Research Article

On an Integral-Type Operator from Zygmund-Type Spaces to Mixed-Norm Spaces on the Unit Ball

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The boundedness and compactness of an integral-type operator recently introduced by the author from Zygmund-type spaces to the mixed-norm space on the unit ball are characterized here.

1. Introduction

Let $\mathbb{B} = \{z \in \mathbb{C}^n : |z| < 1\}$ be the open unit ball in $\mathbb{C}^n$, $\partial\mathbb{B}$ its boundary, dV_N the normalized volume measure on $\mathbb{B}$, and $H(\mathbb{B})$ the class of all holomorphic functions on $\mathbb{B}$. Strictly positive, bounded, continuous functions on $\mathbb{B}$ are called *weights*.

For an $f \in H(\mathbb{B})$ with the Taylor expansion $f(z) = \sum_{|\beta| \geq 0} a_\beta z^\beta$, let

$$\Re f(z) = \sum_{|\beta| \geq 0} |\beta| a_\beta z^\beta \quad (1.1)$$

be the radial derivative of f , where $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ is a multi-index, $|\beta| = \beta_1 + \dots + \beta_n$ and $z^\beta = z_1^{\beta_1} \dots z_n^{\beta_n}$.

A positive, continuous function v on the interval $[0, 1)$ is called normal [1] if there are $\delta \in [0, 1)$ and a and b , $0 < a < b$ such that

$$\begin{aligned} \frac{v(r)}{(1-r)^a} \text{ is decreasing on } [\delta, 1), \quad \lim_{r \rightarrow 1} \frac{v(r)}{(1-r)^a} = 0, \\ \frac{v(r)}{(1-r)^b} \text{ is increasing on } [\delta, 1), \quad \lim_{r \rightarrow 1} \frac{v(r)}{(1-r)^b} = \infty. \end{aligned} \quad (1.2)$$

If we say that a function $\nu : \mathbb{B} \rightarrow [0, \infty)$ is normal, we also assume that it is radial, that is, $\nu(z) = \nu(|z|)$, $z \in \mathbb{B}$.

Let μ be a weight. By $\mathcal{Z}_\mu(\mathbb{B}) = \mathcal{Z}_\mu$, we denote the class of all $f \in H(\mathbb{B})$ such that

$$z(f) := \sup_{z \in \mathbb{B}} \mu(z) |\Re^2 f(z)| < \infty, \quad (1.3)$$

and call it the *Zygmund-type class*. The quantity $z(f)$ is a seminorm. A norm on $\mathcal{Z}_\mu$ can be introduced by $\|f\|_{\mathcal{Z}} = |f(0)| + z(f)$. Zygmund-type class with this norm will be called the *Zygmund-type space*.

The *little Zygmund-type space* on $\mathbb{B}$, denoted by $\mathcal{Z}_{\mu,0}(\mathbb{B}) = \mathcal{Z}_{\mu,0}$, is the closed subspace of $\mathcal{Z}_\mu$ consisting of functions f satisfying the following condition

$$\lim_{|z| \rightarrow 1} \mu(z) |\Re^2 f(z)| = 0. \quad (1.4)$$

For $0 < p, q < \infty$, and ϕ normal, the mixed-norm space $H(p, q, \phi)(\mathbb{B}) = H(p, q, \phi)$ consists of all functions $f \in H(\mathbb{B})$ such that

$$\|f\|_{H(p,q,\phi)} = \left(\int_0^1 M_q^p(f, r) \frac{\phi^p(r)}{1-r} dr \right)^{1/p} < \infty, \quad (1.5)$$

where

$$M_q(f, r) = \left(\int_{\partial \mathbb{B}} |f(r\zeta)|^q d\sigma(\zeta) \right)^{1/q}, \quad (1.6)$$

and $d\sigma$ is the normalized surface measure on $\partial \mathbb{B}$. For $p = q$, $\phi(r) = (1 - r^2)^{(\alpha+1)/p}$, and $\alpha > -1$, the space is equivalent with the weighted Bergman space $A_\alpha^p(\mathbb{B})$.

In [2], the present author has introduced products of integral and composition operators on $H(\mathbb{B})$ as follows (see also [3–5]). Assume $g \in H(\mathbb{B})$, $g(0) = 0$, and φ is a holomorphic self-map of $\mathbb{B}$, then we define an operator on $H(\mathbb{B})$ by

$$P_\varphi^g(f)(z) = \int_0^1 f(\varphi(tz)) g(tz) \frac{dt}{t}, \quad f \in H(\mathbb{B}), \quad z \in \mathbb{B}. \quad (1.7)$$

The operator is an extension of the operator introduced in [6]. Here, we continue to study operator P_φ^g by characterizing the boundedness and compactness of the operator between Zygmund-type spaces and the mixed-norm space. For some results on related integral-type operators mostly in $\mathbb{C}^n$, see, for example, [3, 6–27] and the references therein.

In this paper, constants are denoted by C ; they are positive and may differ from one occurrence to the other. The notation $a \leq b$ means that there is a positive constant C such that $a \leq Cb$. If both $a \leq b$ and $b \leq a$ hold, then one says that $a \asymp b$.

2. Auxiliary Results

In this section, we quote several lemmas which are used in the proofs of the main results.

The first lemma was proved in [2].

Lemma 2.1. *Assume that φ is a holomorphic self-map of $\mathbb{B}$, $g \in H(\mathbb{B})$, and $g(0) = 0$. Then, for every $f \in H(\mathbb{B})$ it holds*

$$\Re \left[P_{\varphi}^g(f) \right](z) = f(\varphi(z))g(z). \quad (2.1)$$

The next Schwartz-type characterization of compactness [28] is proved in a standard way (see, e.g., the proof of the corresponding lemma in [11]), hence we omit its proof.

Lemma 2.2. *Assume $p, q > 0$, φ is a holomorphic self-map of $\mathbb{B}$, $g \in H(\mathbb{B})$, $g(0) = 0$, ϕ is normal, and μ is a weight. Then, the operator $P_{\varphi}^g : \mathcal{Z}_{\mu}$ (or $\mathcal{Z}_{\mu,0}$) $\rightarrow H(p, q, \phi)$ is compact if and only if for every bounded sequence $(f_k)_{k \in \mathbb{N}} \subset \mathcal{Z}_{\mu}$ (or $\mathcal{Z}_{\mu,0}$) converging to 0 uniformly on compacts of $\mathbb{B}$ we have $\lim_{k \rightarrow \infty} \|P_{\varphi}^g f_k\|_{H(p,q,\phi)} = 0$.*

The next lemma is folklore and can be found, for example, in [6] (one-dimensional case for standard power weights is due to Flett [29, Theorems 6 and 7]).

Lemma 2.3. *Assume that $0 < p, q < \infty$, ϕ is normal, and $m \in \mathbb{N}$. Then, the following asymptotic relationship holds for every $f \in H(\mathbb{B})$,*

$$\int_0^1 M_q^p(f, r) \frac{\phi^p(r)}{1-r} dr \asymp |f(0)|^p + \int_0^1 M_q^p(\Re^m f, r) (1-r)^{mp} \frac{\phi^p(r)}{1-r} dr. \quad (2.2)$$

Lemma 2.4. *Assume that μ is normal and $f \in \mathcal{Z}_{\mu}$. Then,*

$$|f(z)| \leq C \|f\|_{\mathcal{Z}_{\mu}} \left(1 + \int_0^{|z|} \int_0^t \frac{ds}{\mu(s)} dt \right), \quad z \in \mathbb{B}. \quad (2.3)$$

Moreover, if

$$\int_0^1 \int_0^t \frac{ds}{\mu(s)} dt < \infty, \quad (2.4)$$

then

$$|f(z)| \leq C \|f\|_{\mathcal{Z}_{\mu}}, \quad (2.5)$$

for any $z \in \mathbb{B}$.

Proof. By Lemma 2.3.1 in [21] applied to $\Re f$ we have that

$$|\Re f(z)| \leq C \|f\|_{\mathcal{Z}_\mu} \left(1 + \int_0^{|z|} \frac{ds}{\mu(s)} \right). \quad (2.6)$$

Hence, for $|z| \geq 1/2$, we have that

$$\left| f(z) - f\left(\frac{z}{2}\right) \right| \leq \int_{1/2}^1 |\Re f(tz)| \frac{dt}{t} \leq C \|f\|_{\mathcal{Z}_\mu} \int_{1/2}^1 \left(1 + \int_0^{t|z|} \frac{ds}{\mu(s)} \right) \frac{d(t|z|)}{|z|}, \quad (2.7)$$

so that

$$|f(z)| \leq M_\infty\left(f, \frac{1}{2}\right) + C \|f\|_{\mathcal{Z}_\mu} \left(1 + \int_0^{|z|} \int_0^t \frac{ds}{\mu(s)} dt \right), \quad (2.8)$$

where $M_\infty(f, 1/2) = \max_{|z| \leq 1/2} |f(z)|$.

If $|z| \leq 1/2$, then by the mean value property of the function $f(z) - f(0)$ (see [30]), Jensen's inequality, and Parseval's formula, we obtain

$$\begin{aligned} \max_{|z| \leq 1/2} |f(z) - f(0)|^2 &\leq 4^n \int_{|z| \leq 3/4} |f(w) - f(0)|^2 dV_N(w) \\ &\leq 4^n \int_{|z| \leq 3/4} |\Re f(w)|^2 dV_N(w) \\ &\leq 3^n \max_{|z| \leq 3/4} |\Re f(z)|^2. \end{aligned} \quad (2.9)$$

From (2.9) and (2.6), we obtain

$$\begin{aligned} M_\infty(f, 1/2) &\leq |f(0)| + \left(\sqrt{3}\right)^n \max_{|z| \leq 3/4} |\Re f(z)| \\ &\leq |f(0)| + \left(\sqrt{3}\right)^n C \|f\|_{\mathcal{Z}_\mu} \left(1 + \int_0^{3/4} \frac{ds}{\mu(s)} \right) \\ &\leq C \|f\|_{\mathcal{Z}_\mu}. \end{aligned} \quad (2.10)$$

From (2.8) and (2.10), (2.3) follows, from which by (2.4) the second statement follows. $\square$

Lemma 2.5. Assume μ is normal and (2.4) holds. Then, for every bounded sequence $(f_k)_{k \in \mathbb{N}} \subset \mathcal{Z}_\mu$ converging to 0 uniformly on compacts of $\mathbb{B}$, we have that

$$\lim_{k \rightarrow \infty} \sup_{z \in \mathbb{B}} |f_k(z)| = 0. \quad (2.11)$$

Proof. From (2.4), we have that for every $\varepsilon > 0$, there is a $\delta \in (0, \min\{\varepsilon, 1/2\})$ such that

$$\int_{(1-\delta)|z|}^{|z|} \int_0^t \frac{ds}{\mu(s)} dt < \varepsilon, \quad (2.12)$$

for $|z| > 1 - \delta$.

Hence, from (2.12) it follows that for each $k \in \mathbb{N}$ and $|z| \geq 1 - \delta$

$$\begin{aligned} |f_k(z) - f_k((1-\delta)z)| &\leq \int_{1-\delta}^1 |\Re f_k(tz)| \frac{dt}{t} \\ &\leq C \|f_k\|_{\mathcal{Z}_\mu} \int_{1-\delta}^1 \left(1 + \int_0^{t|z|} \frac{ds}{\mu(s)} \right) dt \\ &\leq C \|f_k\|_{\mathcal{Z}_\mu} \left(\varepsilon + \int_{(1-\delta)|z|}^{|z|} \int_0^t \frac{ds}{\mu(s)} dt \right). \end{aligned} \quad (2.13)$$

From (2.12) and (2.13), we obtain

$$|f_k(z)| \leq \sup_{|w| \leq 1-\delta} |f_k(w)| + 2C\varepsilon \sup_{k \in \mathbb{N}} \|f_k\|_{\mathcal{Z}_\mu}. \quad (2.14)$$

Letting $k \rightarrow \infty$ in this inequality, using the assumption that f_k converges to 0 on the compact $|w| \leq 1 - \delta$, and using the fact that ε is an arbitrary positive number, the lemma follows. $\square$

3. The Boundedness and Compactness of $P_\varphi^g : \mathcal{Z}_\mu$ (or $\mathcal{Z}_{\mu,0}$) $\rightarrow H(p, q, \phi)$

The boundedness and compactness of the operator $P_\varphi^g : \mathcal{Z}_\mu$ (or $\mathcal{Z}_{\mu,0}$) $\rightarrow H(p, q, \phi)$ are characterized in this section.

Theorem 3.1. Assume that $p, q > 0$, φ is a holomorphic self-map of $\mathbb{B}$, $g \in H(\mathbb{B})$, $g(0) = 0$, ϕ and μ are normal, and μ satisfies condition (2.4). Let

$$G(z) = \int_0^1 g(tz) \frac{dt}{t}. \quad (3.1)$$

Then, the following statements are equivalent:

- (a) $P_\varphi^g : \mathcal{Z}_{\mu,0} \rightarrow H(p, q, \phi)$ is bounded;
- (b) $P_\varphi^g : \mathcal{Z}_\mu \rightarrow H(p, q, \phi)$ is bounded;
- (c) $P_\varphi^g : \mathcal{Z}_{\mu,0} \rightarrow H(p, q, \phi)$ is compact;
- (d) $P_\varphi^g : \mathcal{Z}_\mu \rightarrow H(p, q, \phi)$ is compact;
- (e) $G \in H(p, q, \phi)$.

Moreover, if $P_\varphi^g : \mathcal{Z}_\mu \rightarrow H(p, q, \phi)$ is bounded, then the following asymptotic relations hold:

$$\|P_\varphi^g\|_{\mathcal{Z}_\mu \rightarrow H(p, q, \phi)} \asymp \|P_\varphi^g\|_{\mathcal{Z}_{\mu,0} \rightarrow H(p, q, \phi)} \asymp \|G\|_{H(p, q, \phi)}. \quad (3.2)$$

Proof. The implications (d) $\Rightarrow$ (b), (b) $\Rightarrow$ (a), (d) $\Rightarrow$ (c), and (c) $\Rightarrow$ (a) are obvious.

(a) $\Rightarrow$ (e) Since $P_\varphi^g : \mathcal{Z}_{\mu,0} \rightarrow H(p, q, \phi)$ is bounded and $f(z) \equiv 1 \in \mathcal{Z}_{\mu,0}$, by Lemma 2.1 we have that $G(z) = P_\varphi^g(1)(z) \in H(p, q, \phi)$. Moreover,

$$\|G\|_{H(p,q,\phi)} = \|P_\varphi^g(1)\|_{H(p,q,\phi)} \leq \|P_\varphi^g\|_{\mathcal{Z}_{\mu,0} \rightarrow H(p,q,\phi)}. \quad (3.3)$$

(e) $\Rightarrow$ (d) Assume that $(f_k)_{k \in \mathbb{N}} \subset \mathcal{Z}_\mu$ is a bounded sequence converging to 0 uniformly on compacts of $\mathbb{B}$. Then, by Lemmas 2.1, 2.3, and 2.5, we have

$$\begin{aligned} \|P_\varphi^g f_k\|_{H(p,q,\phi)} &\asymp \left| P_\varphi^g f_k(0) \right| + \left(\int_0^1 M_q^p(g f_k \circ \varphi, r) \frac{\phi^p(r)}{(1-r)^{1-p}} dr \right)^{1/p} \\ &\leq C \|G\|_{H(p,q,\phi)} \sup_{z \in \mathbb{B}} |f_k(z)| \rightarrow 0, \quad \text{as } k \rightarrow \infty, \end{aligned} \quad (3.4)$$

which along with Lemma 2.2 implies the compactness of $P_\varphi^g : \mathcal{Z}_\mu \rightarrow H(p, q, \phi)$.

From (2.4) and by Lemmas 2.3 and 2.4, we have

$$\begin{aligned} \|P_\varphi^g f\|_{H(p,q,\phi)} &\leq C \left(\int_0^1 M_q^p(g f \circ \varphi, r) \frac{\phi^p(r)}{(1-r)^{1-p}} dr \right)^{1/p} \\ &\leq C \|f\|_{\mathcal{Z}_\mu} \left(\int_0^1 M_q^p(g, r) \frac{\phi^p(r)}{(1-r)^{1-p}} dr \right)^{1/p} \\ &\leq C \|f\|_{\mathcal{Z}_\mu} \|G\|_{H(p,q,\phi)}. \end{aligned} \quad (3.5)$$

This, together with (3.3) and the inequality

$$\|P_\varphi^g\|_{\mathcal{Z}_{\mu,0} \rightarrow H(p,q,\phi)} \leq \|P_\varphi^g\|_{\mathcal{Z}_\mu \rightarrow H(p,q,\phi)}, \quad (3.6)$$

implies the asymptotic relations in (3.2), as desired. $\square$

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