u(t) - u^0(t)\right\rangle \Delta t, \quad \forall u \in \mathcal{U}_{\text{ad}}. \end{aligned} \tag{4.21}$$

Hence we have derived the necessary condition (4.1).

(v) Now, we can claim that $\psi \in \mathrm{AC}([0,\sigma(T)]_{\mathbb{T}}, \mathbb{R})$ and the last part of necessary conditions are true. Using Lemma 2.7, (4.12), and (4.13) as well as (4.20), we obtain

$$\begin{aligned} T_2(y) &= \int_{[0,\sigma(T))_{\mathbb{T}}} \left\langle l_x\left(x^0(t), u^0(t)\right), y(t)\right\rangle \Delta t \\ &= \int_{[0,\sigma(T))_{\mathbb{T}}} \left\langle y^\Delta(t) - p(t)y(t), \psi^\sigma(t)\right\rangle \Delta t \\ &= \int_{[0,\sigma(T))_{\mathbb{T}}} \left\langle y^\Delta(t), \psi^\sigma(t)\right\rangle \Delta t - \int_{[0,\sigma(T))_{\mathbb{T}}} \left\langle p(t)y(t), \psi^\sigma(t)\right\rangle \Delta t \\ &= y(\sigma(T))\psi(\sigma(T)) - \int_{[0,\sigma(T))_{\mathbb{T}}} \left\langle y(t), \psi^\Delta(t)\right\rangle \Delta t - \int_{[0,\sigma(T))_{\mathbb{T}}} \left\langle y(t), p(t)\psi^\sigma(t)\right\rangle \Delta t \\ &= y(\sigma(T))\psi(\sigma(T)) - \int_{[0,\sigma(T))_{\mathbb{T}}} \left\langle y(t), \psi^\Delta(t) + p(t)\psi^\sigma(t)\right\rangle \Delta t. \end{aligned} \tag{4.22}$$

From the first and the last equalities, we have

$$y(\sigma(T))\psi(\sigma(T)) - \int_{[0,\sigma(T))_{\mathbb{T}}} \left\langle y(t), \psi^\Delta(t) + p(t)\psi^\sigma(t) + l_x\left(x^0(t), u^0(t)\right)\right\rangle \Delta t = 0. \tag{4.23}$$

Hence, similar to Theorem 3.2, one may choose ψ^{σ} as the solution of the following backward problem:

$$\begin{gathered}\psi^{\Delta}(t) = -p(t)\psi^{\sigma}(t) - l_x\left(x^0(t), u^0(t)\right), \quad \text{for } \Delta\text{-a.e. } t \in [\sigma(0), T]_{\mathbb{T}}, \\ \psi(\sigma(T)) = 0.\end{gathered} \tag{4.24}$$

This completes the proof. □

Remark 4.2. If the control set $\mathcal{U} = \mathbb{R}$, then (4.1) reduces to

$$H_u\left(x^0(t), \psi^{\sigma}(t), u^0(t)\right) = 0, \qquad \Delta\text{-a.e. on } [0, T]_{\mathbb{T}}. \tag{4.25}$$

5. Example (A Model in Economics)

In this section, for illustration, we will apply Theorem 4.1 to the following economics model. This model had been discussed by the method of Nabla version calculus of variation on time scales (see [4, 13]). We briefly present it here. A consumer is seeking to maximize his lifetime utility subject to certain constraints. During each period in his life, a consumer has to make a decision regarding how much to consume and how much to spend. Utility is the value function of the consumer that one wants to maximize. It can depend on numerous variables, in this simple example, it depends only on the consumption of some generic production C. Utility function $u(C)$ abides by the Law of Diminishing Marginal Utility, that is to say, $u'(C) > 0$ and $u''(C) < 0$.

5.1. Discrete Time Model

A representative consumer has to make decisions not just about one period but about the sequence of $\mathbf{C}'\mathbf{s}$: $\mathbf{C}_0, \mathbf{C}_1, \ldots, \mathbf{C}_{\mathbf{T}}$. The problem is to find a consumption path that would maximize lifetime utility $\mathbf{U}$ as follows:

$$\max \mathbf{U}(\mathbf{C}) = \sum_{\mathbf{t}=0}^{\mathbf{T}} \left(\frac{\mathbf{1}}{\mathbf{1}+\boldsymbol{\delta}}\right)^{\mathbf{t}} \mathbf{u}(\mathbf{C}_{\mathbf{t}}), \tag{5.1}$$

where $\mathbf{C}_{\mathbf{t}}$ is the consumption during period $\mathbf{t}$, $\mathbf{u}$ is one-period utility, and $0 < \boldsymbol{\delta} < 1$ is the (constant) discount rate. We assume that the future consumption is less than the current consumption, so we discount the future at the rate $\boldsymbol{\delta}$. The consumer is limited by the budget constraints:

$$\mathbf{A}_{\mathbf{t}+1} = (1+\mathbf{r})\mathbf{A}_{\mathbf{t}} + \mathbf{Y}_{\mathbf{t}} - \mathbf{C}_{\mathbf{t}}, \qquad \mathbf{A}_{\mathbf{T}}\left(\frac{1}{1+\mathbf{r}}\right)^{\mathbf{T}} \geq 0, \tag{5.2}$$

where $\mathbf{A}_{\mathbf{t}+1}$ is the amount of assets held at the beginning of period $\mathbf{t}+1$, $\mathbf{Y}_{\mathbf{t}}$ is the income received in period $\mathbf{t}$, and $\mathbf{r}$ is the constant interest rate. $\mathbf{A}_{\mathbf{T}}(1/(1+\mathbf{r}))^{\mathbf{T}} \geq 0$ that can be interpreted as "we are not allowed to borrow without limit."

5.2. Continuous Time Model

The same problem can be solved in a continuous time case, where lifetime utility is the sum of instantaneous utilities:

$$\max U(C) = \int_0^T u(C(t))e^{-\delta t}dt \tag{5.3}$$

with respect to the path $\{C(t), t \in [0,T]\}$ subject to the constraint

$$A'(t) = A(t)r + Y(t) - C(t). \tag{5.4}$$

5.3. Time Scale Calculus Model

A consumer receives income at one time point, asset holdings are adjusted at a different time point, and consumption takes place at another time point. Consumption and saving decisions can be modeled to occur with arbitrary, time-varying frequency. Hence, the time scale version of this model can de described by

$$\max U(C) = \int_{[0,\sigma(T))_{\mathbb{T}}} u(C(t))\widehat{e}_{-\delta}(t,0)\Delta t, \tag{5.5}$$

subject to the budget constraint

$$A^{\Delta}(t) = rA(t) + Y(t) - C(t), \quad t \in [0,\rho(T)], \tag{5.6}$$

where $\widehat{e}_{-\delta}(t,0)$ is the Nabla exponential function of $-\delta$,

$$\widehat{e}_{-\delta}(t,0) := \exp\left(\int_{[0,t)_{\mathbb{T}}} \widehat{\xi}_{\nu(\tau)}(-\delta)\nabla\tau\right). \tag{5.7}$$

Note that (see [14] for more details)

$$\widehat{e}_{-\delta}(t,0) := \begin{cases} e^{-\delta t} & \text{if } \mathbb{T} = \mathbb{R}, \\ \left(\dfrac{1}{1+\delta}\right)^t & \text{if } \mathbb{T} = \mathbb{Z}. \end{cases} \tag{5.8}$$

Now, we use Theorem 4.1 to solve this model. The Hamiltonian

$$H(A,\psi^{\sigma},C) = -u(C(t))\widehat{e}_{-\delta}(t,0) + \psi^{\sigma}(t)(rA(t) + Y(t) - C(t)). \tag{5.9}$$

Optimal consumption satisfies the following necessary conditions:

$$\begin{gathered} -u'(C(t))\widehat{e}_{-\delta}(t,0) - \varphi^{\sigma}(t) = 0, \\ \varphi^{\Delta}(t) = -r\varphi^{\sigma}(t). \end{gathered} \tag{5.10}$$

If $\mathbb{T} = \mathbb{R}$, then (5.8), (5.10) imply

$$C'(t) = (\delta - r)\frac{u'(C(t))}{u''(C(t))}. \tag{5.11}$$

Due to $u'(C) > 0$ and $u''(C) < 0$, it shows that $C_t' > 0$ if $r > \delta$. Hence, the consumer will wait to consume.

If $\mathbb{T} = \mathbb{Z}$, then (5.8), (5.10) imply

$$u'(C_t) = \frac{1+r}{1+\delta}u'(C_{t+1}). \tag{5.12}$$

It follows from $u'(C) > 0$ and $u''(C) < 0$ that if $u'(C_{t+1}) < u'(C_t)$, then $C_{t+1} > C_t$. Therefore if the interest rate r is higher than the future's discount rate δ, the consumer will wait to consume until next periods. Therefore, we obtain the same results as [4].

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