Research Article

Sharp Power Mean Bounds for the Combination of Seiffert and Geometric Means

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We answer the question: for $\alpha \in (0, 1)$, what are the greatest value p and the least value q such that the double inequality $M_p(a, b) < P^\alpha(a, b)G^{1-\alpha}(a, b) < M_q(a, b)$ holds for all $a, b > 0$ with $a \neq b$. Here, $M_p(a, b)$, $P(a, b)$, and $G(a, b)$ denote the power of order p , Seiffert, and geometric means of two positive numbers a and b , respectively.

1. Introduction

For $p \in \mathbb{R}$, the power mean $M_p(a, b)$ of order p and the Seiffert mean $P(a, b)$ of two positive numbers a and b are defined by

$$M_p(a, b) = \begin{cases} \left(\frac{a^p + b^p}{2} \right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0, \end{cases} \quad (1.1)$$

$$P(a, b) = \begin{cases} \frac{a - b}{4 \arctan(\sqrt{a/b}) - \pi}, & a \neq b, \\ a, & a = b. \end{cases}$$

The main properties of the power mean are given in [1]. It is well known that $M_p(a, b)$ is strictly increasing with respect to p for fixed $a, b > 0$ with $a \neq b$. Recently, the power mean has been the subject of intensive research. In particular, many remarkable inequalities for the power mean can be found in the literature [2–16].

The Seiffert mean P was introduced by Seiffert in [17], it can be rewritten in the following symmetric form (see [18, (2.4)]):

$$P(a, b) = \begin{cases} \frac{a-b}{2 \arcsin((a-b)/(a+b))}, & a \neq b, \\ a, & a = b. \end{cases} \quad (1.2)$$

Let $A(a, b) = (1/2)(a + b)$, $G(a, b) = \sqrt{ab}$, $L(a, b) = \begin{cases} (b-a)/(\log b - \log a), & b \neq a, \\ a, & b = a, \end{cases}$ $H(a, b) = 2ab/(a + b)$ and $I(a, b) = \begin{cases} 1/e(b^b/a^a)^{1/(b-a)}, & b \neq a, \\ a, & b = a, \end{cases}$ be the arithmetic, geometric, logarithmic, harmonic, and identric means of two positive numbers a and b , respectively. Then it is well known that

$$\begin{aligned} \min\{a, b\} < H(a, b) = M_{-1}(a, b) < G(a, b) = M_0(a, b) \\ < L(a, b) < I(a, b) < A(a, b) = M_1(a, b) < \max\{a, b\} \end{aligned} \quad (1.3)$$

for all $a, b > 0$ with $a \neq b$.

In [9], Alzer and Janous presented the sharp power mean bounds for the sum $(2/3)A(a, b) + (1/3)G(a, b)$ as follows:

$$M_{\log 2 / \log 3}(a, b) < \frac{2}{3}A(a, b) + \frac{1}{3}G(a, b) < M_{2/3}(a, b) \quad (1.4)$$

for all $a, b > 0$ with $a \neq b$.

In [17], Seiffert proved that

$$L(a, b) < P(a, b) < I(a, b) \quad (1.5)$$

for all $a, b > 0$ with $a \neq b$.

The following power mean bounds for the Seiffert mean was given by Jagers [19]:

$$M_{1/2}(a, b) < P(a, b) < M_{2/3}(a, b) \quad (1.6)$$

for all $a, b > 0$ with $a \neq b$.

In [20, 21], the authors presented the bounds for the Seiffert mean P in terms of A and G as follows:

$$\begin{aligned} P(a, b) &> \frac{3A(a, b)G(a, b)}{A(a, b) + 2G(a, b)}, \\ \frac{1}{2}A(a, b) + \frac{1}{2}G(a, b) &< P(a, b) < \frac{2}{3}A(a, b) + \frac{1}{3}G(a, b), \\ P(a, b) &> A^{1/3}(a, b)G^{2/3}(a, b) \end{aligned} \quad (1.7)$$

for all $a, b > 0$ with $a \neq b$.

The following sharp lower power mean bounds for $(1/3)G(a, b) + (2/3)H(a, b)$, $(2/3)G(a, b) + (1/3)H(a, b)$, and $P(a, b)$ can be found in [4, 6]:

$$\begin{aligned}\frac{1}{3}G(a, b) + \frac{2}{3}H(a, b) &> M_{-2/3}(a, b), \\ \frac{2}{3}G(a, b) + \frac{1}{3}H(a, b) &> M_{-1/3}(a, b), \\ P(a, b) &> M_{\log 2 / \log \pi}(a, b)\end{aligned}\tag{1.8}$$

for all $a, b > 0$ with $a \neq b$.

The purpose of this paper is to answer the question: for $\alpha \in (0, 1)$, what are the greatest value p and the least value q such that the double inequality $M_p(a, b) < P^\alpha(a, b)G^{1-\alpha}(a, b) < M_q(a, b)$ holds for all $a, b > 0$ with $a \neq b$.

2. Lemmas

In order to prove our main result, we need several lemmas which we present in this section.

Lemma 2.1. *Let $\lambda \in (0, 1/3)$, $x \in [1, \infty)$ and $h(x) = (1-3\lambda)x^{2\lambda+1} - (1+3\lambda)x^{2\lambda} - (1+3\lambda)x + (1-3\lambda)$. Then there exists $x_0 > 1$ such that $h(x) < 0$ for $x \in [1, x_0)$, $h(x) > 0$ for $x \in (x_0, \infty)$ and $h(x_0) = 0$.*

Proof. Simple computations lead to

$$h(1) = -12\lambda < 0, \tag{2.1}$$

$$\lim_{x \rightarrow +\infty} h(x) = +\infty, \tag{2.2}$$

$$\begin{aligned}h'(x) &= (1-3\lambda)(2\lambda+1)x^{2\lambda} - 2\lambda(1+3\lambda)x^{2\lambda-1} - (1+3\lambda), \\ h'(1) &= -6\lambda(1+2\lambda) < 0,\end{aligned}\tag{2.3}$$

$$\lim_{x \rightarrow +\infty} h'(x) = +\infty, \tag{2.4}$$

$$h''(x) = 2\lambda x^{2\lambda-2}[(1+2\lambda)(1-3\lambda)x + (1-2\lambda)(1+3\lambda)] > 0. \tag{2.5}$$

Inequality (2.5) implies that $h'(x)$ is strictly increasing in $[1, \infty)$. Then (2.3) and (2.4) lead to that there exists $x_1 > 1$ such that $h'(x) < 0$ for $x \in [1, x_1)$ and $h'(x) > 0$ for $x \in (x_1, \infty)$. Hence, $h(x)$ is strictly decreasing in $[1, x_1]$ and strictly increasing in $[x_1, \infty)$.

Therefore, Lemma 2.1 follows from (2.1) and (2.2) together with the monotonicity of $h(x)$. $\square$

Lemma 2.2. *If $\lambda \in (1/\sqrt{10}, 1/3)$, then the following statements are true:*

- (1) $300\lambda^4 + 324\lambda^3 + 29\lambda^2 - 45\lambda - 8 < 0$;
- (2) $-1176\lambda^5 - 24\lambda^4 + 114\lambda^3 + 10\lambda^2 - 3\lambda - 1 < 0$;
- (3) $-24\lambda^5 + 72\lambda^4 + 178\lambda^3 + 6\lambda^2 - 25\lambda - 3 < 0$.

Proof. Simple computations lead to

- (1) $300\lambda^4 + 324\lambda^3 + 29\lambda^2 - 45\lambda - 8 < 300 \times (1/3)^4 + 324 \times (1/3)^3 + 29 \times (1/3)^2 - 45/\sqrt{10} - 8 = (590 - 243\sqrt{10})/54 < 0;$
- (2) $-1176\lambda^5 - 24\lambda^4 + 114\lambda^3 + 10\lambda^2 - 3\lambda - 1 < -1176 \times (1/\sqrt{10})^5 - 24 \times (1/\sqrt{10})^4 + 114 \times (1/3)^3 + 10 \times (1/3)^2 - 3/\sqrt{10} - 1 = (3070 - 1107\sqrt{10})/750 < 0;$
- (3) $-24\lambda^5 + 72\lambda^4 + 178\lambda^3 + 6\lambda^2 - 25\lambda - 3 < -24 \times (1/\sqrt{10})^5 + 72 \times (1/3)^4 + 178 \times (1/3)^3 + 6 \times (1/3)^2 - 25/\sqrt{10} - 3 = (34750 - 17037\sqrt{10})/6750 < 0.$ $\square$

Lemma 2.3. *If $\alpha \in (0, 3/\sqrt{10}]$, then*

$$I^\alpha(a, b)G^{1-\alpha}(a, b) < M_{2\alpha/3}(a, b) \quad (2.6)$$

holds for all $a, b > 0$ with $a \neq b$.

Proof. Without loss of generality, we assume that $a > b$. Let $t = a/b > 1$ and $\beta = \alpha/3 \in (0, 1/\sqrt{10}]$. Then

$$\begin{aligned} & \log[M_{2\alpha/3}(a, b)] - \log[I^\alpha(a, b)G^{1-\alpha}(a, b)] \\ &= \log[M_{2\beta}(a, b)] - \log[I^{3\beta}(a, b)G^{1-3\beta}(a, b)] \\ &= \frac{1}{2\beta} \log \frac{1+t^{2\beta}}{2} - \frac{3\beta t}{t-1} \log t - \frac{1-3\beta}{2} \log t + 3\beta. \end{aligned} \quad (2.7)$$

Let

$$f(t) = \frac{1}{2\beta} \log \frac{1+t^{2\beta}}{2} - \frac{3\beta t}{t-1} \log t - \frac{1-3\beta}{2} \log t + 3\beta. \quad (2.8)$$

Then simple computations lead to

$$\lim_{t \rightarrow 1} f(t) = 0, \quad (2.9)$$

$$f'(t) = \frac{g(t)}{(t-1)^2}, \quad (2.10)$$

where $g(t) = (t^{2\beta+1} - 2t^{2\beta} + t^{2\beta-1}) / (1 + t^{2\beta}) - 3\beta(t-1) + 3\beta \log t - (1-3\beta)(t-2+1/t)/2$,

$$\begin{aligned} g(1) &= 0, \\ g'(t) &= \frac{g_1(t)}{2t^2(1+t^{2\beta})^2}, \end{aligned} \quad (2.11)$$

where $g_1(t) = (1-3\beta)t^{4\beta+2} + 6\beta t^{4\beta+1} - (1+3\beta)t^{4\beta} - 2\beta t^{2\beta+2} + 4\beta t^{2\beta+1} - 2\beta t^{2\beta} - (1+3\beta)t^2 + 6\beta t + (1-3\beta)$,

$$g_1(1) = 0, \quad (2.12)$$

$$g_1'(1) = 0, \quad (2.13)$$

$$g_1''(1) = 0. \quad (2.14)$$

Let $g_2(t) = g_1'''(t) / (8\beta t^{2\beta-3})$ and $g_3(t) = g_2'''(t) / (2\beta t^{2\beta-3})$. Then

$$\begin{aligned} g_2(t) &= (1-3\beta)(2\beta+1)(4\beta+1)t^{2\beta+2} + 3\beta(4\beta+1)(4\beta-1)t^{2\beta+1} \\ &\quad - (1+3\beta)(4\beta-1)(2\beta-1)t^{2\beta} - \beta(\beta+1)(2\beta+1)t^2 \\ &\quad + \beta(2\beta+1)(2\beta-1)t - \beta(\beta-1)(2\beta-1), \end{aligned} \quad (2.15)$$

$$g_2(1) = 0, \quad (2.16)$$

$$g_2'(1) = 2(1 - \sqrt{10}\beta)(1 + \sqrt{10}\beta) \geq 0, \quad (2.17)$$

$$g_2''(1) = 2(6\beta+1)(1 - \sqrt{10}\beta)(1 + \sqrt{10}\beta) \geq 0, \quad (2.18)$$

$$\begin{aligned} g_3(t) &= 2(1-3\beta)(2\beta+1)^2(4\beta+1)(\beta+1)t^2 + 3\beta(4\beta+1)(4\beta-1) \\ &\quad \times (2\beta+1)(2\beta-1)t - 2(1+3\beta)(4\beta-1)(2\beta-1)^2(\beta-1), \end{aligned} \quad (2.19)$$

$$g_3(1) = 9\beta(3 - 28\beta^2) > 0, \quad (2.20)$$

$$g_3'(t) = (2\beta+1)(4\beta+1)[4(1-3\beta)(2\beta+1)(\beta+1)t + 3\beta(4\beta-1)(2\beta-1)], \quad (2.21)$$

$$g_3'(1) = (2\beta+1)(4\beta+1)(4 + 3\beta - 46\beta^2), \quad (2.22)$$

$$\min_{\beta \in [0, 1/10]} (4 + 3\beta - 46\beta^2) = \frac{3\sqrt{10} - 6}{10} > 0. \quad (2.23)$$

From (2.21)–(2.23) we clearly see that $g_3'(t) > g_3'(1) \geq (3\sqrt{10} - 6)/10(2\beta+1)(4\beta+1) > 0$ for $t \in (1, \infty)$, hence $g_3(t)$ is strictly increasing in $[1, \infty)$. Then (2.20) implies that $g_3(t) > 0$ for $t \in (1, \infty)$, hence $g_2''(t)$ is strictly increasing in $[1, \infty)$.

It follows from (2.18) and the monotonicity of $g_2''(t)$ that $g_2''(t) > 0$ for $t \in (1, \infty)$, hence $g_2'(t)$ is strictly increasing in $[1, \infty)$. Then (2.17) implies that $g_2'(t) > 0$ for $t \in (1, \infty)$, therefore $g_2(t)$ is strictly increasing in $[1, \infty)$.

Equation (2.16) and the monotonicity of $g_2(t)$ lead to that $g_2(t) > 0$ for $t \in (1, \infty)$, so $g_1''(t)$ is strictly increasing in $[1, \infty)$.

From (2.9)–(2.14) and the monotonicity of $g_1''(t)$ we can deduce that

$$f(t) > 0 \quad (2.24)$$

for $t \in (1, \infty)$.

Therefore, Lemma 2.3 follows from (2.7) and (2.8) together with (2.24). $\square$

Remark 2.4. In [22, Theorem 3.1], the authors proved that

$$I^{1/2}(a, b)G^{1/2}(a, b) < M_{1/3}(a, b), \quad (2.25)$$

$$I^{2/3}(a, b)G^{1/3}(a, b) < M_{4/9}(a, b), \quad (2.26)$$

$$I^{1/3}(a, b)G^{2/3}(a, b) < M_{2/9}(a, b) \quad (2.27)$$

for all $a, b > 0$ with $a \neq b$.

Obviously, (2.6) is the generalization of (2.25)–(2.27).

Remark 2.5. If $\alpha \in (3/\sqrt{10}, 1)$, then $\beta = \alpha/3 \in (1/\sqrt{10}, 1/3)$ and (2.17) leads to

$$g'_2(1) = 2(1 - \sqrt{10}\beta)(1 + \sqrt{10}\beta) < 0. \quad (2.28)$$

Inequality (2.28) and the continuity of $g'_2(t)$ imply that there exists $\delta > 0$ such that

$$g'_2(t) < 0 \quad (2.29)$$

for $t \in [1, 1 + \delta)$.

From (2.29) and (2.9)–(2.16) we can deduce that

$$f(t) < 0 \quad (2.30)$$

for $t \in (1, 1 + \delta)$.

Equations (2.7) and (2.8) together with (2.30) lead to

$$I^\alpha(a, b)G^{1-\alpha}(a, b) > M_{2\alpha/3}(a, b) \quad (2.31)$$

for all $a, b > 0$ with $a/b \in (1, 1 + \delta) \cup (1/(1 + \delta), 1)$.

Therefore, $\alpha_0 = 3/\sqrt{10}$ is the largest value in $(0, 1)$ such that inequality (2.6) holds for $\alpha \in (0, \alpha_0]$.

3. Main Result

Theorem 3.1. *If $\alpha \in (0, 1)$, then*

$$M_0(a, b) < P^\alpha(a, b)G^{1-\alpha}(a, b) < M_{2\alpha/3}(a, b) \quad (3.1)$$

holds for all $a, b > 0$ with $a \neq b$, and $M_0(a, b)$ and $M_{2\alpha/3}(a, b)$ are the best possible lower and upper power mean bounds for the product $P^\alpha(a, b)G^{1-\alpha}(a, b)$.

Proof. For all $a, b > 0$ with $a \neq b$, from (1.3), (1.5) and Lemma 2.1 we clearly see that

$$P^\alpha(a, b)G^{1-\alpha}(a, b) > M_0(a, b) \quad (3.2)$$

for all $\alpha \in (0, 1)$, and

$$P^\alpha(a, b)G^{1-\alpha}(a, b) < M_{2\alpha/3}(a, b) \quad (3.3)$$

for $\alpha \in (0, 3/\sqrt{10}]$.

Next, we prove that (3.3) is also true for $\alpha \in (3/\sqrt{10}, 1)$ and all $a, b > 0$ with $a \neq b$.

Without loss of generality, we assume that $a > b$. Let $t = a/b > 1$ and $\beta = \alpha/3 \in (1/\sqrt{10}, 1/3)$. Then (1.1) leads to

$$\begin{aligned} & \log[M_{2\alpha/3}(a, b)] - \log[P^\alpha(a, b)G^{1-\alpha}(a, b)] \\ &= \log[M_{2\beta}(a, b)] - \log[P^{3\beta}(a, b)G^{1-3\beta}(a, b)] \\ &= \frac{1}{2\beta} \log \frac{1+t^{4\beta}}{2} - 3\beta \log \frac{t^2-1}{4 \arctan t - \pi} - (1-3\beta) \log t. \end{aligned} \quad (3.4)$$

Let

$$F(t) = \frac{1}{2\beta} \log \frac{1+t^{4\beta}}{2} - 3\beta \log \frac{t^2-1}{4 \arctan t - \pi} - (1-3\beta) \log t. \quad (3.5)$$

Then simple computations lead to

$$\lim_{t \rightarrow 1} F(t) = 0, \quad (3.6)$$

$$F'(t) = \frac{(1-3\beta)t^{4\beta+2} - (1+3\beta)t^{4\beta} - (1+3\beta)t^2 + (1-3\beta)}{t(t^2-1)(t^{4\beta}+1)} + \frac{12\beta}{(t^2+1)(4 \arctan t - \pi)}. \quad (3.7)$$

From Lemma 2.1 we know that there exists $\lambda_0 > 1$ such that

$$(1-3\beta)\lambda_0^{4\beta+2} - (1+3\beta)\lambda_0^{4\beta} - (1+3\beta)\lambda_0^2 + (1-3\beta) = 0, \quad (3.8)$$

$$(1-3\beta)t^{4\beta+2} - (1+3\beta)t^{4\beta} - (1+3\beta)t^2 + (1-3\beta) < 0 \quad (3.9)$$

for $t \in [1, \lambda_0)$, and

$$(1-3\beta)t^{4\beta+2} - (1+3\beta)t^{4\beta} - (1+3\beta)t^2 + (1-3\beta) > 0 \quad (3.10)$$

for $t \in (\lambda_0, \infty)$.

We divide two cases to prove that

$$F'(t) > 0 \quad (3.11)$$

for $t > 1$. □

Case 1. $t \in [\lambda_0, \infty)$. Then (3.11) follows from (3.7) and (3.8) together with (3.10).

Case 2. $t \in (1, \lambda_0)$. Then (3.7) can be written as

$$F'(t) = \frac{(1-3\beta)t^{4\beta+2} - (1+3\beta)t^{4\beta} - (1+3\beta)t^2 + (1-3\beta)}{t(t^2-1)(t^{4\beta}+1)(4\arctan t - \pi)} F_1(t), \quad (3.12)$$

where

$$F_1(t) = 4\arctan t + \frac{12\beta t(t^2-1)(t^{4\beta}+1)}{(t^2+1)[(1-3\beta)t^{4\beta+2} - (1+3\beta)t^{4\beta} - (1+3\beta)t^2 + (1-3\beta)]} - \pi. \quad (3.13)$$

Let $x = t^2 \in (1, \lambda_0^2)$, then (3.13) leads to

$$F_1(1) = 0, \quad (3.14)$$

$$F'_1(t) = \frac{4F_2(x)}{(t^2+1)^2[(1-3\beta)t^{4\beta+2} - (1+3\beta)t^{4\beta} - (1+3\beta)t^2 + (1-3\beta)]^2},$$

where

$$\begin{aligned} F_2(x) = & (18\beta^2 - 9\beta + 1)x^{4\beta+3} - (18\beta^2 - 3\beta + 1)x^{4\beta+2} \\ & - (18\beta^2 + 3\beta + 1)x^{4\beta+1} + (18\beta^2 + 9\beta + 1)x^{4\beta} \\ & + 2(6\beta^2 - 1)x^{2\beta+3} - 2(6\beta^2 - 1)x^{2\beta+2} \\ & - 2(6\beta^2 - 1)x^{2\beta+1} + 2(6\beta^2 - 1)x^{2\beta} \\ & + (18\beta^2 + 9\beta + 1)x^3 - (18\beta^2 + 3\beta + 1)x^2 \\ & - (18\beta^2 - 3\beta + 1)x + (18\beta^2 - 9\beta + 1). \end{aligned} \quad (3.15)$$

Let $F_3(x) = (x^{4-2\beta}/8\beta)F_2^{(4)}(x)$, then (3.15) leads to

$$\begin{aligned} F_2(1) &= 0, \\ F_2'(1) &= 0, \\ F_2''(1) &= 0, \\ F_2'''(1) &= 0, \end{aligned} \tag{3.16}$$

$$\begin{aligned} F_3(x) &= \left(18\beta^2 - 9\beta + 1\right)(4\beta + 1)(4\beta + 3)(2\beta + 1)x^{2\beta+3} \\ &\quad - \left(18\beta^2 - 3\beta + 1\right)(2\beta + 1)(4\beta + 1)(4\beta - 1)x^{2\beta+2} \\ &\quad - \left(18\beta^2 + 3\beta + 1\right)(4\beta + 1)(4\beta - 1)(2\beta - 1)x^{2\beta+1} \\ &\quad + \left(18\beta^2 + 9\beta + 1\right)(4\beta - 1)(2\beta - 1)(4\beta - 3)x^{2\beta} \\ &\quad + \left(6\beta^2 - 1\right)(2\beta + 3)(\beta + 1)(2\beta + 1)x^3 \\ &\quad - \left(6\beta^2 - 1\right)(\beta + 1)(2\beta + 1)(2\beta - 1)x^2 \\ &\quad - \left(6\beta^2 - 1\right)(2\beta + 1)(2\beta - 1)(\beta - 1)x \\ &\quad + \left(6\beta^2 - 1\right)(2\beta - 1)(\beta - 1)(2\beta - 3). \end{aligned} \tag{3.17}$$

From $\beta \in (1/\sqrt{10}, 1/3)$ and $x > 1$, we clearly see that $(18\beta^2 - 9\beta + 1)(4\beta + 1)(4\beta + 3)(2\beta + 1)x^{2\beta+3} < (18\beta^2 - 9\beta + 1)(4\beta + 1)(4\beta + 3)(2\beta + 1)x^3$, $-(18\beta^2 - 3\beta + 1)(2\beta + 1)(4\beta + 1)(4\beta - 1)x^{2\beta+2} < -(18\beta^2 - 3\beta + 1)(2\beta + 1)(4\beta + 1)(4\beta - 1)x^2$, $-(18\beta^2 + 3\beta + 1)(4\beta + 1)(4\beta - 1)(2\beta - 1)x^{2\beta+1} < -(18\beta^2 + 3\beta + 1)(4\beta + 1)(4\beta - 1)(2\beta - 1)x^2$ and $(18\beta^2 + 9\beta + 1)(4\beta - 1)(2\beta - 1)(4\beta - 3)x^{2\beta} < (18\beta^2 + 9\beta + 1)(4\beta - 1)(2\beta - 1)(4\beta - 3)x$. These inequalities and Lemma 2.2 lead to

$$\begin{aligned} F_3(x) &< \left(18\beta^2 - 9\beta + 1\right)(4\beta + 1)(4\beta + 3)(2\beta + 1)x^3 \\ &\quad - \left(18\beta^2 - 3\beta + 1\right)(2\beta + 1)(4\beta + 1)(4\beta - 1)x^2 \\ &\quad - \left(18\beta^2 + 3\beta + 1\right)(4\beta + 1)(4\beta - 1)(2\beta - 1)x^2 \\ &\quad + \left(18\beta^2 + 9\beta + 1\right)(4\beta - 1)(2\beta - 1)(4\beta - 3)x \\ &\quad + \left(6\beta^2 - 1\right)(2\beta + 3)(\beta + 1)(2\beta + 1)x^3 \\ &\quad - \left(6\beta^2 - 1\right)(\beta + 1)(2\beta + 1)(2\beta - 1)x^2 \\ &\quad - \left(6\beta^2 - 1\right)(2\beta + 1)(2\beta - 1)(\beta - 1)x \\ &\quad + \left(6\beta^2 - 1\right)(2\beta - 1)(\beta - 1)(2\beta - 3) \end{aligned}$$

$$\begin{aligned}
&= 2\beta(300\beta^4 + 324\beta^3 + 29\beta^2 - 45\beta - 8)x^3 \\
&\quad + (-1176\beta^5 - 24\beta^4 + 114\beta^3 + 10\beta^2 - 3\beta - 1)x^2 \\
&\quad + 2(276\beta^5 - 276\beta^4 + 3\beta^3 + 43\beta^2 - 3\beta - 1)x \\
&\quad + (24\beta^5 - 72\beta^4 + 62\beta^3 - 6\beta^2 - 11\beta + 3) \\
&< 2\beta(300\beta^4 + 324\beta^3 + 29\beta^2 - 45\beta - 8)x \\
&\quad + (-1176\beta^5 - 24\beta^4 + 114\beta^3 + 10\beta^2 - 3\beta - 1)x \\
&\quad + 2(276\beta^5 - 276\beta^4 + 3\beta^3 + 43\beta^2 - 3\beta - 1)x \\
&\quad + 24\beta^5 - 72\beta^4 + 62\beta^3 - 6\beta^2 - 11\beta + 3 \\
&= (-24\beta^5 + 72\beta^4 + 178\beta^3 + 6\beta^2 - 25\beta - 3)x \\
&\quad + 24\beta^5 - 72\beta^4 + 62\beta^3 - 6\beta^2 - 11\beta + 3 \\
&< -24\beta^5 + 72\beta^4 + 178\beta^3 + 6\beta^2 - 25\beta - 3 \\
&\quad + 24\beta^5 - 72\beta^4 + 62\beta^3 - 6\beta^2 - 11\beta + 3 \\
&= 12\beta(20\beta^2 - 3) < 0.
\end{aligned} \tag{3.18}$$

From (3.16)–(3.18) we can deduce that

$$F_2(x) < 0 \tag{3.19}$$

for $x \in (1, \lambda_0^2)$.

Equation (3.14) together with (3.19) imply that

$$F_1(t) < 0 \tag{3.20}$$

for $t \in (1, \lambda_0)$.

Therefore, (3.11) follows from (3.9) and (3.12) together with (3.20).

It follows from (3.4)–(3.6) and (3.11) that

$$P^\alpha(a, b)G^{1-\alpha}(a, b) < M_{2\alpha/3}(a, b) \tag{3.21}$$

for $\alpha \in (3/\sqrt{10}, 1)$ and all $a, b > 0$ with $a \neq b$.

At last, we prove that $M_0(a, b)$ and $M_{2\alpha/3}(a, b)$ are the best possible lower- and upper-power mean bounds for the product $P^\alpha(a, b)G^{1-\alpha}(a, b)$, respectively.

For any $0 < \varepsilon < (2/3)\alpha$ and $x > 0$, from (1.1) one has

$$\begin{aligned} & \lim_{x \rightarrow +\infty} \frac{M_\varepsilon(1, x)}{P^\alpha(1, x)G^{1-\alpha}(1, x)} \\ &= \lim_{x \rightarrow +\infty} \left[\frac{((1+x^{-\varepsilon})/2)^{1/\varepsilon}}{((1-x^{-1})/(4 \arctan \sqrt{x} - \pi))^{\alpha}} x^{(1-\alpha)/2} \right] = +\infty, \end{aligned} \quad (3.22)$$

$$\begin{aligned} & \left[P^\alpha(1, 1+x)G^{1-\alpha}(1, 1+x) \right]^{(2\alpha/3)-\varepsilon} - [M_{(2\alpha/3)-\varepsilon}(1, 1+x)]^{(2\alpha/3)-\varepsilon} \\ &= \frac{J(x)}{\left(4 \arctan \sqrt{1+x} - \pi\right)^{\alpha(2\alpha-3\varepsilon)/3}}, \end{aligned} \quad (3.23)$$

where $J(x) = x^{\alpha(2\alpha-3\varepsilon)/3}(1+x)^{(1-\alpha)(2\alpha-3\varepsilon)/6} - (1/2)(4 \arctan \sqrt{1+x} - \pi)^{\alpha(2\alpha-3\varepsilon)/3}[1 + (1+x)^{(2\alpha-3\varepsilon)/3}]$.

Let $x \rightarrow 0$, making use of the Taylor expansion we get

$$J(x) = \frac{1}{24}\varepsilon(2\alpha-3\varepsilon)x^{(1/3)\alpha(2\alpha-3\varepsilon)+2} + o\left(x^{(1/3)\alpha(2\alpha-3\varepsilon)+2}\right). \quad (3.24)$$

Equation (3.22) implies that for any $0 < \varepsilon < (2/3)\alpha$ there exists $X = X(\varepsilon, \alpha) > 1$ such that $M_\varepsilon(1, x) > P^\alpha(1, x)G^{1-\alpha}(1, x)$ for $x \in (X, \infty)$.

Equations (3.23) and (3.24) imply that for any $0 < \varepsilon < (2/3)\alpha$ there exists $\delta = \delta(\varepsilon, \alpha) > 0$ such that $P^\alpha(1, 1+x)G^{1-\alpha}(1, 1+x) > M_{2\alpha/3-\varepsilon}(1, 1+x)$ for $x \in (0, \delta)$.

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