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Research Article

Razumikhin Stability Theorem for Fractional Systems with Delay

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Fractional calculus techniques and methods started to be applied successfully during the last decades in several fields of science and engineering. In this paper we studied the stability of fractional-order nonlinear time-delay systems for Riemann-Liouville and Caputo derivatives and we extended Razumikhin theorem for the fractional nonlinear time-delay systems.

1. Introduction

Fractional calculus is an emerging field with various valuable applications in science and engineering [1–6].

Fractional calculus is a good candidate to solve the dynamics of complex systems. During the last years fractional calculus was subjected to an intense debate. The fractional differential equations started to play an important role in modeling anomalous diffusion, processes having long-range dependence, and so on. Several open problems remain unsolved or there were partially solved with this type of calculus. Among those kinds of problems we mention the question of stability which is interest in nonlinear science and control theory. Also, the problem of time-delay system has been discussed over many years. For a survey the reader can check the study in [7]. Time delay is very often encountered in different technical systems, for example, electric, pneumatic, and hydraulic networks, chemical processes, and

long transmission lines. The existence of pure time delay, regardless of its presence in a control and/or state, may cause undesirable system transient response, or, generally, even an instability. Numerous reports have been published on this matter, with particular emphasis on the application of Lyapunov's second method [8, 9].

In recent years, considerable attention has been paid to control systems whose processes and/or controllers are of fractional order. This is mainly due to the fact that many real-world physical systems are well characterized by fractional-order differential equations, that is, equations involving noninteger-order derivatives. In particular, it has been shown that viscoelastic materials having memory and hereditary effects [10] and dynamical processes such as semi-infinite lossy RC transmission [11], mass diffusion, and heat conduction [12], can be more adequately modeled by fractional-order models than integer-order models. Moreover, with the success in the synthesis of real noninteger differentiator and the emergence of new electrical circuit element called "fractance" [13], fractional-order controllers [14, 15] including fractional-order PID controllers [14] have been proposed to enhance the robustness and performance of control systems.

Some literatures published about stability of fractional-order linear time-delay systems [16–19]. In the base of Lyapunov's second method, some work has been done in the field of stability of fractional-order nonlinear systems without delay [20, 21]. But it seems that a few attentions have been paid to the stability of fractional-order nonlinear time-delay systems.

When the system involves time delay, it should be regarded as a functional differential equation (FDE). In this case, analysis of the stability relies on the Lyapunov-Krasovskii functional [22, 23]. However, the Razumikhin stability theory is more widely used to prove the stability of time-delay systems [22, 23], since the construction of Lyapunov-Krasovskii functional is more difficult than that of Lyapunov-Razumikhin function.

The purpose of this paper is to develop the Razumikhin theorem for fractional-order nonlinear time-delay systems.

The manuscript is organized as follows. In Section 2 some basic definitions of fractional calculus are mentioned. Section 3 is devoted to fractional nonlinear time-delay systems. Section 4 presents the generalization of the fractional Razumikhin theorem when both fractional derivatives and delay are present.

2. Preliminaries and Definitions

In the fractional calculus the Riemann-Liouville and Caputo fractional derivatives are defined respectively [1–3]:

$$\begin{aligned} {}_{t_0}D_t^q x(t) &= \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \left(\int_{t_0}^{t} \frac{x(s)}{(t-s)^{q+1-n}} ds \right), \quad (n-1 < q \le n), \\ {}_{t_0}^{c}D_t^q x(t) &= \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} \frac{x^{(n)}(s)}{(t-s)^{q+1-n}} ds, \quad (n-1 < q \le n), \end{aligned} \tag{2.1}$$

where $x(t)$ is an arbitrary differentiable function, $n \in \mathbb{N}$, ${}_{t_0}D_t^q$ and ${}_{t_0}^{c}D_t^q$ are the Riemann-Liouville and Caputo fractional derivatives of order q on $[t_0, t]$, respectively, and $\Gamma(\cdot)$ denotes the Gamma function.

For $0 < q \leq 1$ we have

$$
\begin{aligned}
{}_{t_0}D_t^q x(t) &= \frac{1}{\Gamma(1-q)} \frac{d}{dt}\left(\int_{t_0}^{t} \frac{x(s)}{(t-s)^q} ds\right), \quad (0 < q \leq 1), \\
{}_{t_0}^{c}D_t^q x(t) &= \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} \frac{x'(s)}{(t-s)^q} ds, \quad (0 < q \leq 1).
\end{aligned}
\tag{2.2}
$$

Some properties of Riemann-Liouville and Caputo derivatives are recalled below [1–3].

Property 1. When $0 < q \leq 1$, we have

$$
{}_{t_0}^{c}D_t^q x(t) = {}_{t_0}D_t^q x(t) - \frac{x(t_0)}{\Gamma(1-q)}(t-t_0)^{-q}. \tag{2.3}
$$

In particular, if $x(t_0) = 0$, we have

$$
{}_{t_0}^{c}D_t^q x(t) = {}_{t_0}D_t^q x(t). \tag{2.4}
$$

Property 2. For any $v > -1$, we have

$$
{}_{t_0}D_t^q (t-t_0)^v = \frac{\Gamma(1+v)}{\Gamma(1+v-q)}(t-t_0)^{v-q}. \tag{2.5}
$$

In particular, if $0 < q < 1$, $v > 0$, and $x(t) = (t-t_0)^v$, then from Property 1, we have

$$
{}_{t_0}^{c}D_t^q (t-t_0)^v = \frac{\Gamma(1+v)}{\Gamma(1+v-q)}(t-t_0)^{v-q}. \tag{2.6}
$$

Property 3.

$$
{}_{t_0}^{c}D_t^q \left(ax(t) + \mathrm{by}(t)\right) = a\, {}_{t_0}^{c}D_t^q x(t) + b\, {}_{t_0}^{c}D_t^q y(t), \tag{2.7}
$$

where a and b are arbitrary constants.

Property 4. From the definition of Caputo's derivative when $0 < q \leq 1$ we have

$$
{}_{t_0}I_t^q\, {}_{t_0}^{c}D_t^q x(t) = x(t) - x(t_0), \tag{2.8}
$$

where

$$
{}_{t_0}I_t^q f(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s) ds. \tag{2.9}
$$

3. Fractional Nonlinear Time-Delay System

Let $C([a,b],\mathbb{R}^n)$ be the set of continuous functions mapping the interval $[a,b]$ to $\mathbb{R}^n$. In many situations, one may wish to identify a *maximum time delay* r of a system. In this case, we are often interested in the set of continuous function mapping $[-r,0]$ to $\mathbb{R}^n$, for which we simplify the notation to $C = C([-r,0],\mathbb{R}^n)$. For any $A > 0$ and any continuous function of time $x \in C([t_0 - r, t_0 + A],\mathbb{R}^n)$, $t_0 \le t \le t_0 + A$, let $x_t(\theta) \in C$ be a segment of function ψ defined as $x_t(\theta) = x_t(t+\theta)$, $-r \le \theta \le 0$.

Consider fractional nonlinear time-delay system

$$ {}_{t_0}^{c}D_t^q x(t) = f(t, x_t), \tag{3.1} $$

where $x(t) \in \mathbb{R}^n$, $0 < q \le 1$, and $f : \mathbb{R} \times C \to \mathbb{R}^n$. As such, to determine the future evolution of the state, it is necessary to specify the initial state variables $x(t)$ in a time interval of length r, say from $t_0 - r$ to t_0, that is,

$$ x_{t_0} = \phi, \tag{3.2} $$

where $\phi \in C$ is given. In other words $x(t_0 + \theta) = \phi(\theta)$, $-r \le \theta \le 0$.

Definition 3.1. Suppose that $f(t,0) = 0$ for all $t \in \mathbb{R}$. The solution $x = 0$ of (3.1) is said to be stable if for any $t_0 \in \mathbb{R}$, $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon, t_0)$ such that $\|\phi\| < \delta$ implies that $\|x_t(t_0,\phi)\| < \varepsilon$ for $t \ge t_0$. The solution $x = 0$ of (3.1) is said to be asymptotically stable if it is stable and there is a $\delta_a = \delta_a(t_0) > 0$ such that $\|\phi\| < \delta_a$ implies that $x_t(t_0,\phi) \to 0$ as $t \to \infty$. The solution $x = 0$ is said to be uniformly stable if the number δ in the definition is independent of t_0. The solution $x = 0$ of (3.1) is uniformly asymptotically stable if it is uniformly stable and there is a $\delta_a > 0$ such that, for every $\eta > 0$, there is a $T(\eta)$ such that $\|\phi\| < \delta_a$ implies that $\|x_t(t_0,\phi)\| < \varepsilon$ for $t \ge t_0 + T(\eta)$ for every $t_0 \in \mathbb{R}$ [17].

4. Fractional Razumikhin Theorem

As in the study of systems without delay, an effective method for determining the stability of a time-delay system is Lyapunov method. Since in a time-delay system the "state" at time t required the value of $x(t)$ in the interval $[t-r,t]$, that is, x_t, it is natural to expect that, for a time-delay system, corresponding Lyapunov function be a functional $V(t,x_t)$ depending on x_t, which also should measure the deviation of x_t from the trivial solution 0.

Let $V(t,\phi)$ be differentiable, and let $x_t(\tau,\phi)$ be the solution of (3.1) at time t with initial condition $x_\tau = \phi$. Then we calculate the Caputo derivative of $V(t,x_t)$ with respect to t and evaluate it at $t = \tau$ as follow, respectively:

$$ {}_{t_0}D_t^q V(\tau,\phi) = {}_{t_0}D_t^q V(t, x_t(\tau,\phi))\Big|_{t=\tau, x_t=\phi} = \frac{1}{\Gamma(1-q)} \frac{d}{dt}\left(\int_{t_0}^{t} \frac{V(s,x_s)}{(t-s)^q} ds\right)\Bigg|_{t=\tau, x_t=\phi}, \tag{4.1} $$

$$ {}_{t_0}^{c}D_t^q V(\tau,\phi) = {}_{t_0}^{c}D_t^q V(t, x_t(\tau,\phi))\Big|_{t=\tau, x_t=\phi} = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} \frac{V'(s,x_s)}{(t-s)^q} ds\Bigg|_{t=\tau, x_t=\phi}, \tag{4.2} $$

where $0 < q < 1$.

Theorem 4.1. *Suppose that $f : \mathbb{R}\times C \to \mathbb{R}^n$ in* (3.1) *maps $\mathbb{R}\times$(bounded sets in C) into bounded sets in $\mathbb{R}^n$, and $\alpha_1, \alpha_2, \alpha_3 : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ are continuous nondecreasing functions, $\alpha_1(s), \alpha_2(s)$ are positive for $s > 0$, and $\alpha_1(0) = \alpha_2(0) = 0$, α_2 strictly increasing. If there exists a continuously differentiable function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that*

$$\alpha_1(\|x\|) \le V(t,x) \le \alpha_2(\|x\|), \quad \text{for } t \in \mathbb{R},\ x \in \mathbb{R}^n, \tag{4.3}$$

and the Caputo fractional derivative of V along the solution $x(t)$ of (3.1) *satisfies*

$${}_{t_0}^{c}D_t^q V(t,x(t)) \le -\alpha_3(\|x(t)\|), \quad \text{for } 0 < q \le 1, \text{ whenever } V(t+\theta, x(t+\theta)) \le V(t,x(t)) \tag{4.4}$$

for $0 < q \le 1$ and $\theta \in [-r, 0]$, then system (3.1) *is uniformly stable.*

If, in addition, $\alpha_3(s) > 0$ for $s > 0$ and there exists a continuous nondecreasing function $p(s) > s$ for $s > 0$ such that condition (4.4) *is strengthened to*

$${}_{t_0}^{c}D_t^q V(t,x(t)) \le -\alpha_3(\|x(t)\|), \quad \text{if } V(t+\theta, x(t+\theta)) \le p(V(t,x(t))) \tag{4.5}$$

for $0 < q \le 1$ and $\theta \in [-r, 0]$, then system (3.1) *is uniformly asymptotically stable.If in addition $\lim_{s\to\infty}\alpha_1(s) = \infty$, then system* (3.1) *is globally uniformly asymptotically stable.*

The integer-order derivative version of this theorem can be found in [22, 23].

Proof. To prove uniform stability, for any given $\varepsilon > 0$, let $0 < \alpha_2(\delta) < \alpha_1(\varepsilon)$. Then for any given t_0 and ϕ, with $\|\phi\| < \delta$, we have $V(t_0+\theta, \phi(\theta)) \le \alpha_2(\delta) < \alpha_1(\varepsilon)$ for $\theta \in [-r, 0]$. Let x be solution of (3.1) with initial condition $x_{t_0} = \phi$. According to (4.4), as t increases, whenever $V(t,x(t)) = \alpha_2(\delta)$ and $V(t+\theta, x(t+\theta)) \le \alpha_2(\delta)$ for $\theta \in [-r, 0]$, ${}_{t_0}^{c}D_t^q V(t,x(t)) \le 0$, therefore, by Property 4 and (4.3), $V(t,x(t)) \le V(t_0, x(t_0)) \le \alpha_2(\delta)$ for all $t \ge t_0$. Due to the continuity of $V(t,x(t))$, it is therefore impossible for $V(t,x(t))$ to exceed $\alpha_2(\delta)$. In other words, we have $V(t,x(t)) \le \alpha_2(\delta) < \alpha_1(\varepsilon)$ for $t \ge t_0 - r$, but this implies that $\|x(t)\| \le \varepsilon$ for $t \ge t_0 - r$.

To complete the proof of the theorem, suppose that $\delta > 0$, $H > 0$ are such that $\alpha_2(\delta) = \alpha_1(H)$. Such numbers always exist by our hypotheses on α_1 and α_2. In fact, since $\alpha_2(0) = 0$ and $0 < \alpha_1(s) \le \alpha_2(s)$ for $s > 0$, one can preassign H and then determine a δ such that the desired relation is satisfied. If $\alpha_1(s) \to \infty$ as $s \to \infty$, then one can fix δ arbitrarily and determine H such that $\alpha_2(\delta) = \alpha_1(H)$. This remark and reasoning that follows will prove the uniform asymptotic stability of $x = 0$ as well as the fact that $x = 0$ is a globally uniformly asymptotically stable.

If $\alpha_2(\delta) = \alpha_1(H)$, the same argument as in the proof of uniform stability shows that $\|\phi\| \le \delta$ implies that $\|x(t)\| \le H$, $V(t,x(t)) < \alpha_2(\delta)$ for $t \ge t_0 - r$. Suppose that $0 < \eta \le H$ is arbitrary. We need to show that there is a number $\bar{t} = \bar{t}(\eta, \delta)$ such that for any $t_0 \ge 0$ and $\|\phi\| \le \delta$ the solution $x(t)$ of (3.1) satisfies $\|x(t)\| \le \eta$, $t \ge t_0 + \bar{t} + r$. This will be true if we show that $V(t,x(t)) \le \alpha_1(\eta)$, for $t \ge t_0 + \bar{t}$.

From the properties of function $p(s)$, there is a number $a > 0$ such that $p(s) - s > a$ for $\alpha_1(\eta) \le s \le \alpha_2(\delta)$. Let N be the first positive integer such that $\alpha_1(\eta) + Na \ge \alpha_2(\delta)$, and let $\gamma = \inf_{\alpha_2^{-1}(\alpha_1(\eta)) \le s \le H} \alpha_3(s)$ and $T = (N\alpha_2(\delta)\Gamma(1+q)/\gamma)^{1/q}$.

We now show that $V(t,x(t)) \le \alpha_1(\eta)$ for all $t \ge t_0 + T$. First, we show that $V(t,x(t)) \le \alpha_1(\eta) + (N-1)a$ for $t \ge t_0 + (\alpha_2(\delta)\Gamma(1+q)/\gamma)^{1/q}$. If $\alpha_1(\eta) + (N-1)a < V(t,x(t))$, for $t_0 \le t \le t_0 + (\alpha_2(\delta)\Gamma(q+1)/\gamma)^{1/q}$, then, since $V(t,x(t)) \le \alpha_2(\delta)$ for all $t \ge t_0 - r$, it follows that

$$p(V(t,x(t))) > V(t,x(t)) + a \ge \alpha_1(\eta) + Na \ge \alpha_2(\delta) \ge V(t+\theta, x(t+\theta)) \tag{4.6}$$

for $t_0 - r \le t \le t_0 + (\alpha_2(\delta)\Gamma(1+q)/\gamma)^{1/q}$.

Hypothesis (4.5) implies that

$${}_{t_0}^{c}D_t^q V(t,x(t)) \le -\alpha_3(\|x(t)\|) \le -\gamma \tag{4.7}$$

for $t_0 \le t \le t_0 + (\alpha_2(\delta)\Gamma(1+q)/\gamma)^{1/q}$. Consequently,

$${}_{t_0}^{c}D_t^q \left(V(t,x(t)) + \gamma \frac{(t-t_0)^q}{\Gamma(1+q)} \right) \le 0, \tag{4.8}$$

and hence by Property 4, for all

$$t_0 \le t \le t_0 + \left(\alpha_2(\delta)\Gamma(1+q)/\gamma\right)^{1/q} \tag{4.9a}$$

we have

$$V(t,x(t)) \le V(t_0,x(t_0)) - \gamma \frac{(t-t_0)^q}{\Gamma(1+q)} \le \alpha_2(\delta) - \gamma \frac{(t-t_0)^q}{\Gamma(1+q)} \tag{4.9b}$$

on the same interval. The positive property (4.2) of V implies that $V(t,x(t)) \le \alpha_1(\eta) + (N-1)a$ at $t_1 = t_0 + (\alpha_2(\delta)\Gamma(1+q)/\gamma)^{1/q}$. But this implies that $V(t,x(t)) \le \alpha_1(\eta) + (N-1)a$ for all $t \ge t_0 + (\alpha_2(\delta)\Gamma(1+q)/\gamma)^{1/q}$, since ${}_{t_0}^{c}D_t^q V(t,x(t))$ is negative by condition (4.5), and therefore $(d/dt)(V(t,x(t)))$ is negative when $V(t,x(t)) = \alpha_1(\eta) + (N-1)a$.

Now, let $\bar{t}_j = (j\alpha_2(\delta)\Gamma(1+q)/\gamma)^{1/q}$, $j = 1,\ldots,N$, $\bar{t}_0 = 0$, and assume that, for some integer $k \ge 1$, in the interval $\bar{t}_{k-1} - r \le t - t_0 \le \bar{t}_k$, we have

$$\alpha_1(\eta) + (N-k)a \le V(t,x(t)) \le \alpha_1(\eta) + (N-k+1)a. \tag{4.10}$$

By the same type reasoning as above, we have

$${}_{t_0}^{c}D_t^q V(t,x(t)) \le -\gamma \tag{4.11}$$

for $\bar{t}_{k-1} \le t - t_0 \le \bar{t}_k$, and we have

$$ {}_{t_0}^{c}D_t^q \left(V(t, x(t)) + \gamma \frac{(t-t_0)^q}{\Gamma(1+q)} \right) \le 0, \tag{4.12} $$

$$ V(t, x(t)) + \gamma \frac{(t-t_0)^q}{\Gamma(1+q)} \le V\left(t_0 + \bar{t}_{k-1}, x\left(t_0 + \bar{t}_{k-1}\right)\right) + \gamma \frac{\bar{t}_{k-1}^q}{\Gamma(1+q)}, \tag{4.13} $$

and we have

$$ \begin{aligned} V(t, x(t)) &\le V\left(t_0 + \bar{t}_{k-1}, x\left(t_0 + \bar{t}_{k-1}\right)\right) + \gamma \frac{\bar{t}_{k-1}^q}{\Gamma(1+q)} - \gamma \frac{(t-t_0)^q}{\Gamma(1+q)} \\ &\le \alpha_2(\delta) - \gamma \frac{(t-t_0)^q}{\Gamma(1+q)} + (k-1)\alpha_2(\delta) \\ &= k\alpha_2(\delta) - \gamma \frac{(t-t_0)^q}{\Gamma(1+q)} \le 0 \end{aligned} \tag{4.14} $$

if $t - t_0 \ge (k\Gamma(1+q)\alpha_2(\delta)/\gamma)^{1/q}$. Consequently,

$$ V\left(t_0 + \bar{t}_k, x\left(t_0 + \bar{t}_k\right)\right) \le \alpha_1(\eta) + (N-k)a \tag{4.15} $$

and, finally, $V(t, x(t)) \le \alpha_1(\eta) + (N-k)a$ for $t \ge t_0 + \bar{t}_k$. This completes the induction and we have $V(t, x(t)) \le \alpha_1(\eta)$ for all $t \ge t_0 + (N\alpha_2(\delta)\Gamma(1+q)/\gamma)^{1/q}$. This proves the theorem. □

Lemma 4.2. *Let $q \in (0,1)$ and $V(t_0) \ge 0$, then*

$$ {}_{t_0}^{c}D_t^q V(t) \le {}_{t_0}D_t^q V(t). \tag{4.16} $$

Proof. By using Property 1 we have ${}_{t_0}^{c}D_t^q V(t) = {}_{t_0}D_t^q V(t) - V(t_0)(t-t_0)^{-q}/\Gamma(1-q)$. Because $q \in (0,1)$ and $V(t_0) \ge 0$, we obtained that ${}_{t_0}^{c}D_t^q V(t) \le {}_{t_0}D_t^q V(t)$. □

Theorem 4.3. *Suppose that the assumptions in Theorem 4.1 are satisfied except replacing ${}_{t_0}^{c}D_t^q$ by ${}_{t_0}D_t^q$, then one has the same result for uniform stability, uniform asymptotic stability, and global uniform asymptotic stability.*

Proof. It follows from Lemma 4.2 and $V(t, x(t)) \ge 0$ that ${}_{t_0}^{c}D_t^q V(t, x(t)) \le {}_{t_0}D_t^q V(t, x(t))$, which implies that ${}_{t_0}^{c}D_t^q V(x_t(\varphi)) \le {}_{t_0}D_t^q V(x_t(\varphi)) \le -w(x_t(\varphi))$ for all $t \ge t_0$. Following the same proof as in Theorem 4.1 yields uniform stability, uniform asymptotic stability, and global uniform asymptotic stability. □

5. Conclusion

The combination of the fractional calculus and delay techniques seems to describe better the dynamics of the complex systems, namely, because both theories take into account the memory effects. Having in mind these aspects, in this paper we generalized the fractional Razumikhin theorem in presence of Caputo fractional derivative and delay. By using the Caputo and the Riemann-Liouville we have proved two corresponding theorems. The obtained theorems contain as particular case the fractional calculus version as well as the time-delay one.

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