Research Article

On an Integral-Type Operator Acting between Bloch-Type Spaces on the Unit Ball

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Let \( B \) denote the open unit ball of \( \mathbb{C}^n \). For a holomorphic self-map \( \varphi \) of \( B \) and a holomorphic function \( g \) in \( B \) with \( g(0) = 0 \), we define the following integral-type operator:

\[
I_g \varphi f(z) = \int_0^1 \Re f(\varphi(tz))g(tz)(dt/t), \quad z \in B.
\]

Here \( \Re f \) denotes the radial derivative of a holomorphic function \( f \) in \( B \). We study the boundedness and compactness of the operator between Bloch-type spaces \( B_\omega \) and \( B_\mu \), where \( \omega \) is a normal weight function and \( \mu \) is a weight function. Also we consider the operator between the little Bloch-type spaces \( B_{\omega,0} \) and \( B_{\mu,0} \).

1. Introduction

Let \( B \) denote the open unit ball of the \( n \)-dimensional complex vector space \( \mathbb{C}^n \) and \( H(B) \) the space of all holomorphic functions on \( B \). For \( f \in H(B) \) with the Taylor expansion \( f(z) = \sum_{|\gamma| \geq 0} a_\gamma z^\gamma \), let

\[
\Re f(z) = \sum_{|\gamma| \geq 0} |\gamma| a_\gamma z^\gamma
\]

be the radial derivative of \( f \), where \( \gamma = (\gamma_1, \ldots, \gamma_n) \) is a multi-index, \( |\gamma| = \gamma_1 + \cdots + \gamma_n \), and \( z^\gamma = z_1^{\gamma_1} \cdots z_n^{\gamma_n} \). It is well known that

\[
\Re f(z) = \sum_{j=1}^n \overline{z_j} \frac{\partial f}{\partial z_j}(z) = \langle \nabla f(z), \overline{z} \rangle,
\]

where \( \nabla \) is the usual gradient on \( \mathbb{C}^n \).
Let $\varphi$ be a holomorphic self-map of $\mathbb{B}$ and $g \in H(\mathbb{B})$ with $g(0) = 0$. Then $\varphi$ and $g$ define an operator $I_\varphi^g$ on $H(\mathbb{B})$ as follows:

$$I_\varphi^g f(z) = \int_0^1 \Re f(\varphi(tz)) g(tz) \frac{dt}{t}, \quad f \in H(\mathbb{B}), \ z \in \mathbb{B}. \quad (1.3)$$

The following important formula involving $\Re$ and $I_\varphi^g f$ was proved, for example, in [1]

$$\Re \left[ I_\varphi^g f \right](z) = \Re f(\varphi(z)) g(z), \quad z \in \mathbb{B}. \quad (1.4)$$

Motivated by papers [2, 3], operators $I_\varphi^g$ were introduced by the first author of the present paper and Zhu in [1, 4–6], where its boundedness and compactness from the $\alpha$-Bloch space, the Zygmund space, the mixed-norm space, and the generalized weighted Bergman space into the Bloch-type space on the unit ball are studied. In our previous work [7], we studied the boundedness and compactness of $I_\varphi^g$ acting between weighted-type spaces. For related operators on $\mathbb{C}^n$ see, for example, [8–21] and the references therein.

Let $\omega$ be a strictly positive continuous function on $\mathbb{B}$ (weight). If $\omega(z) = \omega(|z|)$ for every $z \in \mathbb{B}$, we call it radial weight. A weight $\omega$ is called normal ([9, 22]) if it is radial and there are $a$ and $b$, $0 < a < b < \infty$ such that $\omega(r)/(1-r)^a$ is decreasing on $[0, 1)$, $\omega(r)/(1-r)^b$ is increasing on $[0, 1)$,

$$\lim_{r \to 1^-} \frac{\omega(r)}{(1-r)^a} = 0, \quad \lim_{r \to 1^-} \frac{\omega(r)}{(1-r)^b} = \infty. \quad (1.5)$$

A radial weight $\omega$ is called typical if it is nonincreasing with respect to $|z|$ and $\omega(z) \to 0$ as $|z| \to 1$. If $\omega$ is normal, then by the monotonicity of $\omega(r)/(1-r)^a$, for $0 \leq r_1 < r < 1$, we have that

$$\omega(r) = (1-r)^a \frac{\omega(r)}{(1-r)^a} < (1-r)^a \frac{\omega(r_1)}{(1-r_1)^a} < \omega(r_1), \quad (1.6)$$

that is, $\omega$ is decreasing on $[0, 1)$. On the other hand, from the first equality in (1.5), we have that for any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$0 < \omega(r) < \varepsilon(1-r)^a, \quad (\delta < r < 1), \quad (1.7)$$

which implies $\lim_{r \to 1^-} \omega(r) = 0$. Hence every normal weight $\omega$ is also typical.

For a weight $\omega$, the associated weight $\tilde{\omega}$ ([23]) is defined by

$$\tilde{\omega}(z) := \frac{1}{\sup \left\{ |f(z)| : f \in H^\infty_{\omega}, \|f\|_{H^\infty_{\omega}} \leq 1 \right\}}, \quad z \in \mathbb{B}. \quad (1.8)$$
Here $H^\infty_\omega$ denotes the weighted-type space consisting of all $f \in H(\mathbb{B})$ with

$$
\|f\|_{H^\infty_\omega} = \sup_{z \in \mathbb{B}} \omega(z)|f(z)| < \infty
$$

(1.9)

(see, e.g., [23, 24]). Associated weights assist us in studying of weighted-type spaces of holomorphic functions. It is known that associated weights are also continuous, $0 < \omega \leq \tilde{\omega}$, and for each $z \in \mathbb{B}$, we can find an $f_z \in H^\infty_\omega$, $\|f_z\|_{H^\infty_\omega} < 1$ such that $f_z(z) = 1/\tilde{\omega}(z)$. Let $H^\infty_{\omega,0}$ be the little weighted-type space, that is, the space of all $f \in H(\mathbb{B})$ such that $\omega(z)|f(z)| \to 0$ as $|z| \to 1^-$. If $\omega$ is typical, then the unit ball $B_{H^\infty_\omega}$ is the closure of $B_{H^\infty_{\omega,0}}$ for the compact open topology. Hence we have

$$
\tilde{\omega}(z) = \sup \left\{ \frac{1}{|f(z)|} : f \in H^\infty_{\omega,0}, \|f\|_{H^\infty_\omega} \leq 1 \right\}
$$

(1.10)

and so for each $z \in \mathbb{B}$, we can choose an $f_z \in B_{H^\infty_{\omega,0}}$ such that $f_z(z) = 1/\tilde{\omega}(z)$. A weight $\omega$ is called essential if it satisfies that $\tilde{\omega} \leq C\omega$ for some positive constant $C$. By the arguments in [25], we see that a normal weight function is also essential. For some examples of essential weights, see, for example, [25]. Related results can also be found in [22, 26].

The Bloch-type space $B_\omega$ is the space of all holomorphic functions $f$ on $\mathbb{B}$ such that

$$
b_\omega(f) = \sup_{z \in \mathbb{B}} \omega(z)|\Re f(z)| < \infty,
$$

(1.11)

where $\omega$ is a weight (see, e.g., [20]). The little Bloch-type space $B_{\omega,0}$ consists of all $f \in H(\mathbb{B})$ such that

$$
\lim_{|z| \to 1^-} \omega(z)|\Re f(z)| = 0.
$$

(1.12)

Both spaces $B_\omega$ and $B_{\omega,0}$ are Banach spaces with the norm

$$
\|f\|_{B_\omega} = \|f(0)\| + b_\omega(f),
$$

(1.13)

and $B_{\omega,0}$ is a closed subspace of $B_\omega$. When $\omega(r) = 1 - r^2$, the space $B_\omega$ is a classical Bloch space.

The purpose of this paper is to characterize the boundedness and compactness of the operators $I^*_\varphi : B_\omega \to B_\mu$ and $I^*_\varphi : B_{\omega,0} \to B_{\mu,0}$.

Throughout this paper, we assume that $\varphi$ is a holomorphic self-map of $\mathbb{B}$ and $g \in H(\mathbb{B})$ with $g(0) = 0$. Furthermore, some constants are denoted by $C$; they are positive and may differ from one occurrence to the other. The notation $a \leq b$ means that there exists a positive constant $C$ such that $a \leq Cb$. Moreover, if both $a \leq b$ and $b \leq a$ hold, then one says that $a \asymp b$.

2. Auxiliary Results

Here we formulate and prove some auxiliary results which are used in the proofs of the main ones.
The following lemma was proved in [20, Theorem 2.1].

**Lemma 2.1.** Let \( \omega \) be a normal weight function and \( f \in H(\mathbb{B}) \). Then \( f \in \mathcal{B}_\omega \) if and only if
\[
\|f\|_{\mathcal{B}_\omega} = |f(0)| + \sup_{z \in \overline{\mathbb{B}}} \omega(z)|\nabla f(z)|.
\]

Moreover, \( f \in \mathcal{B}_{\omega,0} \) if and only if \( \lim_{|z| \to 1^-} \omega(z)|\nabla f(z)| = 0 \).

As an application of Lemma 2.1, we have the following result.

**Lemma 2.2.** Let \( \omega \) be a normal weight function and \( f \in \mathcal{B}_\omega \). Then \( f \in \mathcal{B}_{\omega,0} \) if and only if it holds that \( \lim_{r \to 1^-} \|f_r - f\|_{\mathcal{B}_\omega} = 0 \), where \( f_r(z) = f(rz) \).

**Proof.** Take an \( f \in \mathcal{B}_{\omega,0} \). For a fixed \( \varepsilon > 0 \), by Lemma 2.1, we can choose a \( \delta_0 \in (0,1) \) such that
\[
\omega(z)|\nabla f(z)| < \frac{\varepsilon}{2}
\]
for any \( z \in \mathbb{B} \setminus \delta_0 \overline{\mathbb{B}} \). Since \( (\partial f_r/\partial z_j)(z) = r(\partial f/\partial z_j)(rz) \) for \( j \in \{1,\ldots,n\} \), \( r \in (0,1) \), and \( z \in \mathbb{B} \), we have
\[
\|f_r - f\|_{\mathcal{B}_\omega} = \sup_{z \in \overline{\mathbb{B}}} \omega(z)|r\nabla f(rz) - \nabla f(z)|
\leq \sup_{z \in \mathbb{B} \setminus \delta_0 \overline{\mathbb{B}}} \omega(z)|r\nabla f(rz) - \nabla f(z)|
+ \sup_{z \in \mathbb{B} \setminus \delta_0 \overline{\mathbb{B}}} \omega(z)|r\nabla f(rz) - \nabla f(z)|.
\]

Since
\[
\max_{|z| \leq \delta_0} |r\nabla f(rz) - \nabla f(z)| \to 0, \quad \text{as} \quad r \to 1^-,
\]
we see that the second term in (2.3) converges to 0 as \( r \to 1^- \).

If \( r \in (\delta_0,1) \) and \( z \in \mathbb{B} \setminus \delta_0 \overline{\mathbb{B}} \), then by (2.2) we have
\[
\omega(rz)|\nabla f(rz)| < \frac{\varepsilon}{2}.
\]

By (1.6) we have that \( \omega(z) \leq \omega(rz) \) for \( r,|z| \in [0,1) \).

Hence we have
\[
\sup_{z \in \mathbb{B} \setminus \delta_0 \overline{\mathbb{B}}} \omega(z)|r\nabla f(rz) - \nabla f(z)| \leq \sup_{z \in \mathbb{B} \setminus \delta_0 \overline{\mathbb{B}}} \omega(rz)|\nabla f(rz)| + \sup_{z \in \mathbb{B} \setminus \delta_0 \overline{\mathbb{B}}} \omega(z)|\nabla f(z)| < \varepsilon
\]
for all \( r \in (\delta_0,1) \). This proves that \( \lim_{r \to 1^-} \|f_r - f\|_{\mathcal{B}_\omega} = 0 \) whenever \( f \in \mathcal{B}_{\omega,0} \).
Corollary 2.3. Let \( \omega \) be a normal weight function. Then the set of all holomorphic polynomials is dense in \( \mathcal{B}_{\omega,0} \).

Proof. For the homogeneous expansion \( f = \sum_{k=0}^{\infty} F_k \) of an \( f \in \mathcal{B}_{\omega,0} \), we set \( P_j = \sum_{k=0}^{j} F_k \) for each \( j \in \mathbb{N} \). Since \( P_j \to f \) uniformly on compact subsets of \( \mathbb{B} \) as \( j \to \infty \), we see that \( \Re[P_j] \to \Re[f] \) uniformly on \( \mathbb{B} \) for any \( r \in (0,1) \). Moreover, we have

\[
\left\| P_r - f \right\|_{\mathcal{B}_{\omega}} \leq \left\| P_r - f_r \right\|_{\mathcal{B}_{\omega}} + \left\| f_r - f \right\|_{\mathcal{B}_{\omega}} \leq \sup_{z \in \partial \mathbb{B}} \omega(z) \sup_{z \in \mathbb{B}} \left| \Re[P_r](z) - \Re[f_r](z) \right| + \left\| f_r - f \right\|_{\mathcal{B}_{\omega}}.
\]

Combining this with Lemma 2.2, we get the desired result.

The following lemma can be found in [1, Lemma 3]. Its proof is similar to the proof of the corresponding one-dimensional result in [27], for the case of the little Bloch space \( \mathcal{B}_{(1-r),0} \). Hence we omit the proof.

Lemma 2.4. A closed subset \( K \) in \( \mathcal{B}_{\omega,0} \) is compact if and only if it is bounded and

\[
\lim_{|z| \to 1^-} \sup_{j \in K} \omega(z) |\Re f(z)| = 0.
\]

The following lemma is very useful for estimating the norm of the Bloch-type space.

Lemma 2.5. Assume that \( m \) is a positive integer and \( \omega \) is normal. Then for every \( f \in H(\mathbb{B}) \),

\[
\sup_{z \in \mathbb{B}} \omega(z) |f(z)| \leq |f(0)| + \sup_{z \in \mathbb{B}} (1 - |z|)^m \omega(z) |\Re f(z)|.
\]

Proof. For the details of the proof, we can refer [9] or [28].
3. The Boundedness of Operator $I_\omega^\phi$

In this section we consider the boundedness of the operator $I_\omega^\phi : \mathcal{B}_\omega \to \mathcal{B}_\mu$ or $I_\omega^\phi : \mathcal{B}_{\omega,0} \to \mathcal{B}_{\mu,0}$.

**Theorem 3.1.** Let $\omega$ be a normal weight function and $\mu$ a weight function. Then the following conditions are equivalent:

(a) $I_\omega^\phi : \mathcal{B}_\omega \to \mathcal{B}_\mu$ is bounded;

(b) $I_\omega^\phi : \mathcal{B}_{\omega,0} \to \mathcal{B}_\mu$ is bounded;

(c) $\phi$ and $g$ satisfy

$$\sup_{z \in \mathcal{B}} \frac{\mu(z)|g(z)||\varphi(z)|}{\tilde{\omega}(\varphi(z))} < \infty. \quad (3.1)$$

Moreover, if $I_\omega^\phi : \mathcal{B}_\omega \to \mathcal{B}_\mu$ is bounded, then

$$\left\| I_\omega^\phi \right\|_{\mathcal{B}_\omega \to \mathcal{B}_\mu} \geq \sup_{z \in \mathcal{B}} \frac{\mu(z)|g(z)||\varphi(z)|}{\tilde{\omega}(\varphi(z))}. \quad (3.2)$$

**Proof.** The implication (a) $\Rightarrow$ (b) is clear, so we only prove (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (a).

(b) $\Rightarrow$ (c): assume that $I_\omega^\phi : \mathcal{B}_{\omega,0} \to \mathcal{B}_\mu$ is bounded and fix $z \in \mathcal{B}$. We may assume that $\varphi(z) \neq 0$. For $w := \varphi(z)$, there exists $h_w \in H_{\omega,0}^\infty$ such that $\|h_w\|_{H_{\omega,0}^\infty} \leq 1$ and $h_w(\omega) = 1/\tilde{\omega}(\omega)$. We define the function $f_w$ as follows:

$$f_w(v) = \int_0^1 h_w(tv) \frac{(tv,w)}{|w|} \frac{dt}{t}, \quad v \in \mathcal{B}. \quad (3.3)$$

Since $\Re f_w(v) = h_w(v)(\langle v, w \rangle / |w|)$, we see that $f_w \in \mathcal{B}_{\omega,0}$ and $\|f_w\|_{\mathcal{B}_\omega} \leq 1$. Hence, by (1.4), we have

$$\left\| I_\omega^\phi f \right\|_{\mathcal{B}_\omega} \geq \left\| I_\omega^\phi f \right\|_{\mathcal{B}_\omega} \geq \mu(z)|g(z)||\Re f_w(\varphi(z))| = \frac{\mu(z)|g(z)||\varphi(z)|}{\tilde{\omega}(\varphi(z))}, \quad (3.4)$$

and so condition (3.1) is true.

(c) $\Rightarrow$ (a): we assume (3.1) and take an $f \in \mathcal{B}_\omega$. Since $\omega$ is an essential weight (due to its normality), (1.4) gives

$$\mu(z)|\Re I_\omega^\phi f(z)| \leq \mu(z)|g(z)||\Re f(\varphi(z))| \leq \mu(z)|g(z)||\varphi(z)| \frac{\tilde{\omega}(\varphi(z))|\nabla f(\varphi(z))|}{\tilde{\omega}(\varphi(z))} \leq \mu(z)|g(z)||\varphi(z)| \sup_{w \in \mathcal{B}} \omega(w)|\nabla f(w)|. \quad (3.5)$$
for any \( z \in \mathbb{B} \). By Lemma 2.1, we have \( \sup_{w \in \mathbb{B}} \omega(w) |\nabla f(w)| \leq \|f\|_{B_\mu} \), and so we obtain
\[
\left\| I_\varphi^g f \right\|_{B_\mu} \leq \sup_{z \in \mathbb{B}} \frac{\mu(z) |g(z)| |\varphi(z)|}{\omega(\varphi(z))} \|f\|_{B_\mu}.
\] (3.6)
This implies that \( I_\varphi^g : B_\omega \to B_\mu \) is bounded. The relation (3.2) follows from (3.4) and (3.6). This completes the proof. \( \square \)

**Theorem 3.2.** Let \( \omega \) be a normal weight function and \( \mu \) a weight function. Then the following conditions are equivalent:

(a) \( I_\varphi^g : B_{\omega,0} \to B_{\mu,0} \) is bounded;

(b) \( \varphi \) and \( g \) satisfy
\[
\lim_{|z| \to 1^+} \mu(z) |g(z)| |\varphi(z)| = 0, \quad \sup_{z \in \mathbb{B}} \frac{\mu(z) |g(z)| |\varphi(z)|}{\omega(\varphi(z))} < \infty.
\] (3.7)

**Proof.** (a) \( \Rightarrow \) (b): as in the proof of Theorem 3.1, for fixed \( z \in \mathbb{B} \) and \( w = \varphi(z) \), we see that \( \varphi \) and \( g \) satisfy the condition
\[
\sup_{z \in \mathbb{B}} \frac{\mu(z) |g(z)| |\varphi(z)|}{\omega(\varphi(z))} < \infty.
\] (3.8)
On the other hand, since the normality of \( \omega \) implies that the function \( \pi_j(z) := z_j \) \( (1 \leq j \leq n) \) belongs to \( B_{\omega,0} \), we obtain that \( \mu(z) |g(z)| |\varphi_j(z)| \to 0 \) for each \( j \), and so \( \mu(z) |g(z)| |\varphi(z)| \to 0 \) as \( |z| \to 1^+ \).

(b) \( \Rightarrow \) (a): the assumption \( \lim_{|z| \to 1^+} \mu(z) |g(z)| |\varphi(z)| = 0 \) shows that \( I_\varphi^g p \in B_{\mu,0} \) for any polynomial \( p \). For each \( f \in B_{\omega,0} \), by Corollary 2.3, we can choose a sequence of polynomials \( \{p_j\}_{j \in \mathbb{N}} \) such that \( \|f - p_j\|_{B_\omega} \to 0 \) as \( j \to \infty \). Furthermore, the assumption
\[
\sup_{z \in \mathbb{B}} \frac{\mu(z) |g(z)| |\varphi(z)|}{\omega(\varphi(z))} < \infty
\] (3.9)
shows that \( I_\varphi^g : B_{\omega,0} \to B_{\mu} \) is bounded by Theorem 3.1. Thus we obtain
\[
0 \leq \left\| I_\varphi^g f - I_\varphi^g p_j \right\|_{B_\mu} \leq \left\| I_\varphi^g \right\|_{B_{\omega,0} \to B_{\mu}} \|f - p_j\|_{B_\omega} \to 0 \quad (\text{as } j \to \infty).
\] (3.10)
Since \( I_\varphi^g f \in B_{\mu} \), \( \{I_\varphi^g p_j\}_{j \in \mathbb{N}} \subset B_{\mu,0} \), and \( B_{\mu,0} \) is closed in \( B_{\mu} \), we have \( I_\varphi^g f \in B_{\mu,0} \) for any \( f \in B_{\omega,0} \). Hence \( I_\varphi^g (B_{\omega,0}) \subseteq B_{\mu,0} \) which means that \( I_\varphi^g : B_{\omega,0} \to B_{\mu,0} \) is bounded. The proof is accomplished. \( \square \)

The following corollary is an immediate consequence of Theorems 3.1 and 3.2.

**Corollary 3.3.** Let \( \omega \) be a normal weight function and \( \mu \) a weight function. Then \( I_\varphi^g : B_{\omega,0} \to B_{\mu,0} \) is bounded if and only if \( \lim_{|z| \to 1^+} \mu(z) |g(z)| |\varphi(z)| = 0 \) and \( I_\varphi^g : B_\omega \to B_\mu \) is bounded.
4. The Compactness of Operator $I^S_\varphi$

In this section we characterize the compactness of $I^S_\varphi : B_\varphi \to B_\mu$ or $I^S_\varphi : B_{\varphi,0} \to B_{\mu,0}$. To do this, we need the following standard lemma (see, e.g., [13, Lemma 3]).

**Lemma 4.1.** Let $\omega$ and $\mu$ be weight functions. Suppose that the operator $I^S_\varphi : B_\varphi$ (or $B_{\varphi,0}$) $\to B_\mu$ is bounded. Then $I^S_\varphi : B_\omega$ (or $B_{\omega,0}$) $\to B_\mu$ is compact if and only if for every bounded sequence $\{f_j\}_{j \in \mathbb{N}}$ in $B_\omega$ (or $B_{\omega,0}$) which converges to 0 uniformly on compact subsets of $B$, $\|I^S_\varphi f_j\|_{B_\mu} \to 0$ as $j \to \infty$.

**Theorem 4.2.** Let $\omega$ and $\mu$ be weight functions. Suppose that $\varphi$ is a holomorphic self-map of $B$ such that $\|\varphi\|_\infty < 1$ and the operator $I^S_\varphi : B_\omega \to B_\mu$ is bounded. Then $I^S_\varphi : B_\omega \to B_\mu$ is compact. Here $\|\varphi\|_\infty$ denotes the supremum $\sup_{z \in \overline{B}} |\varphi(z)|$.

**Proof.** Since $\|\varphi\|_\infty < 1$, we see that $|\varphi(z)| \leq r$ for some $r \in (0,1)$ and any $z \in B$. From the proof of Theorem 3.1, we see that the boundedness of $I^S_\varphi : B_\omega \to B_\mu$ implies

$$M := \sup_{z \in B} \frac{\mu(z) |g(z)||\varphi(z)|}{\tilde{\omega}(\varphi(z))} < \infty. \quad (4.1)$$

Thus we obtain that

$$\sup_{z \in B} \mu(z) |g(z)||\varphi(z)| \leq M \sup_{w \in \overline{B}} \tilde{\omega}(w) < \infty. \quad (4.2)$$

Take a bounded sequence $\{f_j\}_{j \in \mathbb{N}}$ in $B_\omega$ such that $f_j \to 0$ uniformly on compact subsets of $B$ as $j \to \infty$. By (1.4), we have

$$\|I^S_\varphi f_j\|_{B_\mu} = \sup_{z \in B} \mu(z) |g(z)||\nabla f_j(\varphi(z))|$$

$$\leq \sup_{z \in B} \mu(z) |g(z)||\varphi(z)||\nabla f_j(\varphi(z))|$$

$$\leq M \sup_{w \in \overline{B}} \tilde{\omega}(w) \sup_{w \in \overline{B}} \|\nabla f_j(w)\|. \quad (4.3)$$

Since $\partial f_j/\partial z_k$ ($1 \leq k \leq n$) also converges to 0 uniformly on $\overline{B}$ as $j \to \infty$, (4.2) and (4.3) show that $\|I^S_\varphi f_j\|_{B_\mu} \to 0$ as $j \to \infty$. From Lemma 4.1, it follows that $I^S_\varphi : B_\omega \to B_\mu$ is compact, and so we get the assertions. □

**Lemma 4.3.** Suppose that $\omega$ is a weight function. Then there exists a sequence $\{f_k\}_{k \in \mathbb{N}}$ in the closed unit ball of $B_\omega$ such that $f_k \to 0$ uniformly on compact subsets of $B$ as $k \to \infty$.

**Proof.** Let $\{w_k\}_{k \in \mathbb{N}} \subset B$ with $|w_k| \to 1^-$ as $k \to \infty$. For each $w_k$, there exists $h_k := h_{w_k} \in H^\infty_\omega$ such that $\|h_k\|_{H^\infty_\omega} \leq 1$ and $h_k(w_k) = 1/\tilde{\omega}(w_k)$. We define $f_k$ as follows:

$$f_k(z) = \int_0^1 h_k(tz) \left\{ \frac{(t z, w_k)}{|w_k|} \right\}^{1/(1-|w_k|)} \frac{dt}{t}, \quad z \in B. \quad (4.4)$$
Since \( f_k(0) = 0 \) and \( |\Re f_k(z)| \leq |h_k(z)| \), we have \( \{f_k\}_{k \in \mathbb{N}} \subset \mathcal{B}_\omega \) and \( \|f_k\|_{\mathcal{B}_\omega} \leq 1 \) for each \( k \in \mathbb{N} \).

For any compact subset \( \mathcal{K} \) of \( \mathbb{B} \), we can choose an \( r \in (0, 1) \) such that \( \mathcal{K} \subset r\overline{\mathbb{B}} \). Hence we obtain that for any \( z \in \mathcal{K} \)

\[
|f_k(z)| \leq \int_0^1 |h_k(tz)| t^{1/|\omega(t)|} dt \leq \|h_k\|_{H^\omega} \int_0^1 \frac{1}{\omega(tz)} t^{1/|\omega(t)|} dt \leq \max_{w \in r\overline{\mathbb{B}}} \frac{1}{\omega(w)} (1 - |\omega_k|).
\]

(4.5)

From the above inequality, it follows that \( f_k \) converges to 0 uniformly on compact subsets of \( \mathbb{B} \) as \( k \to \infty \). This completes the proof. \( \square \)

**Remark 4.4.** If we assume that \( \omega \) is typical in Lemma 4.3, then we can choose \( h_k \in H^\omega_{\omega,0} \). In this case, hence, we see that \( f_k \) belongs to \( \mathcal{B}_{\omega,0} \) for each \( k \in \mathbb{N} \).

**Theorem 4.5.** Let \( \omega \) be a normal weight function and \( \mu \) a weight function. Suppose that the operator \( I^g_\psi : \mathcal{B}_\omega \to \mathcal{B}_\mu \) is bounded and \( \|\varphi\|_{\infty} = 1 \). Then the following conditions are equivalent:

(a) \( I^g_\psi : \mathcal{B}_\omega \to \mathcal{B}_\mu \) is compact;

(b) \( I^g_\psi : \mathcal{B}_{\omega,0} \to \mathcal{B}_\mu \) is compact;

(c) \( \varphi \) and \( g \) satisfy

\[
\lim_{|\varphi(z)| \to 1^-} \frac{\mu(z)|g(z)| |\varphi(z)|}{\hat{\omega}(\varphi(z))} = 0.
\]

(4.6)

**Proof.** (a) \( \Rightarrow \) (b): this implication is obvious.

(b) \( \Rightarrow \) (c): take a sequence \( \{z_k\}_{k \in \mathbb{N}} \) in \( \mathbb{B} \) with \( |\varphi(z_k)| \to 1^- \) as \( k \to \infty \) and put \( w_k = \varphi(z_k) \) for each \( k \). Then, by Remark 4.4 after Lemma 4.3, there exists a sequence \( \{f_k\}_{k \in \mathbb{N}} \) in \( \mathcal{B}_{\omega,0} \) such that \( \sup_{k \in \mathbb{N}} \|f_k\|_{\mathcal{B}_\omega} \leq 1 \) and \( f_k \to 0 \) uniformly on compact subsets of \( \mathbb{B} \) as \( k \to \infty \). By Lemma 4.1, the compactness of \( I^g_\psi : \mathcal{B}_{\omega,0} \to \mathcal{B}_\mu \) implies that \( \|I^g_{\psi f_k}\|_{\mathcal{B}_\mu} \to 0 \) as \( k \to \infty \).

On the other hand, (1.4) gives \( \Re[I^g_{\psi f_k}](z) = \Re f_k(\varphi(z))g(z) \), and so we have

\[
\|I^g_{\psi f_k}\|_{\mathcal{B}_\mu} \geq \mu(z_k) \Re f_k(\varphi(z_k)) |g(z_k)| \geq \mu(z_k) \Re f_k(\varphi(z_k)) |g(z_k)||\varphi(z_k)|.
\]

(4.7)

From the construction (4.4) of \( f_k \), we obtain

\[
\Re f_k(\varphi(z_k)) = \left|\varphi(z_k)\right|^{1/(1-|\varphi(z_k)|)} \hat{\omega}(\varphi(z_k)),
\]

(4.8)

for each \( k \in \mathbb{N} \). Combining this with (4.7), we have

\[
\|I^g_{\psi f_k}\|_{\mathcal{B}_\mu} \geq \mu(z_k) |g(z_k)| |\varphi(z_k)| \hat{\omega}(\varphi(z_k))^{1/(1-|\varphi(z_k)|)} |\varphi(z_k)|^{1/(1-|\varphi(z_k)|)}.
\]

(4.9)
Letting \( k \to \infty \), we have
\[
\lim_{k \to \infty} \frac{\mu(z_k) |g(z_k)| \|\varphi(z_k)\|}{\tilde{\omega}(\varphi(z_k))} = 0,
\] (4.10)
for any sequence \( \{z_k\}_{k \in \mathbb{N}} \) with \( |\varphi(z_k)| \to 1 \). This proves that (4.6) is true.

(c) \( \Rightarrow \) (a): we will prove the following estimate:
\[
\left\| I^\delta_f \right\|_e \leq \limsup_{|\varphi(z)| \to 1} \frac{\mu(z) |g(z)| \|\varphi(z)\|}{\tilde{\omega}(\varphi(z))}.
\] (4.11)
Here \( \|I^\delta_f\|_e \) denotes the essential norm of \( I^\delta_f : \mathcal{B}_\omega \to \mathcal{B}_\mu \), namely,
\[
\left\| I^\delta_f \right\|_e = \inf \left\{ \left\| I^\delta_f + \mathcal{K} \right\|_{\mathcal{B}_\omega \to \mathcal{B}_\mu} : \mathcal{K} : \mathcal{B}_\omega \to \mathcal{B}_\mu \text{ is compact} \right\}.
\] (4.12)

Now we take a sequence \( \{r_i\}_{i \in \mathbb{N}} \subset (0,1) \) which increasingly converges to 1 and put
\[
I^\delta_{r_i} f(z) = \int_0^1 \Re f(r_i \varphi(tz)) g(tz) \frac{dt}{t}.
\] (4.13)
Since \( \|r_i \varphi\|_\infty \leq r_i < 1 \), Theorem 4.2 implies that \( I^\delta_{r_i} : \mathcal{B}_\omega \to \mathcal{B}_\mu \) is compact for each \( i \in \mathbb{N} \). For any \( f \in \mathcal{B}_\omega \) with \( \|f\|_{\mathcal{B}_\omega} \leq 1 \), from (1.4) it follows that
\[
\left\| I^\delta_f - I^\delta_{r_i} f \right\|_{\mathcal{B}_\omega} = \sup_{z \in \mathcal{B}} \mu(z) |g(z)| \|\varphi(z)\| \left| \Re f(\varphi(z)) - \Re f(r_i \varphi(z)) \right|
\leq \sup_{R < |\varphi(z)| \leq 1} \mu(z) |g(z)| \|\varphi(z)\| \left| \Re f(r_i \varphi(z)) \right|
+ \sup_{|\varphi(z)| \leq R} \mu(z) |g(z)| \|\varphi(z)\| \left| \Re f(\varphi(z)) \right|,
\] (4.14)
for some fixed \( R \in (0, 1) \). The essentiality of \( \omega \) and Lemma 2.1 give
\[
\mu(z) |g(z)| \left| \Re f(\varphi(z)) \right| \leq \frac{\mu(z) |g(z)| \|\varphi(z)\|}{\tilde{\omega}(\varphi(z))} \tilde{\omega}(\varphi(z)) \left| \nabla f(\varphi(z)) \right|
\leq \frac{\mu(z) |g(z)| \|\varphi(z)\|}{\tilde{\omega}(\varphi(z))} \sup_{w \in \mathcal{B}} \omega(w) \left| \nabla f(w) \right| \leq \frac{\mu(z) |g(z)| \|\varphi(z)\|}{\tilde{\omega}(\varphi(z))}.
\] (4.15)
Similarly, we also have

\[
\mu(z)|g(z)||\Re f(r_\omega \varphi(z))| \leq \frac{\mu(z)|g(z)||\varphi(z)|}{\tilde{\omega}(r_\omega \varphi(z))},
\]

for each \(l \in \mathbb{N}\). The normality of \(\omega\) implies that

\[
\frac{\omega(r_\omega \varphi(z))}{(1 - |r_\omega \varphi(z)|)^a} \geq \frac{\omega(\varphi(z))}{(1 - |\varphi(z)|)^a},
\]

for each \(l \in \mathbb{N}\) and some \(a > 0\), and so by the essentiality,

\[
\frac{\tilde{\omega}(r_\omega \varphi(z))}{(1 - |r_\omega \varphi(z)|)^a} \geq \frac{\tilde{\omega}(\varphi(z))}{(1 - |\varphi(z)|)^a}.
\]

Thus (4.16) and (4.18) give

\[
\mu(z)|g(z)||\Re f(r_\omega \varphi(z))| \leq \frac{\mu(z)|g(z)||\varphi(z)|}{\tilde{\omega}(\varphi(z))},
\]

for each \(l \in \mathbb{N}\). By (4.15) and (4.19), we obtain

\[
\sup_{R<\|\varphi(z)\|<1} \mu(z)|g(z)||\Re f(\varphi(z)) - \Re f(r_\omega \varphi(z))| \leq \sup_{R<\|\varphi(z)\|<1} \frac{\mu(z)|g(z)||\varphi(z)|}{\tilde{\omega}(\varphi(z))}.
\]

When \(\|\varphi(z)\| \leq R\), by using the mean value theorem, we have

\[
\mu(z)|g(z)||\Re f(\varphi(z)) - \Re f(r_\omega \varphi(z))| \leq (1 - r_1)\mu(z)|g(z)||\varphi(z)|\sup_{|w|\leq R} |\nabla \Re f(w)|
\leq \frac{1 - r_1}{1 - R} \max_{w \in \mathbb{B}} \frac{1}{\omega(w)} \sup_{z \in \mathbb{B}} \mu(z)|g(z)||\varphi(z)|
\]

\[
\times \sup_{w \in \mathbb{B}} \omega(w)(1 - |w|)|\nabla \Re f(w)|.
\]

Since \(\omega(w)(1 - |w|)\) is also normal, by Lemmas 2.1 and 2.5, we have

\[
\sup_{w \in \mathbb{B}} \omega(w)(1 - |w|)|\nabla \Re f(w)| \times \sup_{w \in \mathbb{B}} \omega(w)(1 - |w|)|\Re^2 f(w)|
\leq \sup_{w \in \mathbb{B}} \omega(w)|\Re f(w)|.
\]
Hence we obtain
\[
\sup_{|\varphi(z)| \leq R} \mu(z) \left| g(z) \right| \left| \Re f(\varphi(z)) - \Re f(r_i \varphi(z)) \right| \leq \frac{1 - r_i}{1 - R} \max_{\omega \in \mathbb{B}} \sup_{z \in \mathbb{B}} \mu(z) \left| g(z) \right| \left| \varphi(z) \right|.
\]  
(4.23)

Since the boundedness of \( I^\omega \varphi : \mathcal{B}_\omega \to \mathcal{B}_\mu \) implies \( \sup_{z \in \mathbb{B}} \mu(z) \left| g(z) \right| \left| \varphi(z) \right| < \infty \), letting \( l \to \infty \) in the above inequality, we have
\[
\sup_{\| f \|_{\mathcal{B}_\mu} \leq 1} \sup_{|\varphi(z)| \leq R} \mu(z) \left| g(z) \right| \left| \Re f(\varphi(z)) - \Re f(r_i \varphi(z)) \right| \to 0.
\]  
(4.24)

By using (4.14), (4.20), and (4.24) and letting \( R \to 1^- \), we obtain the desired estimate
\[
\left\| I^\omega \varphi \right\|_e \leq \lim_{|z| \to 1^-} \sup_{|\varphi(z)| \to 1^-} \frac{\mu(z) \left| g(z) \right| \left| \varphi(z) \right|}{\tilde{\omega}(\varphi(z))}.
\]  
(4.25)

So if condition (4.6) is true, then \( \| I^\omega \varphi \|_e = 0 \), which means that \( I^\omega \varphi : \mathcal{B}_\omega \to \mathcal{B}_\mu \) is compact. Our proof is accomplished. \( \square \)

**Theorem 4.6.** Let \( \omega \) be a normal weight function and \( \mu \) a weight function. Suppose that the operator \( I^\omega \varphi : \mathcal{B}_{\omega,0} \to \mathcal{B}_{\mu,0} \) is bounded. Then \( I^\omega \varphi : \mathcal{B}_{\omega,0} \to \mathcal{B}_{\mu,0} \) is compact if and only if
\[
\lim_{|z| \to 1^-} \frac{\mu(z) \left| g(z) \right| \left| \varphi(z) \right|}{\tilde{\omega}(\varphi(z))} = 0.
\]  
(4.26)

**Proof.** Suppose that (4.26) holds. For any \( f \in \mathcal{B}_{\omega,0} \), by Lemma 2.1 and (1.4), we have
\[
\mu(z) \left| \Re I^\omega \varphi \tilde{f} \right|(z) = \mu(z) \left| \Re f(\varphi(z)) \right| \left| g(z) \right|
\]
\[
\leq \mu(z) \left| g(z) \right| \left| \varphi(z) \right| \left| \nabla f(\varphi(z)) \right|
\]
\[
\leq \frac{\mu(z) \left| g(z) \right| \left| \varphi(z) \right|}{\tilde{\omega}(\varphi(z))} \omega(\varphi(z)) \left| \nabla f(\varphi(z)) \right|
\]
\[
\leq \frac{\mu(z) \left| g(z) \right| \left| \varphi(z) \right|}{\tilde{\omega}(\varphi(z))} \| f \|_{\mathcal{B}_\omega}.
\]  
(4.27)

Combining this with (4.26), we obtain
\[
\lim_{|z| \to 1^-} \sup_{\| f \|_{\mathcal{B}_\mu} \leq 1} \mu(z) \left| \Re I^\omega \varphi \tilde{f} \right|(z) = 0.
\]  
(4.28)

Hence it follows from Lemma 2.4 that \( I^\omega \varphi : \mathcal{B}_{\omega,0} \to \mathcal{B}_{\mu,0} \) is compact.
Conversely, we assume that $I^k_q : B_{\omega,0} \to B_{\mu,0}$ is compact. By Theorem 3.2, we see that

$$\lim_{|z| \to 1^{-}} \mu(z) |g(z)| |\varphi(z)| = 0. \quad (4.29)$$

Thus this implies (4.26) if $||\varphi||_\infty < 1$.

Now assume $||\varphi||_\infty = 1$. We claim that

$$\limsup_{|z| \to 1^{-}} \frac{\mu(z) |g(z)| |\varphi(z)|}{\omega_1(\varphi(z))} = \limsup_{|\varphi(z)| \to 1^{-}} \frac{\mu(z) |g(z)| |\varphi(z)|}{\omega_1(\varphi(z))}. \quad (4.30)$$

Further assume that $\{z_k\}_{k \in \mathbb{N}}$ is a sequence in $\mathbb{B}$ such that

$$\limsup_{|z| \to 1^{-}} \frac{\mu(z) |g(z)| |\varphi(z)|}{\omega_1(\varphi(z))} = \lim_{k \to \infty} \frac{\mu(z_k) |g(z_k)| |\varphi(z_k)|}{\omega_1(\varphi(z_k))}. \quad (4.31)$$

If $\sup_{k \in \mathbb{N}} |\varphi(z_k)| < 1$, then from this and (4.29) we have that both limits in (4.30) are equal to zero. If $\sup_{k \in \mathbb{N}} |\varphi(z_k)| = 1$, then there is a subsequence $\{\varphi(z_k)\}_{k \in \mathbb{N}}$ such that $|\varphi(z_k)| \to 1^{-}$ as $l \to \infty$. Hence we have

$$\limsup_{|z| \to 1^{-}} \frac{\mu(z) |g(z)| |\varphi(z)|}{\omega_1(\varphi(z))} \leq \limsup_{|\varphi(z)| \to 1^{-}} \frac{\mu(z) |g(z)| |\varphi(z)|}{\omega_1(\varphi(z))}, \quad (4.32)$$

and so (4.30) holds.

Since $I^k_q : B_{\omega,0} \to B_{\mu}$ is also compact, by Theorem 4.5, we see that the second limit in (4.30) is equal to zero, so that (4.26) holds. This completes the proof. \qed

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**References**


