Research Article

q -Bernstein Polynomials Associated with q -Stirling Numbers and Carlitz's q -Bernoulli Numbers

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Recently, Kim (2011) introduced q -Bernstein polynomials which are different q -Bernstein polynomials of Phillips (1997). In this paper, we give a p -adic q -integral representation for q -Bernstein type polynomials and investigate some interesting identities of q -Bernstein type polynomials associated with q -extensions of the binomial distribution, q -Stirling numbers, and Carlitz's q -Bernoulli numbers.

1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ denote the ring of p -adic integers, the field of p -adic rational numbers, the complex number field, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$.

When one talks of q -extensions, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p -adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ then one normally assumes $|q| < 1$, and if $q \in \mathbb{C}_p$ then one normally assumes $|1 - q|_p < 1$.

The q -bosonic natural numbers are defined by $[n]_q = (1 - q^n)/(1 - q) = 1 + q + q^2 + \cdots + q^{n-1}$ for $n \in \mathbb{N}$, and the q -factorial is defined by $[n]_q! = [n]_q [n-1]_q \cdots [2]_q [1]_q$ (see [1–3]). For the q -extension of binomial coefficients, we use the following notation in the form of

$$\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q! [k]_q!} = \frac{[n]_q [n-1]_q \cdots [n-k+1]_q}{[k]_q!}. \quad (1.1)$$

Let $C[0,1]$ denote the set of continuous functions on the real interval $[0,1]$. The Bernstein operator for $f \in C[0,1]$ is defined by

$$\mathbb{B}_n(f | x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=0}^n f\left(\frac{k}{n}\right) B_{k,n}(x), \quad (1.2)$$

where $n, k \in \mathbb{Z}_+$. The polynomials $B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ are called Bernstein polynomials of degree n (see [4–8]). For $f \in C[0,1]$, q -Bernstein type operator of order n for f is defined by

$$\mathbb{B}_{n,q}(f | x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} [x]_q^k [1-x]_{1/q}^{n-k} = \sum_{k=0}^n f\left(\frac{k}{n}\right) B_{k,n}(x, q), \quad (1.3)$$

where $n, k \in \mathbb{Z}_+$. Here $B_{k,n}(x, q) = \binom{n}{k} [x]_q^k [1-x]_{1/q}^{n-k}$ are called q -Bernstein type polynomials of degree n (see [9]).

We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and write $f \in \text{UD}(\mathbb{Z}_p)$, if the difference quotient $F_f(x, y) = (f(x) - f(y)) / (x - y)$ has a limit $f'(a)$ as $(x, y) \rightarrow (a, a)$. For $f \in \text{UD}(\mathbb{Z}_p)$, the p -adic q -integral on $\mathbb{Z}_p$ is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x \quad (1.4)$$

(see [10]). Carlitz's q -Bernoulli numbers can be represented by a p -adic q -integral on $\mathbb{Z}_p$ as follows:

$$\int_{\mathbb{Z}_p} [x]_q^n d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} [x]_q^n q^x = \beta_{n,q} \quad (1.5)$$

(see [10, 11]). The k th order factorial of $[x]_q$ is defined by

$$[x]_{k,q} = [x]_q [x-1]_q \cdots [x-k+1]_q = \frac{(1-q^x)(1-q^{x-1}) \cdots (1-q^{x-k+1})}{(1-q)^k} \quad (1.6)$$

and is called the q -factorial of x of order k (see [10]).

In this paper, we give a p -adic q -integral representation for q -Bernstein type polynomials and derive some interesting identities for the q -Bernstein type polynomials associated with the q -extension of binomial distributions, q -Stirling numbers, and Carlitz's q -Bernoulli numbers.

2. q -Bernstein Polynomials

In this section, we assume that $0 < q < 1$. Let $\mathbb{P}_{n,q} = \{\sum_i a_i [x]_q^i \mid a_i \in \mathbb{R}\}$ be the space of q -polynomials of degree less than or equal to n .

We claim that the q -Bernstein type polynomials of degree n defined by (1.3) are a basis for $\mathbb{P}_{n,q}$.

First, we see that the q -Bernstein type polynomials of degree n span the space of q -polynomials. That is, any q -polynomials of degree less than or equal to n can be written as a linear combination of the q -Bernstein type polynomials of degree n .

For $n, k \in \mathbb{Z}_+$ and $x \in [0, 1]$, we have

$$B_{k,n}(x, q) = \sum_{l=k}^n \binom{n}{l} \binom{l}{k} (-1)^{l-k} [x]_q^l \quad (2.1)$$

(see [9]). If there exist constants $C_0, C_1, \dots, C_n$ such that $C_0 B_{0,n}(x, q) + C_1 B_{1,n}(x, q) + \dots + C_n B_{n,n}(x, q) = 0$ holds for all x , then we can derive the following equation from (2.1):

$$\begin{aligned} 0 &= C_0 B_{0,n}(x, q) + C_1 B_{1,n}(x, q) + \dots + C_n B_{n,n}(x, q) \\ &= C_0 \sum_{i=0}^n (-1)^i \binom{n}{i} \binom{i}{0} [x]_q^i + C_1 \sum_{i=1}^n (-1)^{i-1} \binom{n}{i} \binom{i}{1} [x]_q^i \\ &\quad + \dots + C_n \sum_{i=n}^n (-1)^{i-n} \binom{n}{i} \binom{i}{n} [x]_q^i \\ &= C_0 + \left\{ \sum_{i=0}^1 C_i (-1)^{i-1} \binom{n}{1} \binom{1}{i} \right\} [x]_q + \dots + \left\{ \sum_{i=0}^n C_i (-1)^{i-n} \binom{n}{n} \binom{n}{i} \right\} [x]_q^n. \end{aligned} \quad (2.2)$$

Since the power basis is a linearly independent set, it follows that

$$\begin{aligned} C_0 &= 0, \\ \sum_{i=0}^1 C_i (-1)^{i-1} \binom{n}{1} \binom{1}{i} &= 0, \\ \vdots &\quad \quad \quad \vdots \\ \sum_{i=0}^n C_i (-1)^{i-n} \binom{n}{n} \binom{n}{i} &= 0, \end{aligned} \quad (2.3)$$

which implies that $C_0 = C_1 = \dots = C_n = 0$ (C_0 is clearly zero, substituting this in the second equation gives $C_1 = 0$, substituting these two into the third equation gives $C_2 = 0$, and so on). Hence, we have the following theorem.

Theorem 2.1. *The q -Bernstein type polynomials of degree n are a basis for $\mathbb{P}_{n,q}$.*

Let us consider a q -polynomial $P_q(x) \in \mathbb{P}_{n,q}$ as a linear combination of q -Bernstein type basis functions as follows:

$$P_q(x) = C_0 B_{0,n}(x, q) + C_1 B_{1,n}(x, q) + \cdots + C_n B_{n,n}(x, q). \quad (2.4)$$

We can write (2.4) as a dot product of two values:

$$P_q(x) = (B_{0,n}(x, q), B_{1,n}(x, q), \dots, B_{n,n}(x, q)) \begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_n \end{pmatrix}. \quad (2.5)$$

From (2.5), we can derive the following equation:

$$P_q(x) = \left(1, [x]_q, \dots, [x]_q^n\right) \begin{pmatrix} b_{00} & 0 & 0 & \cdots & 0 \\ b_{10} & b_{11} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n0} & b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_n \end{pmatrix}, \quad (2.6)$$

where the b_{ij} are the coefficients of the power basis that are used to determine the respective q -Bernstein type polynomials.

From (1.3) and (2.1), we note that

$$\begin{aligned} B_{0,2}(x, q) &= [1 - x]_{1/q}^2 = \sum_{l=0}^2 \binom{2}{l} (-1)^l [x]_q^l = 1 - 2[x]_q + [x]_q^2, \\ B_{1,2}(x, q) &= \binom{2}{1} [x]_q [1 - x]_{1/q} = 2[x]_q - 2[x]_q^2, \\ B_{2,2}(x, q) &= \binom{2}{2} [x]_q^2 = [x]_q^2. \end{aligned} \quad (2.7)$$

In the quadratic case ($n = 2$), the matrix representation is

$$P_q(x) = \left(1, [x]_q, [x]_q^2\right) \begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \end{pmatrix}. \quad (2.8)$$

In the cubic case ($n = 3$), the matrix representation is

$$P_q(x) = \begin{pmatrix} 1, [x]_q, [x]_q^2, [x]_q^3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{pmatrix}. \quad (2.9)$$

In many applications of q -Bernstein polynomials, a matrix formulation for the q -Bernstein type polynomials seems to be useful.

Remark 2.2 (see [12]). All results of this section for $q = 1$ are well known in classical case (see Bernstein Polynomials by Joy).

3. q -Bernstein Polynomials, q -Stirling Numbers, and q -Bernoulli Numbers

In this section, we assume that $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$.

For $f \in \text{UD}(\mathbb{Z}_p)$, let us consider the p -adic analogue of q -Bernstein type operator of order n on $\mathbb{Z}_p$ as follows:

$$\mathbb{B}_{n,q}(f \mid x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} [x]_q^k [1-x]_{1/q}^{n-k} = \sum_{k=0}^n f\left(\frac{k}{n}\right) B_{k,n}(x, q). \quad (3.1)$$

Here $B_{k,n}(x, q)$ is the q -Bernstein type polynomials of degree n on $\mathbb{Z}_p$ defined by

$$B_{k,n}(x, q) = \binom{n}{k} [x]_q^k [1-x]_{1/q}^{n-k}, \quad (3.2)$$

for $n, k \in \mathbb{Z}_+$ and $x \in \mathbb{Z}_p$.

Let $(Eh)(x) = h(x+1)$ be the shift operator. Then the q -difference operator is defined by

$$\Delta_q^n := (E - I)_q^n = \prod_{i=1}^n (E - q^{i-1}I), \quad (3.3)$$

where $(Ih)(x) = h(x)$. From (3.3), we derive the following equation:

$$\Delta_q^n f(0) = \sum_{k=0}^n \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} f(n-k). \quad (3.4)$$

By (3.4), we easily see that

$$f(x) = \sum_{n \geq 0} \binom{x}{n}_q \Delta_q^n f(0) \quad (3.5)$$

(see [10, 11]).

The q -Stirling number of the first kind is defined by

$$\prod_{k=1}^n (1 + [k]_q z) = \sum_{k=0}^n S_{1,q}(n, k) z^k, \quad (3.6)$$

and the q -Stirling number of the second kind is also defined by

$$\prod_{k=1}^n \left(\frac{1}{1 + [k]_q z} \right) = \sum_{k=0}^n S_{2,q}(n, k) z^k. \quad (3.7)$$

By (3.3), (3.4), (3.6), and (3.7), we get

$$S_{2,q}(n, k) = \frac{q^{-\binom{k}{2}}}{[k]_q!} \sum_{j=0}^k (-1)^j q^{\binom{j}{2}} \binom{k}{j}_q [k-j]_q^n = \frac{q^{-\binom{k}{2}}}{[k]_q!} \Delta_q^k 0^n, \quad (3.8)$$

for $n, k \in \mathbb{Z}_+$ (see [10, 13]).

From the definition of q -Bernstein type polynomials of degree n on $\mathbb{Z}_p$, we easily see that

$$\int_{\mathbb{Z}_p} B_{k,n}(x, q) d\mu_q(x) = \sum_{l=0}^{n-k} \binom{n-k}{l} \binom{n}{k} (-1)^l \int_{\mathbb{Z}_p} [x]_q^{l+k} d\mu_q(x). \quad (3.9)$$

By (1.5) and (3.9), we obtain the following proposition.

Proposition 3.1. *For $n, k \in \mathbb{Z}_+$, one has*

$$\int_{\mathbb{Z}_p} B_{k,n}(x, q) d\mu_q(x) = \sum_{l=0}^{n-k} \binom{n-k}{l} \binom{n}{k} (-1)^l \beta_{l+k,q}, \quad (3.10)$$

where $\beta_{l+k,q}$ are the $(l+k)$ th Carlitz's q -Bernoulli numbers.

From the definition of q -Bernstein polynomial, we note that

$$\sum_{k=i}^n \frac{\binom{k}{i}}{\binom{n}{i}} B_{k,n}(x, q) = \sum_{k=0}^i q^{\binom{k}{2}} \binom{x}{k}_q [k]_q! S_{2,q}(k, i-k), \quad (3.11)$$

where $i \in \mathbb{N}$. From the definition of q -binomial coefficient, we have

$$\binom{n+1}{k}_q = \binom{n}{k-1}_q + q^k \binom{n}{k}_q = q^{n-k} \binom{n}{k-1}_q + \binom{n}{k}_q. \quad (3.12)$$

By (3.12), we see that

$$\int_{\mathbb{Z}_p} \binom{x}{n}_q d\mu_q(x) = \frac{(-1)^n}{[n+1]_q} q^{(n+1)-\binom{n+1}{2}} \quad (3.13)$$

(see [10, 11]). From (1.5), (3.11), and (3.13), we obtain the following theorem.

Theorem 3.2. For $n, k \in \mathbb{Z}_+$ and $i \in \mathbb{N}$, one has

$$\sum_{k=i}^n \sum_{l=0}^{n-k} \frac{\binom{k}{i}}{\binom{n}{i}} \binom{n-k}{l} \binom{n}{k} (-1)^l \beta_{l+k,q} = \sum_{k=0}^i q^{\binom{k}{2}} [k]_q! S_{2,q}(k, i-k) \frac{(-1)^k}{[k+1]_q} q^{(k+1)-\binom{k+1}{2}}. \quad (3.14)$$

It is easy to see that, for $i \in \mathbb{N}$,

$$\sum_{k=i}^n \frac{\binom{k}{i}}{\binom{n}{i}} B_{k,n}(x, q) = [x]_q^i. \quad (3.15)$$

By (3.11) and (3.15), we easily get

$$[x]_q^i = \sum_{k=0}^i q^{\binom{k}{2}} \binom{x}{k}_q [k]_q! S_{2,q}(k, i-k) \quad (3.16)$$

(see [10]). Thus, we have

$$\begin{aligned} \int_{\mathbb{Z}_p} [x]_q^i d\mu_q(x) &= \sum_{k=0}^i q^{\binom{k}{2}} [k]_q! S_{2,q}(k, i-k) \int_{\mathbb{Z}_p} \binom{x}{k}_q d\mu_q(x) \\ &= q \sum_{k=0}^i [k]_q! S_{2,q}(k, i-k) \frac{(-1)^k}{[k+1]_q}. \end{aligned} \quad (3.17)$$

By (1.5) and (3.17), we obtain the following corollary.

Corollary 3.3. For $n, k \in \mathbb{Z}_+$ and $i \in \mathbb{N}$, one has

$$\beta_{i,q} = q \sum_{k=0}^i [k]_q! S_{2,q}(k, i-k) \frac{(-1)^k}{[k+1]_q}. \quad (3.18)$$

It is known that

$$S_{2,q}(n, k) = \frac{1}{(1-q)^k} \sum_{j=0}^k (-1)^{k-j} \binom{k+n}{k-j}_q \binom{j+n}{j}_q \quad (3.19)$$

(see [10]) and

$$\binom{n}{k}_q = \sum_{j=0}^n \binom{n}{j}_q (q-1)^{j-k} S_{2,q}(k, j-k). \quad (3.20)$$

By a simple calculation, we have that

$$\begin{aligned} q^{nx} &= \sum_{k=0}^n (q-1)^k q^{\binom{k}{2}} \binom{n}{k}_q [x]_{k,q} \\ &= \sum_{m=0}^n \left\{ \sum_{k=m}^n (q-1)^k \binom{n}{k}_q S_{1,q}(k, m) \right\} [x]_q^m, \\ q^{nx} &= \sum_{m=0}^n \binom{n}{m}_q (q-1)^m [x]_q^m. \end{aligned} \quad (3.21)$$

From (3.21), we note that

$$\binom{n}{m}_q = \sum_{k=m}^n (q-1)^{-m+k} \binom{n}{k}_q S_{1,q}(k, m) \quad (3.22)$$

(see [10]).

Thus, we obtain the following proposition.

Proposition 3.4. For $n, k \in \mathbb{Z}_+$, one has

$$B_{k,n}(x, q) = \binom{n}{k}_q [x]_q^k [1-x]_{1/q}^{n-k} = \sum_{m=k}^n (q-1)^{-k+m} \binom{n}{m}_q S_{1,q}(m, k) [x]_q^k [1-x]_{1/q}^{n-k}. \quad (3.23)$$

From the definition of the q -Stirling numbers of the first kind, we get

$$q^{\binom{n}{2}} \binom{x}{n}_q [n]_{q^{-1}}! = [x]_{n,q} q^{\binom{n}{2}} = \sum_{k=0}^n S_{1,q}(n, k) [x]_q^k. \quad (3.24)$$

By (3.11) and (3.24), we obtain the following theorem.

Theorem 3.5. For $n, k \in \mathbb{Z}_+$ and $i \in \mathbb{N}$, one has

$$\sum_{k=i}^n \frac{\binom{k}{i}}{\binom{n}{i}} B_{k,n}(x, q) = \sum_{k=0}^i \sum_{l=0}^k S_{1,q}(k, l) S_{2,q}(k, i-k) [x]_q^l. \quad (3.25)$$

By (3.15) and Theorem 3.5, we obtain the following corollary.

Corollary 3.6. For $i \in \mathbb{Z}_+$, one has

$$\beta_{i,q} = \sum_{k=0}^i \sum_{l=0}^k S_{1,q}(k, l) S_{2,q}(k, i-k) \beta_{l,q}. \quad (3.26)$$

The q -Bernoulli polynomials of order $k \in \mathbb{Z}_+$ are defined by

$$\beta_{n,q}^{(k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^n \binom{n}{i} (-1)^i q^{ix} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{l=1}^k (k-l+i)x_l} d\mu_q(x_1) \cdots d\mu_q(x_k). \quad (3.27)$$

Thus, we have

$$\beta_{n,q}^{(k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^n \frac{(-1)^i \binom{n}{i} (i+k) \cdots (i+1)}{[i+k]_q \cdots [i+1]_q} q^{ix} \quad (3.28)$$

(see [10]). The inverse q -Bernoulli polynomials of order k are defined by

$$\beta_{n,q}^{(-k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^n \frac{(-1)^i \binom{n}{i} q^{ix}}{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{l=1}^k (k-l+i)x_l} d\mu_q(x_1) \cdots d\mu_q(x_k)}. \quad (3.29)$$

In the special case $x = 0$, $\beta_{n,q}^{(k)}(0) = \beta_{n,q}^{(k)}$ are called the n th q -Bernoulli numbers of order k , and $\beta_{n,q}^{(-k)}(0) = \beta_{n,q}^{(-k)}$ are also called the inverse q -Bernoulli numbers of order k (see [10]).

From (3.29), we have

$$\begin{aligned} \beta_{k,q}^{(-n)} &= \frac{1}{(1-q)^k} \sum_{j=0}^k (-1)^j \binom{k}{j} \frac{[j+n]_q \cdots [j+1]_q}{(j+n) \cdots (j+1)} \\ &= \frac{1}{(1-q)^k} \sum_{j=0}^k (-1)^j \frac{\binom{k+n}{n-j}}{\binom{k+n}{n}} \binom{j+n}{n}_q \frac{[n]_q!}{n!} \\ &= \frac{[n]_q!}{\binom{k+n}{n} n!} \left\{ \frac{1}{(1-q)^k} \sum_{j=0}^k (-1)^j \binom{k+n}{n-j} \binom{j+n}{n}_q \right\}. \end{aligned} \quad (3.30)$$

By (3.19) and (3.30), we get

$$\frac{n!}{[n]_q!} \binom{k+n}{n} \beta_{k,q}^{(-n)} = S_{2,q}(n, k). \quad (3.31)$$

Therefore, by (3.11) and (3.31), we obtain the following theorem.

Theorem 3.7. For $i, n, k \in \mathbb{Z}_+$, one has

$$\sum_{k=i}^n \frac{\binom{k}{i}}{\binom{n}{i}} B_{k,n}(x, q) = \sum_{k=0}^i q^{\binom{k}{2}} k! \binom{i}{k} \binom{x}{k}_q \beta_{i-k,q}^{(-k)}. \quad (3.32)$$

It is easy to show that

$$q^{\binom{n}{2}} \binom{x}{n}_q = \frac{1}{[n]_q!} \prod_{k=0}^{n-1} ([x]_q - [k]_q) = \frac{1}{[n]_q!} \sum_{k=0}^n (-1)^k [x]_q^{n-k} S_{1,q}(n-1, k). \quad (3.33)$$

Thus, we have that

$$\sum_{k=i}^n \frac{\binom{k}{i}}{\binom{n}{i}} B_{k,n}(x, q) = \sum_{k=0}^i \sum_{j=0}^k (-1)^j [x]_q^{k-j} S_{1,q}(k-1, j) \frac{k!}{[k]_q!} \binom{i}{k} \beta_{i-k,q}^{(-k)}, \quad (3.34)$$

where $n, k, i \in \mathbb{Z}_+$.

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