Research Article

On $P$- and $p$-Convexity of Banach Spaces

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We show that every $U$-space and every Banach space $X$ satisfying $\delta_X(1) > 0$ are $P(3)$-convex, and we study the nonuniform version of $P$-convexity, which we call $p$-convexity.

1. Introduction

Kottman introduced in 1970 the concept of $P$-convexity in [1]. He proved that every $P$-convex space is reflexive and also that $P$-convexity follows from uniform convexity, as well as from uniform smoothness. In this paper we study conditions which guarantee the $P$-convexity of a Banach space and generalize the result of Kottman concerning uniform convexity in two different ways: every $U$-space and every Banach space $X$ satisfying $\delta_X(1) > 0$ are $P(3)$-convex. There are many convexity conditions of Banach spaces which have a uniform and also a nonuniform version, for example, strictly convexity is the nonuniform version of uniform convexity, smoothness is the nonuniform version of uniform smoothness, and a $u$-space is the nonuniform version of a $U$-space, among others. We also define the concept of $p$-convexity, which is the nonuniform version of $P$-convexity and obtain some interesting results.

2. $P$-Convex Banach Spaces

Throughout this paper we adopt the following notation. $(X, \| \cdot \|)$ will be a Banach space and when there is no possible confusion, we simply write $X$. The unit ball $\{ x \in X : \| x \| \leq 1 \}$ and the unit sphere $\{ x \in X : \| x \| = 1 \}$ are denoted, respectively, by $B_X$ and $S_X$. $B(y, r)$ will denote the closed ball with center $y$ and radius $r$. The topological dual space of $X$ is denoted by $X^*$. 
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2.1. \(P\)-Convexity

The next concept was given by Kottman in [1].

**Definition 2.1.** Let \(X\) be a Banach space. For each \(n \in \mathbb{N}\) let

\[
P(n, X) = \sup \{ r > 0 : \text{there exist } n \text{ disjoint balls of radius } r \text{ in } B_X \}.
\]

(2.1)

It is easy to see that \(P(n, X) \leq 1/2\) for \(n \geq 2\).

**Definition 2.2.** \(X\) is said to be \(P\)-convex if \(P(n, X) < 1/2\) for some \(n \in \mathbb{N}\).

The following lemma was proved in [1].

**Lemma 2.3.** Let \(X\) be a Banach space and \(n \in \mathbb{N}\). Then \(P(n, X) < 1/2\) if and only if there exists \(\varepsilon > 0\) such that for any \(x_1, x_2, \ldots, x_n \in S_X\)

\[
\min \{ \|x_i - x_j\| : 1 \leq i, j \leq n, \ i \neq j \} \leq 2 - \varepsilon.
\]

(2.2)

That is, \(X\) is \(P\)-convex if and only if \(X\) satisfies condition (2.2) for some \(n \in \mathbb{N}\) and some \(\varepsilon > 0\).

**Definition 2.4.** Given \(n \in \mathbb{N}\) and \(\varepsilon > 0\) we say that \(X\) is \(P(\varepsilon, n)\)-convex if \(X\) satisfies (2.2). For each \(n \in \mathbb{N}\), \(X\) is said to be \(P(n)\)-convex if it is \(P(\varepsilon, n)\)-convex for some \(\varepsilon > 0\).

2.2. \(P\)-Convexity and the Coefficient of Convexity

In [1], Kottman proved that if \(X\) is a Banach space satisfying the condition \(\delta_X(2/3) > 0\), then \(X\) is \(P(3)\)-convex, where \(\delta_X\) is the modulus of convexity. In this section we give a result which improves this condition, and we show that this assumption is sharp.

We recall the following concepts introduced by J. A. Clarkson in 1936.

**Definition 2.5.** The modulus of convexity of a Banach space \(X\) is the function \(\delta_X : [0, 2] \to [0, 1]\) defined by

\[
\delta_X(\varepsilon) = \inf \left\{ 1 - \frac{\| x + y \|}{2} : x, y \in B_X, \| x - y \| \geq \varepsilon \right\}.
\]

(2.3)

The coefficient of convexity of a Banach space \(X\) is the number \(\varepsilon_0(X)\) defined as

\[
\varepsilon_0(X) = \sup \{ \varepsilon \in [0, 2] : \delta_X(\varepsilon) = 0 \}.
\]

(2.4)

We also need the following definition given by R. C. James in 1964.

**Definition 2.6.** \(X\) is said to be uniformly nonsquare if there exists \(\alpha > 0\) such that for all \(\xi, \eta \in S_X\)

\[
\min \{ \|\xi - \eta\|, \|\xi + \eta\| \} \leq 2 - \alpha.
\]

(2.5)

In order to prove our theorem we need two known results which can be found in [2].
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Lemma 2.7 (Goebel-Kirk). Let $X$ be a Banach space. For each $\epsilon \in [\varepsilon_0(X), 2]$, one has the equality $\delta_X(2 - 2\delta_X(\varepsilon)) = 1 - \varepsilon/2$.

Lemma 2.8 (Ullán). Let $X$ be a Banach space. For each $0 \leq \varepsilon_2 \leq \varepsilon_1 < 2$ the following inequality holds: $\delta_X(\varepsilon_1) - \delta_X(\varepsilon_2) \leq (\varepsilon_1 - \varepsilon_2)/(2 - \varepsilon_1)$.

Using these lemmas we obtain:

Theorem 2.9. Let $X$ be a Banach space which satisfies $\delta_X(1) > 0$, that is, $\varepsilon_0(X) < 1$. Then $X$ is $P(3)$-convex. Moreover, there exists a Banach space $X$ with $\varepsilon_0(X) = 1$ which is not $P(3)$-convex.

Proof. Let $t_0 = 2 - \sqrt{2 - \varepsilon_0(X)}$. Clearly $\varepsilon_0(X) < t_0 < 1$. Let $x, y, z \in S_X$, and suppose that $\|x - y\| > 2 - 2\delta_X(t_0)$ and $\|x - z\| > 2 - 2\delta_X(t_0)$. By Lemma 2.7, we have

$$\frac{\|x + y\|}{2} \leq 1 - \delta_X(2 - 2\delta_X(t_0)) = 1 - \left(1 - \frac{t_0}{2}\right) = \frac{t_0}{2}. \quad (2.6)$$

Similarly $\|(x + z)/2\| \leq t_0/2$. Hence we get

$$\|z - y\| \leq \|z + x\| + \|x + y\| \leq 2t_0. \quad (2.7)$$

Finally, from Lemma 2.8 it follows that

$$\delta_X(t_0) = \delta_X(t_0) - \delta_X(\varepsilon_0(X)) \leq \frac{t_0 - \varepsilon_0(X)}{2 - t_0} = \sqrt{2 - \varepsilon_0(X)} - 1 = 1 - t_0. \quad (2.8)$$

Then $\|y - z\| \leq 2t_0 \leq 2 - 2\delta_X(t_0)$, and thus $X$ is $P(3)$-convex.

Now consider for each $1 < p < \infty$ the space $l_{p,\infty}$ defined as follows. Each element $x = [x_i] \in l_p$ may be represented as $x = x^+ - x^-$, where the respective $i$th components of $x^+$ and $x^-$ are given by $(x^+)_i = \max\{x_i, 0\}$ and $(x^-)_i = \max\{-x_i, 0\}$. Set $\|x\|_{l_p,\infty} = \max\{\|x^+\|_p, \|x^-\|_p\}$ where $\| \cdot \|_p$ stands for the $l_p$-norm. The space $l_{p,\infty} = (l_p, \| \cdot \|_{l_p,\infty})$ satisfies $\varepsilon_0(l_{p,\infty}) = 1$ (see [3]). On the other hand let $x_1 = e_1 - e_3, x_2 = -e_1 + e_2, x_3 = -e_2 + e_3 \in S_{l_{p,\infty}}$, where $\{e_i\}$ is the canonical basis in $l_p$. These points satisfy that $\|x_i - x_j\|_{l_p,\infty} = 2, i \neq j$. Thus $l_{p,\infty}$ is not $P(3,2)$-convex.

It is known that if a Banach space $X$ satisfies $\varepsilon_0(X) < 1$, then $X$ has normal structure as well as $P(3)$-convexity. The space $X = l_{p,\infty}$ is an example of a Banach space with $\varepsilon_0(X) = 1$ which does not have normal structure (see [3]) and is not $P(3)$-convex.

Kottman also proved in [1] that every uniformly smooth space is a $P$-convex space. We obtain a generalization of this fact. Before we show this result we recall the next concept.

Definition 2.10. The modulus of smoothness of a Banach space $X$ is the function $\rho_X : [0, \infty) \to [0, \infty)$ defined by

$$\rho_X(t) = \sup \left\{ \frac{1}{2} \left( \|x + ty\| + \|x - ty\| - 2 \right) : x, y \in S_X \right\} \quad (2.9)$$

for each $t \geq 0$. $X$ is called uniformly smooth if $\lim_{t \to 0} \rho_X(t)/t = 0$. 


The proofs of the following lemmas can be found in [4, 5].

**Lemma 2.11.** For every Banach space $X$, one has $\lim_{t \to 0} \rho_X(t)/t = (1/2)\varepsilon_0(X^*)$.

**Lemma 2.12.** Let $X$ be a Banach space. $X$ is $P(3)$-convex if and only if $X^*$ is $P(3)$-convex.

By Theorem 2.9 and by the previous lemmas we deduce the next result.

**Corollary 2.13.** If $X$ is a Banach space satisfying $\lim_{t \to 0} \rho_X(t)/t < 1/2$, then $X$ is $P(3)$-convex.

With respect to $P(4)$-convex spaces we have this result, which is easy to prove.

**Proposition 2.14.** If $X$ is a Banach space $P(\varepsilon, 4)$-convex, then $\varepsilon_0(X) \leq 2 - \varepsilon$, and hence $X$ is uniformly nonsquare.

In fact, in bidimensional normed spaces, $P(4)$-convexity and uniform nonsquareness coincide. The proof of this involves many calculations and can be seen in [6].

Another technical proof (see [6]) shows that if $X$ is a bidimensional normed space, then $X$ is always $P(1,5)$-convex. Hence the space $X = (\mathbb{R}^2, \|\cdot\|_\infty)$ is $P(1,5)$-convex and $\varepsilon_0(X) = 2$, and thus $P(5)$-convexity does not imply uniform squareness.

### 2.3. Relation between $U$-Spaces and $P$-Convex Spaces

In this section we show that $P$-convexity follows from $U$-convexity. The following concept was introduced by Lau in 1978 [7].

**Definition 2.15.** A Banach space $X$ is called a $U$-space if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$x, y \in S_X, \quad f(x - y) > \varepsilon, \quad \text{for some } f \in \nabla(x) \Rightarrow \|x + y/2\| \leq 1 - \delta,$$  \hspace{1cm} (2.10)

where for each $x \in X$

$$\nabla(x) = \{ f \in S_{X^*} : f(x) = \|x\| \}.$$  \hspace{1cm} (2.11)

The modulus of this type of convexity was introduced by Gao in [8] and further studied by Mazcuñán-Navarro [9] and Saejung [10]. The following result is proved in [8].

**Lemma 2.16.** Let $X$ be a Banach space. If $X$ is $U$-space, then $X$ is uniformly nonsquare.

From the above we obtain the next theorem which is a generalization of Kottman’s result, who showed in [1] that $P(3)$-convexity follows from uniform convexity.

**Theorem 2.17.** If $X$ is a $U$-space, then $X$ is $P(3)$-convex.

**Proof.** By Lemma 2.16 we have that there exists $\alpha > 0$ such that for all $\xi, \eta \in S_X$

$$\min\{\|\xi - \eta\|, \|\xi + \eta\|\} \leq 2 - \alpha.$$  \hspace{1cm} (2.12)
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Since $X$ is a $U$-space, for $\varepsilon = \alpha/2$ there exists $\delta > 0$ such that

$$x, y \in S_X, \quad f(x - y) \geq \frac{\alpha}{2}, \quad \text{for some } f \in \nabla(x) \Rightarrow \left\| \frac{x + y}{2} \right\| \leq 1 - \delta. \quad (2.13)$$

We claim that $X$ is $P(\beta, 3)$-convex, where $\beta = \min\{\alpha, \delta\}$. Indeed, proceeding by contradiction, assume that there exist $x, y, z \in S_X$ such that

$$\min\{\|x - y\|, \|x - z\|, \|y - z\|\} > 2 - \beta. \quad (2.14)$$

Define $w = -y$ and $u = -z$, and let $f \in \nabla(w)$. If $f(w - x) \geq \alpha/2$, then

$$\left\| \frac{w + x}{2} \right\| < 1 - \delta. \quad (2.15)$$

Therefore $2 - \delta \leq 2 - \beta < \|x - y\| < 2 - 2\delta$, which is not possible. Hence $f(w - x) < \alpha/2$. Similarly we prove $f(w + u) < \alpha/2$. Also $\|x + u\| = \|x - z\| > 2 - \beta \geq 2 - \alpha$, and hence, by (2.12) we have $f(x - u) \leq \|x - u\| \leq 2 - \alpha$. By the above we have

$$2 = 2f(w) = f(w - x) + f(x - u) + f(u + w) < \frac{\alpha}{2} + 2 - \alpha + \frac{\alpha}{2} = 2 \quad (2.16)$$

which is a contradiction. \qed

\section*{2.4. The Dual Concept of $P$-Convexity}

In [1], Kottman introduces a property which turns out to be the dual concept of $P$-convexity. In this section we characterize the dual of a $P$-convex space in an easier way. We begin by showing Kottman’s characterization.

\textbf{Definition 2.18.} Let $X$ be a Banach space and $\varepsilon > 0$. A convex subset $A$ of $B_X$ is said to be $\varepsilon$-flat if $A \cap (1 - \varepsilon)B_X = \emptyset$. A collection $\mathcal{D}$ of $\varepsilon$-flats is called complemented if for each pair of $\varepsilon$-flats $A$ and $B$ in $\mathcal{D}$ we have that $A \cup B$ has a pair of antipodal points. For any $n \in \mathbb{N}$ we define

$$F(n, X) = \inf\{ \varepsilon > 0 : B_X \text{ has a complemented collection } \mathcal{D} \text{ of } \varepsilon \text{-flats such that } \text{Card} (\mathcal{D}) = n \}. \quad (2.17)$$

\textbf{Theorem 2.19 (Kottman).} Let $X$ be a Banach space and $n \in \mathbb{N}$. Then

(a) $F(n, X^*) = 0 \iff P(n, X) = 1/2$.

(b) $P(n, X^*) = 1/2 \iff F(n, X) = 0$.

Now we define $P$-smoothness and prove that it turns out to be the dual concept of $P$-convexity. The advantage of this characterization is that it uses only simple concepts, and one does not need $\varepsilon$-flats. Besides in the proof of the duality we do not need Helly’s theorem nor the theorem of Hahn-Banach, as Kottman does in Theorem 2.19.
Definition 2.20. Let $X$ be a Banach space and $\delta > 0$. For each $f, g \in X^*$ set $S(f, g, \delta) = \{ x \in B_X : f(x) \geq 1 - \delta, g(x) \geq 1 - \delta \}$. Given $\delta > 0$ and $n \in \mathbb{N}$, $X$ is said to be $P(\delta, n)$-smooth if for each $f_1, f_2, \ldots, f_n \in S_X^*$ there exist $1 \leq i, j \leq n, i \neq j$, such that $S(f_i, -f_j, \delta) = \emptyset$. $X$ is said to be $P(n)$-smooth if it is $P(\delta, n)$-smooth for some $\delta > 0$, and $X$ is said to be $P$-smooth if it is $P(\delta, n)$-smooth for some $\delta > 0$ and some $n \in \mathbb{N}$.

Proposition 2.21. Let $X$ be a Banach space. Then

(a) $X$ is $P(n)$-convex if and only if $X^*$ is $P(n)$-smooth.

(b) $X$ is $P(n)$-smooth if and only if $X^*$ is $P(n)$-convex.

Proof. (a) Let $X$ be a $P(\epsilon, n)$-convex space. Let $x_{1}^{**}, \ldots, x_{n}^{**} \in S_{X^{**}}$. We will show that there exist $1 \leq i, j \leq n, i \neq j$, such that $S(x_{i}^{**}, -x_{j}^{**}, \epsilon/4) = \emptyset$. Since $X$ is $P$-convex, it is also reflexive. Therefore $x_{i}^{**} = j(x_{i})$, $x_{j}^{**} = j(x_{j})$ for some $x_{i}, \ldots, x_{n} \in S_X$, where $j$ is the canonical injection from $X$ to $X^{**}$. By hypothesis, there exist $1 \leq i, j \leq n, i \neq j$, such that $\|x_{i} - x_{j}\| \leq 2 - \epsilon$. Therefore it is enough to prove that

$$\left\{ f \in B_{X^*} : f(x_{i}) \geq 1 - \frac{\epsilon}{4}, -f(x_{j}) \geq 1 - \frac{\epsilon}{4} \right\} = \emptyset.$$  \hfill (2.18)

We proceed by contradiction supposing that there exists $f \in B_X$ such that $f(x_i) \geq 1 - \epsilon/4$ and $-f(x_j) \geq 1 - \epsilon/4$. Then

$$2 - \epsilon \geq \|x_{i} - x_{j}\| \geq f(x_{i}) - f(x_{j}) \geq 2 - \frac{\epsilon}{2},$$  \hfill (2.19)

which is not possible; consequently $X^*$ is $P(\epsilon/4, n)$-smooth.

Now let $X$ be a Banach space such that $X^*$ is $P(\epsilon, n)$-smooth. Let $x_{1}, \ldots, x_{n} \in S_X$. By hypothesis, there exist $1 \leq i, j \leq n, i \neq j$, such that $S(j(x_{i}), -j(x_{j}), \epsilon) = \emptyset$, that is, for each $f \in B_{X^*}$ we have $f(x_{i}) < 1 - \epsilon$ or $-f(x_{j}) < 1 - \epsilon$. We will see that $\|x_{i} - x_{j}\| \leq 2 - \epsilon$. We again proceed by contradiction supposing that $\|x_{i} - x_{j}\| = \|j(x_{i}) - j(x_{j})\| > 2 - \epsilon$. There exists $f \in S_X$ such that $j(x_{i} - x_{j})(f) = f(x_{i}) - f(x_{j}) > 2 - \epsilon$. If $f(x_{i}) < 1 - \epsilon$, then

$$1 = \|f\|\|x_{j}\| \geq -f(x_{j}) > 2 - \epsilon - f(x_{i}) > 1$$  \hfill (2.20)

which is not possible. Similarly if $-f(x_{j}) < 1 - \epsilon$, we obtain a contradiction. Thus $\|x_{i} - x_{j}\| \leq 2 - \epsilon$, and consequently $X$ is $P(\epsilon, n)$-convex. The proof of (b) is analogous to the proof of (a).

Therefore the conditions $X$ is $P(\epsilon)$-smooth and $F(n, X) > 0$ must be equivalent.

3. $p$-Convex Banach Spaces

In this section we introduce the nonuniform version of $P$-convexity and we call it $p$-convexity.
Definition 3.1. Let $X$ be a Banach space and $n \in \mathbb{N}$. $X$ is said to be $p$ ($n$)-convex if for any $x_1, \ldots, x_n \in S_X$, there exist $1 \leq i, j \leq n$, $i \neq j$, such that $\|x_i - x_j\| < 2$. $X$ is said to be $p$-convex if is $p(n)$-convex for some $n \in \mathbb{N}$.

Kottman defined the concept of $P$-convexity in terms of the intersection of balls. We will do something similar to give an equivalent definition of $p$-convexity. It is easy to see that in a normed space any two closed balls of radius 1/2 contained in the unit ball have non empty intersection. If the radius is less than 1/2, for example, in $l_1$ for every $n$ and for every $r < 1/2$, then there exist $n$ closed balls of radius $r$ so that no two of them intersect. In fact let $\{e_i\}_{i=1}^\infty$ be the canonical basis of $l_1$. Then the closed balls of radius $r < 1/2$ centered at the points $(1/2)e_i$, $i \in \mathbb{N}$ are disjoint and contained in the unit ball. However, if $X$ is $p(n)$-convex, we will see that for any $n$ points in the unit ball there exists $r < 1/2$ so that if the $n$ closed balls centered at these $n$ points are contained in the unit ball, there are two different balls with non empty intersection. To prove this we need the following lemma, which was shown in [11].

Lemma 3.2. Let $X$ be a Banach space and $x, y \in X$, $x, y \neq 0$. Then

$$
\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \geq \frac{1}{\min\{\|x\|, \|y\|\}} (\|x - y\| - \|x\| - \|y\|).$

(3.1)

Lemma 3.3. $X$ is a $p(n)$-convex space if and only if for any $y_1, \ldots, y_n \in B_X$ there exists $r \in (0, 1/2)$ such that, if $B(y_i, r) \subset B_X$ for all $i = 1, \ldots, n$, then there are $1 \leq i, j \leq n$, $i \neq j$, so that

$$
B(y_i, r) \cap B(y_j, r) \neq \emptyset.
$$

(3.2)

Proof. Assume that $X$ satisfies condition (3.2), and let $x_1, \ldots, x_n \in S_X$. Let $r \in (0, 1/2)$ be the number which satisfies condition (3.2) for $x_1/2, \ldots, x_n/2$. It is easy to see that $B(x_i/2, r) \subset B_X$ for each $i = 1, \ldots, n$. Therefore there exist $1 \leq i, j \leq n$, $i \neq j$, such that

$$
B\left(\frac{x_i}{2}, r\right) \cap B\left(\frac{x_j}{2}, r\right) \neq \emptyset.
$$

(3.3)

Let

$$
y \in B\left(\frac{x_i}{2}, r\right) \cap B\left(\frac{x_j}{2}, r\right).
$$

(3.4)

We have

$$
\left\| \frac{x_i - x_j}{2} \right\| \leq \left\| \frac{x_i}{2} - y \right\| + \left\| \frac{x_j}{2} - y \right\| < 2r < 1,
$$

(3.5)

and thus $X$ is $p(n)$-convex. Now we suppose that there exist $y_1, \ldots, y_n \in B_X$ such that for any $\rho \in (0, 1/2)$ we have

$$
B\left(\frac{y_i}{2} - \rho\right) \subset B_X
$$

(3.6)
for all $i = 1, \ldots, n$, and
\[
B\left( y_i, \frac{1}{2} - \rho \right) \cap B\left( y_j, \frac{1}{2} - \rho \right) = \emptyset,
\]
for all $i, j = 1, \ldots, n$, $i \neq j$. We verify that $X$ is not $p(n)$-convex in four steps.

(a) Take $\|y_i - y_j\| > 1 - 2\rho$ for any $i, j = 1, \ldots, n$, $i \neq j$.

(b) Take $1/2 - 3\rho < \|y_i\| \leq 1/2 + \rho$, for all $i = 1, \ldots, n$. To verify this claim we note that $\|y_i/\|y_i\| - y_i\| \geq 1/2 - \rho$ for all $i$, because if $\|y_i/\|y_i\| - y_i\| < 1/2 - \rho$ for some $i$, then $y_i/\|y_i\| \in \text{int } B(y_i, 1/2 - \rho) \subset \text{int } B_X$, which is not possible. Hence, as $\|y_i/\|y_i\| - y_i\| = 1 - \|y_i\|$, it follows that $\|y_i\| = 1 - \|y_i/\|y_i\| - y_i\| \leq 1/2 + \rho$, for each $i = 1, \ldots, n$. Now, if $\|y_i\| \leq 1/2 - 3\rho$ for some $i$, we have by (a) that for any $j \neq i$, $1 - 2\rho < \|y_i - y_j\| \leq \|y_i\| + \|y_j\| \leq (1/2 - 3\rho) + (1/2 + \rho) = 1 - 2\rho$ which is not possible.

(c) Take $\|y_i\| - \|y_j\| < 4\rho$, for any $i, j = 1, \ldots, n$, $i \neq j$. Indeed, by (b) we get $-4\rho = (1/2 - 3\rho) - (1/2 + \rho) < \|y_i\| - \|y_j\| < (1/2 + \rho) - (1/2 - 3\rho) = 4\rho$.

(d) From (a), (b), (c), and by Lemma 3.2, we have
\[
\left\| \frac{y_i}{\|y_i\|} - \frac{y_j}{\|y_j\|} \right\| \geq \frac{1}{\|y_i\|} (\|y_i - y_j\| - \|y_i\| - \|y_j\|) > 2 - \frac{16\rho}{1 + 2\rho}
\]
for any $i, j = 1, \ldots, n$, $i \neq j$. Since $\rho > 0$ is arbitrary, as $\rho \to 0$, we obtain $\|y_i/\|y_i\| - y_j/\|y_j\|| = 2$, for all $i, j = 1, \ldots, n$, $i \neq j$, and thus $X$ is not $p(n)$-convex.

Next we give some examples of spaces which are not $p$-convex. The first is not reflexive and the last one is superreflexive.

**Example 3.4.** $c_0$, and consequently, $C[0,1]$ and $l_\infty$ are not $p$-convex spaces. Indeed, let $\{e_i\}_{i=1}^\infty$ be the canonical basis in $c_0$. For each $n \in \mathbb{N}$ we define $u_i = \sum_{j=1}^n \lambda_{i,j} e_j$, where $\lambda_{i,j} = 1$ if $j \neq i$, $\lambda_{i,i} = -1$, and $i = 1, \ldots, n$. Clearly $u_1, \ldots, u_n \in S_{c_0}$, and for each $i \neq j$ we have $\|u_i - u_j\|_\infty = 2$.

**Example 3.5.** Let $X$ denote the space obtained by renorming $l_2$ as follows. For $x = (x_i)_{i \in \mathbb{N}} \in l_2$ set
\[
\|x\| = \max \left\{ \sup_{i,j} \left| x_i - x_j \right|, \left( \sum_{i=1}^\infty x_i^2 \right)^{1/2} \right\}.
\]
Then $\|x\| \leq \|x\| \leq \sqrt{2}\|x\|$, where $\| \cdot \|$ stands for the $l_2$-norm and $X$ is superreflexive. On the other hand, the canonical basis $\{e_n\}_{n}$ in $l_2$ satisfies $\|e_i - e_j\|_\infty = 2$ for each $i \neq j$. Thus $X$ is not $p$-convex.

Now we will mention several properties that imply $p$-convexity.
Recall the following concepts. Let $X$ be a Banach space. $X$ is said to be a $u$-space if it satisfies the following implication:

$$x, y \in S_X, \quad \left\| \frac{x + y}{2} \right\| = 1 \implies \nabla(x) = \nabla(y).$$  \hspace{1cm} (3.10)

$X$ is said to be smooth if for any $x \in S_X$, there exists a unique $f \in S_X$ such that $f(x) = 1$. That is, for each $x \in S_X$, $\nabla(x)$ contains a single point. $X$ is called strictly convex if the following implication holds:

$$\forall x, y \in B_X : x \neq y \implies \left\| \frac{x + y}{2} \right\| < 1.$$  \hspace{1cm} (3.11)

**Proposition 3.6.** Every smooth space, every strictly convex space and every $u$-space are $p(3)$-convex space.

**Proof.** Every smooth space and every strictly convex space are $u$-space. It suffices to show that $p(3)$-convexity follows from being $u$-space. If $X$ is a $u$-space, then for any $x, y \in S_X$ the following inequality holds: $\min\{\|x - y\|, \|x + y\|\} < 2$. Indeed, if we suppose that there exist $x, y \in S_X$ such that $\|x + y\| = \|x - y\| = 2$, then $\nabla(x) = \nabla(y)$ and $\nabla(x) = \nabla(-y)$, which is not possible. Suppose that $X$ is not $p(3)$-convex, and there exist $x, y, z \in S_X$ so that $\|x - y\| = \|y - z\| = \|z - x\| = 2$. Since $(1/2)\|x - y\| = (1/2)\|y - z\| = 1$, we have $\nabla(x) = \nabla(-y) = \nabla(z)$. Let $f \in \nabla(-y)$; then $f(x + z) \leq \|x + z\| < 2$, and

$$2 = f(x) + f(-y) = f(x + z) - f(z) + f(-y) = f(x + z) < 2.$$  \hspace{1cm} (3.12)

Thus $X$ is $p(3)$-convex.

Obviously $P$-convexity implies $p$-convexity; however, a $p$-convex space is not necessarily $P$-convex, even if the space is reflexive as the following example shows.

**Example 3.7.** Let $\{r_k\}_{k=1}^{\infty}$ be a sequence of real numbers such that $r_k > 1$ for each $k \in \mathbb{N}$ and $r_k \downarrow 1$, when $k \to \infty$. Consider the space $X = \sum_{k=1}^{\infty} \otimes_2 I_{r_k}$. It is known that this space is strictly convex, hence it is also $p(3)$-convex. It is also known that $X$ is reflexive. However $X$ is not $P$-convex. Indeed, let $\varepsilon > 0$. We choose $k \in \mathbb{N}$ such that $2 - \varepsilon < 2^{1/r_k}$. If $\{e_i\}_{i=1}^{\infty}$ is the canonical basis of $I_{r_k}$, we have that $\|e_i - e_j\| = 2^{1/r_k} > 2 - \varepsilon$ for all $i, j \in \mathbb{N}$, $i \neq j$, and hence $X$ is not a $P$-convex space.

We have obtained a result which shows a strong relation between $P$-convexity and $p$-convexity with respect to the ultrapower of Banach spaces. We recall the definition and some results regarding ultrapowers which can be found in [4].

A filter $\mathcal{U}$ on $I$ is called an ultrafilter on $I$ if $\mathcal{U}$ is a maximal element from $\mathcal{P}$ with respect to the set inclusion. $\mathcal{U}$ is an ultrafilter on $I$ if and only if for all $A \subset I$ either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$. Let $\{X_i\}_{i \in I}$ be a family of Banach spaces, and let

$$l_\infty(X_i) = \left\{ \{x_i\}_{i \in I} \in \prod_{i \in I} X_i : \sup\{\|x_i\|_{X_i} : i \in I\} < \infty \right\}.$$  \hspace{1cm} (3.13)
If we define \( \| \{ x_i \}_{i \in I} \|_\infty = \sup \{ \| x_i \|_{X_i} : i \in I \} \) for each \( \{ x_i \}_{i \in I} \in l_\infty (X_i) \), then \( \| \cdot \|_\infty \) defines a norm in \( l_\infty (X_i) \), and \( l_\infty (X_i), \| \cdot \|_\infty \) is a Banach space. If \( \mathcal{U} \) is a free ultrafilter on \( I \), then for each \( \{ x_i \}_{i \in I} \in l_\infty (X_i) \) we have \( \lim_{\mathcal{U}} x_i \) always exists and is unique. Let \( \mathcal{U} \) be an ultrafilter on \( I \), and define

\[
\mathcal{A}_\mathcal{U} = \left\{ \{ x_i \} \in l_\infty (X_i) : \lim_{\mathcal{U}} \| x_i \| = 0 \right\}.
\] (3.14)

\( \mathcal{A}_\mathcal{U} \) is a closed subspace of \( l_\infty (X_i) \). The ultraproduct of \( \{ X_i \}_{i \in I} \) with respect to the ultrafilter \( \mathcal{U} \) on \( I \) is the quotient space \( l_\infty (X_i) / \mathcal{A}_\mathcal{U} \) equipped with the quotient norm, which is denoted by \( \{ X_i \}_\mathcal{U} \) and its elements by \( \{ x_i \}_\mathcal{U} \). If \( X_i = X \) for all \( i \in I \), then \( \{ X_i \}_\mathcal{U} = \{ X_i \}_\mathcal{U} \) is called the ultraproduct of \( X \). The quotient norm in \( \{ X_i \}_\mathcal{U} \),

\[
\| \{ x_i \}_\mathcal{U} \| = \inf \{ \| \{ x_i + y_i \}_\mathcal{U} \|_\infty : \{ y_i \}_\mathcal{U} \in \mathcal{A}_\mathcal{U} \},
\] (3.15)

satisfies the equality

\[
\| \{ x_i \}_\mathcal{U} \| = \lim_{\mathcal{U}} \| x_i \|_{X_i}, \quad \text{for each} \{ x_i \}_\mathcal{U} \in \{ X_i \}_\mathcal{U}.
\] (3.16)

If \( \mathcal{U} \) is nontrivial, then \( X \) can be embedded into \( \{ X \}_\mathcal{U} \) isometrically. We will write \( \tilde{X}_i \) instead of \( \{ X_i \}_\mathcal{U} \) and \( \tilde{x} \) instead of \( \{ x_i \}_\mathcal{U} \) unless we need to specify the ultrafilter we are talking about.

It is known that \( X \) is uniformly convex if and only if \( \tilde{X} \) is strictly convex, \( X \) is uniformly smooth if and only if \( \tilde{X} \) is smooth, and \( X \) is a \( \mathcal{U} \)-space if and only if \( \tilde{X} \) is a \( \mathcal{U} \)-space (see [12]). Similarly we obtain the following result.

**Theorem 3.8.** Let \( X \) be a Banach space and \( m \in \mathbb{N} \). The following are equivalent:

(a) \( \tilde{X} \) is \( P(m) \)-convex.

(b) \( X \) is \( P(m) \)-convex,

(c) \( \tilde{X} \) is \( p(m) \)-convex,

**Proof.** (a) \( \Rightarrow \) (b). Let \( \{ x_i^{(n)} \}_n \in \tilde{X}, x_i \in S_{\tilde{X}}, i = 1, \ldots, m \). Since \( \lim_{\mathcal{U}} \| x_i^{(n)} \|_X = \| \tilde{x} \|_\tilde{X} = 1 \) for all \( i \), there exists a subsequence \( \{ x_i^{(n)} \}_k \) of \( \{ x_i^{(m)} \}_n \) such that \( \lim_{k \to \infty} \| x_i^{(n)} \|_X = 1 \) and \( \| x_i^{(n)} \|_X > 0 \), for all \( k \in \mathbb{N} \). Define

\[
y_i^{(m)} = \frac{x_i^{(m)}}{\left\| x_i^{(m)} \right\|_X}, \quad \Gamma_{i,j} = \left\{ k \in \mathbb{N} : \left\| y_i^{(m)} - y_j^{(m)} \right\|_X \leq 2 - \| x_i^{(n)} - x_j^{(n)} \|_X \right\},
\] (3.17)

for each \( i, j = 1, \ldots, m, i \neq j \). We verify that there exist \( 1 \leq i, j \leq m, i \neq j, \) such that \( \Gamma_{i,j} \in \mathcal{U} \). We proceed by contradiction assuming that, \( \Gamma_{i,j} \notin \mathcal{U} \) for all \( i \neq j \). Hence \( \mathbb{N} \setminus \bigcup_{i \neq j} \Gamma_{i,j} \neq \emptyset \), therefore there exists \( k_0 \in \mathbb{N} \setminus \left( \bigcup_{i \neq j} \Gamma_{i,j} \right) \). Thus we have \( \| y_i^{(m)} - y_j^{(m)} \| > 2 - \varepsilon \) for each \( i \neq j \), and \( X \) is not \( P(m) \)-convex, which is a contradiction.
Therefore there exist $1 \leq i, j \leq m, i \neq j$, such that $\Gamma_{ij} \in \mathcal{U}$, and hence $\lim_{\mathcal{U}} \| y^{(n_i)}_i - y^{(n_j)}_j \|_X \leq 2 - \varepsilon$. Finally, note that

$$
\| x^{(n_i)}_i - x^{(n_j)}_j \|_X \leq \| x^{(n_i)}_i - y^{(n_i)}_i \|_X + \| y^{(n_i)}_i - x^{(n_j)}_j \|_X + \| y^{(n_j)}_i - y^{(n_j)}_j \|_X
$$

(3.18)

Therefore $\tilde{X}$ is $P(m)$-convex.

(b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (a). Suppose that $X$ is not $P(m)$-convex. Hence for any $n \in \mathbb{N}$ there exist $x^{(n)}_i, \ldots, x^{(n)}_m \in S_X$ such that $\| x^{(n)}_i - x^{(n)}_j \|_X > 2 - 1/n$ for all $i, j = 1, \ldots, m, i \neq j$. Define $\tilde{x}_i = \{ x^{(n)}_i \}_{\mathcal{U}}$ for each $i = 1, \ldots, m$. Clearly $\tilde{x}_i \in S_{\tilde{X}}$ for all $i$, because $\| \tilde{x}_i \|_{\tilde{X}} = \lim_{\mathcal{U}} \| x^{(n)}_i \|_X = 1$, and also,

$$
\| \tilde{x}_i - \tilde{x}_j \|_{\tilde{X}} = \lim_{\mathcal{U}} \| x^{(n)}_i - x^{(n)}_j \|_X = \lim_{n \to \infty} \| x^{(n)}_i - x^{(n)}_j \|_X = 2,
$$

(3.19)

for each $i \neq j$. Hence $\tilde{X}$ is not $p(m)$-convex.

By the above theorem we can deduce the following known result.

**Corollary 3.9.** If $X$ is $P$-convex, then $X$ is superreflexive.

**Proof.** If $X$ is $P$-convex, then $\tilde{X}$ is $P$-convex and therefore is reflexive. However in ultrapower reflexivity and superreflexivity are equivalent, hence $\tilde{X}$ is superreflexive, and consequently $X$ is superreflexive. □

Now we turn our attention to some results regarding the $p$-convexity and the $P$-convexity of quotient spaces. To prove them we need the following concept.

**Definition 3.10.** A subspace $Y$ of a normed space $X$ is said to be proximinal if for all $x \in X$ there exists $y \in Y$ such that $d(x, Y) = \| x - y \|$. It is easy to see that every proximinal subspace $Y$ of a Banach space $X$ is closed.

**Proposition 3.11.** If $X$ is $p(n)$-convex and $Y$ is a proximinal subspace of $X$, then $X/Y$ is $p(n)$-convex.

**Proof.** Let $q : X \to X/Y$ be the quotient function. By the proximinality of $Y$ we have $q(B_X) = B_{X/Y}$. Let $\tilde{x}_1, \ldots, \tilde{x}_n \in S_{X/Y}$ and $x_1, \ldots, x_n \in S_X$ such that $\tilde{x}_i = q(x_i)$. Since $X$ is $p(n)$-convex, there exist $1 \leq i, j \leq n, i \neq j$, such that $\| x_i - x_j \| < 2$, and consequently $\| \tilde{x}_i - \tilde{x}_j \| < 2$. □

**Corollary 3.12.** Let $X$ be $p(n)$-convex and reflexive. If $Y$ is a closed subspace of $X$, then $X/Y$ is $p(n)$-convex.
Proof. It is shown in [13] that a Banach space $X$ is reflexive if and only if each closed subspace of $X$ is proximinal, and thus the corollary is a consequence of Proposition 3.11.

Similarly we can prove that if $X$ is $P(\varepsilon, n)$-convex and $Y$ is a closed subspace of $X$, then $X/Y$ is $P(\varepsilon, n)$-convex.

We obtain two results involving $q$-direct sums of $p$-convex spaces. Next we will define these sums as in [14] by Saito , et al.

**Definition 3.13.** Set $\Psi = \{q : [0, 1] \rightarrow \mathbb{R} \mid q$ is a continuous convex function, $\max\{1 - t, t\} \leq q(t) \leq 1, \text{ for all } 0 \leq t \leq 1\}$

Let $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ be Banach spaces. For each $q \in \Psi$, one defines the norm $\| \cdot \|_q$ in $X \oplus Y$ as $\|(0, 0)\|_q = 0$ and for each $(x, y) \neq (0, 0)$

$$\|(x, y)\|_q = \left( \|x\|_X + \|y\|_Y \right) q \left( \frac{\|y\|_Y}{\|x\|_X + \|y\|_Y} \right).$$

(3.20)

In [15] it is shown that $(X \oplus Y, \| \cdot \|_q)$ is a Banach space, denoted by $X \oplus_q Y$ called the $q$-direct and sum of $X$ and $Y$.

The proof of the following theorem is similar to the proof of Theorem 3.5 in [16], which shows the corresponding result for $P$-convex spaces.

**Theorem 3.14.** Let $X$ and $Y$ be Banach spaces and $q \in \Psi$. Then $X \oplus_q Y$ is $p$-convex if and only if $X$ and $Y$ are $p$-convex.

In [17] there is a theorem stating several equivalent conditions for strict convexity. We prove a similar result for $p$-convexity.

**Lemma 3.15.** Let $X$ be a Banach space. The next assertions are equivalent.

(a) $X$ is $p(n)$-convex.

(b) For any $q \in (1, \infty)$ and for any $x_1, \ldots, x_n \in X$, not all zero, there exist $1 \leq i, j \leq n, i \neq j$, such that $\|x_i - x_j\| < 2^{(q-1)/q}(\|x_i\|^q + \|x_j\|^q)^{1/q}$.

(c) For some $q \in (1, \infty)$ and for any $x_1, \ldots, x_n \in X$, not all zero, there exist $1 \leq i, j \leq n, i \neq j$, such that $\|x_i - x_j\| < 2^{(q-1)/q}(\|x_i\|^q + \|x_j\|^q)^{1/q}$.

Proof. The implications (b) $\Rightarrow$ (c) $\Rightarrow$ (a) are immediate. We verify (a) $\Rightarrow$ (b). Let $q \in (1, \infty)$ and $x_1, \ldots, x_n \in X$, not all zero. If $x_j = 0$ and $x_i \neq 0$ for some $1 \leq i, j \leq n$, then it is clear that $\|x_i - x_j\| < 2^{(q-1)/q}(\|x_i\|^q + \|x_j\|^q)^{1/q}$. Suppose that $x_1, \ldots, x_n \in X \setminus \{0\}$. There exist $1 \leq i, j \leq n, i \neq j$, such that

$$\left\| \frac{x_i}{\|x_i\|} - \frac{x_j}{\|x_j\|} \right\| < 2. \quad (3.21)$$

If $\|x_j\| < \|x_i\|$ by Lemma 3.2 we get

$$\|x_i - x_j\| \leq \|x_j\| \left\| \frac{x_i}{\|x_i\|} - \frac{x_j}{\|x_j\|} \right\| + \|x_i\| + \|x_j\| < \|x_i\| + \|x_j\|. \quad (3.22)$$
As the function \( t \mapsto t^q \) is convex we obtain that
\[
\left\| \frac{x_i - x_j}{2} \right\|^q < \left( \frac{\|x_i\| + \|x_j\|}{2} \right)^q \leq \frac{1}{2} \left( \|x_i\|^q + \|x_j\|^q \right).
\]  
(3.23)
Thus \( \|x_i - x_j\| < 2^{(q-1)/q} (\|x_i\|^q + \|x_j\|^q)^{1/q} \).

**Proposition 3.16.** Let \( \{X_i\}_{i \in I} \) be a family of \( p(n) \)-convex spaces, where the index set \( I \neq \emptyset \) has any cardinality. Then the space \( X = \bigcup_{i \in I} X_i \) is \( p(n) \)-convex.

**Proof.** Let \( x^{(k)} = \{x^{(k)}_i\}_{i \in I} \in X \), 1 \( \leq k \leq n \), not all zero. Let \( i_0 \in I \) be such that \( x^{(k)}_{i_0} \neq 0 \), for some \( k \in \{1, \ldots, n\} \). As \( X_{i_0} \) is a \( p(n) \)-convex space, we have by the preceding lemma that there exist \( 1 \leq l, m \leq n \) such that
\[
\left\| x^{(l)}_{i_0} - x^{(m)}_{i_0} \right\|^q < 2^{q-1} \left( \left\| x^{(l)}_{i_0} \right\|^q + \left\| x^{(m)}_{i_0} \right\|^q \right).
\]  
(3.24)
By the above we obtain
\[
\left\| x^{(l)} - x^{(m)} \right\|^q = \sum_{i \in I} \left\| x^{(l)}_i - x^{(m)}_i \right\|^q
\]
\[
< \sum_{i \in I} 2^{q-1} \left( \left\| x^{(l)}_i \right\|^q + \left\| x^{(m)}_i \right\|^q \right) = 2^{q-1} \left( \left\| x^{(l)} \right\|^q + \left\| x^{(m)} \right\|^q \right).
\]  
(3.25)
Therefore, by the previous lemma, \( X \) is \( p(n) \)-convex.

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**References**


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