Research Article

Global Behavior of the Difference Equation

$$x_{n+1} = (p + x_{n-1}) / (qx_n + x_{n-1})$$

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We study the following difference equation $x_{n+1} = (p + x_{n-1}) / (qx_n + x_{n-1})$, $n = 0, 1, \dots$, where $p, q \in (0, +\infty)$ and the initial conditions $x_{-1}, x_0 \in (0, +\infty)$. We show that every positive solution of the above equation either converges to a finite limit or to a two cycle, which confirms that the Conjecture 6.10.4 proposed by Kulenović and Ladas (2002) is true.

1. Introduction

Kulenović and Ladas in [1] studied the following difference equation:

$$x_{n+1} = \frac{p + x_{n-1}}{qx_n + x_{n-1}}, \quad n = 0, 1, \dots, \quad (1.1)$$

where $p, q \in (0, +\infty)$ and the initial conditions $x_{-1}, x_0 \in (0, +\infty)$, and they obtained the following theorems.

Theorem A (see [1, Theorem 6.6.2]). *Equation (1.1) has a prime period-two solution*

$$\dots, \phi, \psi, \phi, \psi, \dots \quad (1.2)$$

if and only if $q > 1 + 4p$. Furthermore, when $q > 1 + 4p$, the prime period-two solution is unique and the values of ϕ and ψ are the positive roots of the quadratic equation

$$t^2 - t + \frac{p}{q-1} = 0. \quad (1.3)$$

Theorem B (see [1, Theorem 6.6.4]). Let $\{x_n\}_{n=-1}^{+\infty}$ be a solution of (1.1). Let I be the closed interval with end points 1 and p/q and let J and K be the intervals which are disjoint from I and such that

$$I \cup J \cup K = (0, +\infty). \quad (1.4)$$

Then either all the even terms of the solution lie in J and all odd terms lie in K , or vice-versa, or for some $N \geq 0$,

$$x_n \in I \quad \text{for } n \geq N, \quad (E1)$$

when (E1) holds, except for the length of the first semicycle of the solution, if $p < q$, the length is one; if $p > q$, the length is at most two.

Theorem C (see [1, Theorem 6.6.5]). (a) Assume $q \leq 1 + 4p$. Then the equilibrium $\bar{x} = (1 + \sqrt{1 + 4p(1 + q)}) / (2(1 + q))$ of (1.1) is global attractor.

(b) Assume $q > 1 + 4p$. Then every solution of (1.1) eventually enters and remains in the interval $[p/q, 1]$.

In [1], they proposed the following conjecture.

Conjecture 1 (see [1, Conjecture 6.10.4]). Assume that $p, q \in (0, +\infty)$. Show that every positive solution of (1.1) either converges to a finite limit or to a two cycle.

Gibbons et al. in [2] triggered off the investigation of the second-order difference equations $x_{n+1} = f(x_n, x_{n-1})$ such that the function $f(x, y)$ is increasing in y and decreasing in x . Motivated by [2], Berg [3] and Stević [4] obtained some important results on the existence of monotone solutions of such equations which was later considerably developed in a series of papers [5–14] (for related papers see also [15–19]). The monotonous character of solutions of the equations was explained by Stević in [20]. For some other papers in the area, see also [1, 17–19, 21–26] and the references cited therein. In this paper, we shall confirm that the Conjecture 1 is true. The main idea used in this paper can be found in papers [24, 26].

2. Global behavior of (1.1)

Theorem 2.1. Let $\{x_n\}_{n=-1}^{+\infty}$ be a nonoscillatory solution of (1.1); then $\{x_n\}_{n=-1}^{+\infty}$ converges to the unique positive equilibrium $\bar{x}$ of (1.1).

Proof. Since $\{x_n\}_{n=-1}^{+\infty}$ is a nonoscillatory solution of (1.1), we may assume without loss of generality that there exists $N > 0$ such that $x_n \leq \bar{x}$ for any $n \geq N$. We claim $x_{n+1} \geq x_n$ for any $n \geq N$. Indeed, if $x_{n+1} < x_n$ for some $n \geq N$, then

$$\frac{p}{q\bar{x} + \bar{x}} + \frac{1}{q+1} = \frac{p + \bar{x}}{q\bar{x} + \bar{x}} = \bar{x} \geq x_{n+2} = \frac{p + x_n}{qx_{n+1} + x_n} > \frac{p + x_n}{qx_n + x_n} = \frac{p}{qx_n + x_n} + \frac{1}{q+1}, \quad (2.1)$$

which implies $x_n > \bar{x}$; this is a contradiction. Let $\lim_{n \rightarrow \infty} x_n = a$; then $a = (p + a)/(qa + a)$ and $a = \bar{x}$. The proof is complete. $\square$

In the sequel, let $q > 1 + 4p$ and $\dots, \phi, \psi, \phi, \psi, \dots$ the unique prime period-two solution of (1.1) with $\phi < \psi$. Define $f \in C([\phi, \psi] \times [\phi, \psi], [\phi, \psi])$ by

$$f(x, y) = \frac{p + y}{qx + y} \quad (2.2)$$

for any $x, y \in [\phi, \psi]$ and $g \in C([\phi, \psi], [\phi, \psi])$ by

$$y^* = g(y) = \frac{p + y - y^2}{qy} \quad (2.3)$$

for any $y \in [\phi, \psi]$. Then

$$f(y^*, y) = y. \quad (2.4)$$

Lemma 2.2. *Let $q > 1 + 4p$, then the following statements are true.*

- (i) $f(x, y) > y$ if and only if $x < y^*$.
- (ii) $x > y$ if and only if $x^* < y^*$.
- (iii) If $\bar{x} < y < \psi$, then $f(y, y^*) < y^*$ and $y > y^{**}$. If $\phi < y < \bar{x}$, then $f(y, y^*) > y^*$ and $y^{**} > y$.

Proof. (i) Since f is decreasing in x and $f(y^*, y) = y$, $x < y^*$ if and only if $f(x, y) > f(y^*, y) = y$.

(ii) Since $y^* = g(y)$ is a decreasing function for y , $x > y$ if and only if $x^* < y^*$.

(iii) Since

$$\begin{aligned} f(y, y^*) - y^* &= \frac{p + ((p + y - y^2)/qy)}{qy + ((p + y - y^2)/qy)} - \frac{p + y - y^2}{qy} \\ &= \frac{(q^2 - 1)[y - (1 - \sqrt{1 + 4p + 4pq})/2(q + 1)](y - \phi)(y - \bar{x})(y - \psi)}{qy[(q^2 - 1)y^2 + p + y]}, \end{aligned} \quad (2.5)$$

it follows that

$$\begin{aligned} \bar{x} < y < \psi &\implies f(y, y^*) < y^*, \\ \phi < y < \bar{x} &\implies f(y, y^*) > y^*. \end{aligned} \quad (2.6)$$

By (i), we obtain $y > y^{**}$ if $\bar{x} < y < \psi$ and $y^{**} > y$ if $\phi < y < \bar{x}$. The proof is complete. $\square$

Lemma 2.3. Let $q > 1 + 4p$ and $\{x_n\}_{n=-1}^{+\infty}$ is a positive solution of (1.1); then $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ do exactly one of the following.

- (i) Eventually, they are both monotonically increasing.
- (ii) Eventually, they are both monotonically decreasing.
- (iii) Eventually, one of them is monotonically increasing and the other is monotonically decreasing.

Proof. See [20] (also see [27]). □

Remark 2.4. Stević in [20] noticed the relationship between the monotonicity of the subsequences x_{2n} and x_{2n-1} of solution $\{x_n\}_{n=-1}^{+\infty}$ of a second-order difference equation $x_{n+1} = f(x_n, x_{n-1})$ and the monotonicity of the function $f(x, y)$ in variables x and y . A simple observation shows that Stević's proof works in the general case if the function y/x is replaced by $f(x, y)$. The result was later used for many times by Stević and his collaborators (see, e.g., [21, 23–26]).

Lemma 2.5. Let $q > 1 + 4p$. Assume that there exists some i such that $\psi \geq x_i \geq x_{i+2} > \bar{x} > x_{i+1} \geq \phi$; then $x_{i+1} \geq x_{i+3}$.

Proof. Since $x_{i+2} = f(x_{i+1}, x_i) \leq x_i = f(x_i^*, x_i)$, it follows that $x_{i+1} \geq x_i^*$. By Lemma 2.2(ii), we get $x_i^{**} \geq x_{i+1}^*$, which with Lemma 2.2(iii) implies $x_i \geq x_i^{**} \geq x_{i+1}^*$. Since $f(x, y)$ is increasing in y ($x, y \in [\phi, \psi]$) and $x_i \geq x_{i+1}^*$, it follows that

$$x_{i+2} = f(x_{i+1}, x_i) \geq f(x_{i+1}, x_{i+1}^*). \quad (2.7)$$

By Lemma 2.2(iii), we have $x_{i+2} \geq f(x_{i+1}, x_{i+1}^*) \geq x_{i+1}^*$ as $\bar{x} \geq x_{i+1} \geq \phi$. Thus $x_{i+1} = f(x_{i+1}^*, x_{i+1}) \geq f(x_{i+2}, x_{i+1}) = x_{i+3}$. The proof is complete. □

Theorem 2.6. Let $q > 1 + 4p$ and $\{x_n\}_{n=-1}^{+\infty}$ be an oscillatory solution of (1.1); then $\{x_n\}_{n=-1}^{+\infty}$ converges to the unique prime period-two solution of (1.1).

Proof. It follows from Theorem C(b) that there exists $N > 0$ such that for any $n \geq N$,

$$x_n \in \left[\frac{p}{q}, 1 \right], \quad (2.8)$$

and $x_N \geq \bar{x}$ and $x_{N+1} < \bar{x}$. We assume without loss of generality that

$$x_n \in \left[\frac{p}{q}, 1 \right] \quad \text{for any } n \geq -1, \quad (2.9)$$

and $x_{-1} \geq \bar{x}$ and $x_0 < \bar{x}$. Since

$$h(x, y) = \frac{p+y}{qx+y} \quad \left(x, y \in \left[\frac{p}{q}, 1 \right] \right) \quad (2.10)$$

is decreasing in x and increasing in y , it follows that $x_{2n-1} > \bar{x}$ and $x_{2n} < \bar{x}$ for any $n \geq 1$.

If $x_{2n-1} > \bar{x}$ is eventually increasing or $x_{2n} < \bar{x}$ is eventually decreasing, then it follows from Theorem A that $\lim_{n \rightarrow \infty} x_{2n-1} = \psi$ and $\lim_{n \rightarrow \infty} x_{2n} = \phi$.

If $x_{2n-1} > \bar{x}$ is eventually decreasing and $x_{2n} < \bar{x}$ is eventually increasing, we may assume without loss of generality that $x_{2n} \leq x_{2n+2} < \bar{x} < x_{2n+1} \leq x_{2n-1}$ for any $n \geq 0$. It follows from Lemma 2.5 that $x_{2n} \leq x_{2n+2} \leq \phi < \bar{x} < \psi \leq x_{2n+1} \leq x_{2n-1}$ for any $n \geq 0$. By Theorem A, we obtain $\lim_{n \rightarrow \infty} x_{2n-1} = \psi$ and $\lim_{n \rightarrow \infty} x_{2n} = \phi$. The proof is complete. $\square$

We confirm from Theorems thm1, 2.6, and C(a) that the Conjecture 1 is true.

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