Research Article

Weighted Differentiation Composition Operators from the Mixed-Norm Space to the $n$th Weigthed-Type Space on the Unit Disk

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The boundedness and compactness of the weighted differentiation composition operator from the mixed-norm space to the $n$th weighted-type space on the unit disk are characterized.

1. Introduction

Throughout this paper $\mathbb{D}$ will denote the open unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the class of all holomorphic functions on $\mathbb{D}$, and $H^\infty = H^\infty(\mathbb{D})$ the space of all bounded holomorphic functions on $\mathbb{D}$ with the norm $\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|$.

The mixed norm space $H_{p,q,\gamma} = H_{p,q,\gamma}(\mathbb{D})$, $0 < p, q < \infty$, $-1 < \gamma < \infty$, consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{H_{p,q,\gamma}}^q = \int_0^1 M_p^q(f,r)(1-r)^\gamma dr < \infty,$$  \hspace{1cm} (1.1)

where

$$M_p(f,r) = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}.$$ \hspace{1cm} (1.2)
Abstract and Applied Analysis

A positive continuous function on \( \mathbb{D} \) is called a weight. Let \( \mu(z) \) be a weight and \( n \in \mathbb{N}_0 \). The \( n \)-th weighted-type space on \( \mathbb{D} \), denoted by \( \mathcal{K}^{(n)}_\mu(\mathbb{D}) \), consists of all \( f \in H(\mathbb{D}) \) such that

\[
b_{\mathcal{K}^{(n)}_\mu(\mathbb{D})}(f) := \sup_{z \in \mathbb{D}} \mu(z) \left| f^{(n)}(z) \right| < \infty. \tag{1.3}\]

The space was recently introduced by this author in [1] as an extension of several weighted-type spaces which attracted a lot of attention in the last few decades. For instance, when \( n = 0 \), the space becomes the weighted-type space \( H^\infty_\mu(\mathbb{D}) \) (see, e.g., [2–4]), when \( n = 1 \), the Bloch-type space \( B_\mu(\mathbb{D}) \) (see, e.g., [5–7]), and for \( n = 2 \), the Zygmund-type space \( \mathcal{Z}_\mu(\mathbb{D}) \). Some information on Zygmund-type spaces on \( \mathbb{D} \) and some operators on them can be found, for example, in [8–10] and on the unit ball, for example, in [11, 12].

The quantity \( b_{\mathcal{K}^{(n)}_\mu(\mathbb{D})}(f) \) is a seminorm on the \( n \)-th weighted-type space \( \mathcal{K}^{(n)}_\mu(\mathbb{D}) \) and a norm on \( \mathcal{K}^{(n)}_\mu(\mathbb{D})/\mathbb{P}_{n-1} \), where \( \mathbb{P}_{n-1} \) is the set of all polynomials whose degrees are less than or equal to \( n - 1 \). A natural norm on the \( n \)-th weighted-type space is introduced as follows:

\[
\|f\|_{\mathcal{K}^{(n)}_\mu(\mathbb{D})} = \sum_{j=0}^{n-1} |f^{(j)}(0)| + b_{\mathcal{K}^{(n)}_\mu(\mathbb{D})}(f). \tag{1.4}\]

With this norm the \( n \)-th weighted-type space becomes a Banach space.

The little \( n \)-th weighted-type space, denoted by \( \mathcal{K}^{(n)}_{\mu,0}(\mathbb{D}) \), is a closed subspace of \( \mathcal{K}^{(n)}_\mu(\mathbb{D}) \) consisting of those \( f \) for which

\[
\lim_{|z| \to 1} \mu(z) \left| f^{(n)}(z) \right| = 0. \tag{1.5}\]

An analytic self-map \( \varphi : \mathbb{D} \to \mathbb{D} \) induces the composition operator \( C_\varphi \) on \( H(\mathbb{D}) \), defined by \( C_\varphi(f)(z) = f(\varphi(z)) \) for \( f \in H(\mathbb{D}) \) (see, e.g., [8, 13–16]).

Let \( \varphi \) be an analytic self-map of \( \mathbb{D} \), \( u \in H(\mathbb{D}) \), and \( m \in \mathbb{N} \). Then the weighted differentiation composition operator, denoted by \( D^{m}_{\varphi,u} \), is defined on \( H(\mathbb{D}) \) by

\[
D^{m}_{\varphi,u}f(z) = u(z)f^{(m)}(\varphi(z)), \quad f \in H(\mathbb{D}). \tag{1.6}\]

Recently there has been some interest in studying some particular cases of operator \( D^{m}_{\varphi,u} \) (see, e.g., [17–25]). For some other products of linear operators on spaces of holomorphic functions see also recent papers [11, 26–32].

Here we study the boundedness and compactness of the operator \( D^{m}_{\varphi,u} \) from \( H_{p,q,T} \) to \( n \)-th weighted-type spaces, where \( n \in \mathbb{N} \).

Throughout this paper, constants are denoted by \( C \); they are positive and may differ from one occurrence to the other. The notation \( A \asymp B \) means that there is a positive constant \( C \) such that \( B/C \leq A \leq CB \).
2. Auxiliary Results

Here we quote some auxiliary results which will be used in the proofs of the main results. The first lemma can be proved in a standard way (see, e.g., in [13, Proposition 3.11] or in [15, Lemma 3]).

**Lemma 2.1.** Assume that \( m \in \mathbb{N}_0, n \in \mathbb{N}, p, q > 0, \gamma > -1 \), \( \varphi \) is an analytic self-map of \( D \) and \( u \in H(D) \). Then the operator \( D_{\varphi,u}^m : H_{p,q,\gamma} \rightarrow \mathcal{W}_{n,\mu} \) is compact if and only if \( D_{\varphi,u}^m : H_{p,q,\gamma} \rightarrow \mathcal{W}_{n,\mu} \) is bounded and for any bounded sequence \( (f_k)_{k \in \mathbb{N}} \) in \( H_{p,q,\gamma} \) which converges to zero uniformly on compact subsets of \( D \), \( D_{\varphi,u}^m f_k \rightarrow 0 \) in \( \mathcal{W}_{n,\mu} \) as \( k \rightarrow \infty \).

The next lemma is known, but we give a proof of it for the benefit of the reader.

**Lemma 2.2.** Assume that \( n \in \mathbb{N}_0, 0 < p, q < \infty, -1 < \gamma < \infty \) and \( f \in H_{p,q,\gamma} \). Then there is a positive constant \( C \) independent of \( f \) such that

\[
\left| f^{(n)}(z) \right| \leq C \frac{\|f\|_{H_{p,q,\gamma}}}{(1 - |z|^2)^{(y+1)/q + 1/p + n}}.
\]  

(2.1)

**Proof.** By the monotonicity of the integral means, using the well-known asymptotic formula

\[
\int_0^1 M_p^q(f,r)(1-r)^y dr \approx |f(0)|^q + \int_0^1 M_p^q(f^{(n)},r)(1-r)^{y+mq} dr,
\]

and Theorem 7.2.5 in [33], we have that

\[
\|f\|_{H_{p,q,\gamma}}^q \geq \int_0^1 M_p^q(f^{(n)},r)(1-r)^{y+mq} dr \\
\geq CM_p^q \left( f^{(n)}, \frac{1+|z|}{2} \right)^{y+1+nq} \\
\geq C \left( 1 - |z|^2 \right)^{y+1+nq+q/p} \left| f^{(n)}(z) \right|^q,
\]

(2.3)

from which the result follows. \( \square \)

The following lemma can be found in [34].

**Lemma 2.3.** For \( \beta > -1 \) and \( m > 1 + \beta \) one has

\[
\int_0^1 \frac{(1-r)^\beta}{(1-\rho r)^m} dr \leq C(1-\rho)^{1+\beta-m}, \quad 0 < \rho < 1.
\]

(2.4)

A proof of the next lemma can be found in [35, Lemma 2.3].
Lemma 2.4. Assume $a > 0$ and

$$D_n(a) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a & a + 1 & \cdots & a + n - 1 \\ a(a + 1) & (a + 1)(a + 2) & \cdots & (a + n - 1)(a + n) \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix},$$

Then $D_n(a) = \prod_{j=1}^{n-1} j!$.

The following formula

$$(f \circ \varphi)^{(n)}(z) = \sum_{k=0}^{n} f^{(k)}(\varphi(z)) \sum_{k_1, \ldots, k_n} \frac{n!}{k_1! \cdots k_n!} \prod_{j=1}^{n} \left( \frac{\varphi^{(j)}(z)}{j!} \right)^{k_j},$$

where the second sum is over all nonnegative integers $k_1, k_2, \ldots, k_n$ satisfying $k = k_1 + k_2 + \cdots + k_n$ and $k_1 + 2k_2 + \cdots + nk_n = n$, is attributed to Faà di Bruno [36]. By using Bell polynomials $B_{n,k}(x_1, \ldots, x_{n-1})$ it can be written as follows:

$$(f \circ \varphi)^{(n)}(z) = \sum_{k=0}^{n} f^{(k)}(\varphi(z)) B_{n,k}(\varphi'(z), \varphi''(z), \ldots, \varphi^{(n-k+1)}(z)).$$

For $n \in \mathbb{N}$ the last sum can go from $k = 1$ since $B_{n,0}(\varphi'(z), \varphi''(z), \ldots, \varphi^{(n+1)}(z)) = 0$; however we will keep the summation since for $n = 0$ the only existing term $B_{0,0}$ is equal to 1 and we will use it.

The Leibnitz formula along with (2.6) yields

$$(u(z) g(\varphi(z)))^{(n)} = \sum_{l=0}^{n} C_l^n u^{(n-l)}(z) \sum_{k=0}^{l} g^{(k)}(\varphi(z)) B_{l,k}(\varphi'(z), \ldots, \varphi^{(l-k+1)}(z)).$$

Hence we have the next result.

Lemma 2.5. Assume that $g, u \in H(\mathbb{D})$ and $\varphi$ is an analytic self-map of $\mathbb{D}$. Then

$$(u(z) g(\varphi(z)))^{(n)} = \sum_{k=0}^{n} g^{(k)}(\varphi(z)) \sum_{l=k}^{n} C_l^n u^{(n-l)}(z) B_{l,k}(\varphi'(z), \ldots, \varphi^{(l-k+1)}(z)).$$

3. The Boundedness and Compactness of $D^m_{\varphi,u} : H_{p,q,Y} \to \mathcal{H}^{(n)}_\mu$

This section characterizes the boundedness and compactness of the operator $D^m_{\varphi,u} : H_{p,q,Y} \to \mathcal{H}^{(n)}_\mu$. 
Theorem 3.1. Suppose that \( m, n \in \mathbb{N}, 0 < p, q < \infty, -1 < \gamma < \infty, \varphi \) is an analytic self-map of the unit disk, \( u \in H(D) \), and \( \mu \) is a weight. Then the operator \( D^m_{\varphi,u} : H^p,q,\gamma \rightarrow \mathcal{K}^{(n)}_{\mu} \) is bounded if and only if for each \( k \in \{0, 1, \ldots, n\} \)

\[
I_k := \sup_{z \in D} \frac{\mu(z) \left| \sum_{l=0}^{n} C_l u^{(n-l)}(z) B_{l,k}(\varphi(z), \ldots, \varphi^{(l-k+1)}(z)) \right|}{(1 - |\varphi(z)|^2)^{(\gamma+1)/q+1/p+m+k}} < \infty. \tag{3.1}
\]

Moreover if \( D^m_{\varphi,u} : H^p,q,\gamma \rightarrow \mathcal{K}^{(n)}_{\mu} \) is bounded, then the following asymptotic relation holds

\[
\left\| D^m_{\varphi,u} \right\|_{H^p,q,\gamma \rightarrow \mathcal{K}^{(n)}_{\mu}} \asymp \sum_{k=0}^{n} I_k. \tag{3.2}
\]

Proof. First assume that \( D^m_{\varphi,u} : H^p,q,\gamma \rightarrow \mathcal{K}^{(n)}_{\mu} \) is bounded; then there exists a constant \( C \) such that

\[
\left\| D^m_{\varphi,u} f \right\|_{\mathcal{K}^{(n)}_{\mu}} \leq C \left\| f \right\|_{H^p,q,\gamma}. \tag{3.3}
\]

for all \( f \in H^p,q,\gamma \).

For a fixed \( w \in D \), \( t \geq (\gamma + 1)/q \), and constants \( c_1, \ldots, c_{n+1} \), set

\[
g_w(z) = \sum_{j=1}^{n+1} \frac{c_j}{\prod_{l=0}^{m-1} (j + t + 1/p + l)} \tilde{g}_{w,j}(z), \tag{3.4}
\]

where

\[
\tilde{g}_{w,j}(z) = \frac{(1 - |w|^2)^{j+t-(\gamma+1)/q}}{\left(1 - \overline{w}z\right)^{(1/p+t)/r}}, \quad j = 1, \ldots, n+1. \tag{3.5}
\]

By [33, Theorem 1.4.10], we get

\[
M_p(\tilde{g}_{w,j}, r) \leq C \frac{(1 - |w|^2)^{j+t-(\gamma+1)/q}}{(1 - r|w|)^{j+t}}, \quad j = 1, \ldots, n+1. \tag{3.6}
\]
Applying Lemma 2.3, we have that
\[ \|g_{w_j}\|_{H_{p,q}}^2 = \int_0^1 M_p^2(g_{w_j}, r) (1 - r)^q \, dr \]
\[ \leq C \int_0^1 \frac{(1 - |w|^2)^{q(j+t)-(j+1)}}{(1 - r|w|)^{(j+t)}} (1 - r)^q \, dr \]
\[ \leq C. \] (3.7)

Therefore \( g_w \in H_{p,q}, \) and moreover \( \sup_{w \in \Omega} \|g_w\|_{H_{p,q}} < \infty. \)

Now we show that for each \( s \in \{m, m+1, \ldots, m+n\}, \) there are constants \( c_1, c_2, \ldots, c_{n+1}, \)
such that
\[ g_w^{(s)}(w) = \left( \frac{w}{1 - |w|^2} \right)^{s+1/p+1/p}, \quad g_w^{(0)}(w) = 0, \quad t \in \{m, m+1, m+n\} \setminus \{s\}. \] (3.8)

By differentiating function \( g_w, \) for each \( s \in \{m, m+1, \ldots, m+n\}, \) (3.8) becomes
\[ (t + p^{-1} + m + 1)c_1 + (t + p^{-1} + m + 2)c_2 + \cdots + \left( t + p^{-1} + m + n + 1 \right)c_{n+1} = 0, \]
\[ \vdots \]
\[ \prod_{j=1}^{s-m} \left( t + p^{-1} + m + j \right)c_1 + \cdots + \prod_{j=1}^{s-m} \left( t + p^{-1} + m + n + j \right)c_{n+1} = 1, \] (3.9)
\[ \vdots \]
\[ \prod_{j=1}^{n} \left( t + p^{-1} + m + j \right)c_1 + \cdots + \prod_{j=1}^{n} \left( t + p^{-1} + m + n + j \right)c_{n+1} = 0. \]

Applying Lemma 2.4 with \( a = t + 1/p + m + 1 > 0 \) and where \( n \to n + 1, \) we see that
the determinant of system (3.9) is different from zero, as claimed.

By \( g_{w,k}, k \in \{0, 1, \ldots, n\}, \) denote the corresponding family of functions which satisfy
(3.8) with \( s = m + k. \) Then, for each fixed \( k \in \{0, 1, \ldots, n\}, \) inequality (3.3) along with (2.9)
and (3.8) implies that for each \( \psi(w) \neq 0 \)
\[ \frac{\mu(w)|\psi(w)|^{k+m}}{\left(1 - |\psi(w)|^2\right)^{(y+1)/q+1/p+k+m}} \left| \sum_{l=k}^n C_l^m w^{(n-l)}(w) B_{l,k}(\psi(w), \ldots, \psi^{(l-k+1)}(w)) \right| \]
\[ \leq C \sup_{w \in \Omega} \left\| D_{\psi,w}^m (g_{\psi(w),k}) \right\|_{H_{p,q}} \leq C \left\| D_{\psi,w}^m \right\|_{H_{p,q}} \to \|\psi\|^m. \] (3.10)
From (3.10) it follows that for each } k \in \{0,1,\ldots,n\},
\begin{align*}
\sup_{|\varphi(z)|>1/2} \frac{\mu(z)}{m}\left| \sum_{l=0}^{n} C^n_l \varphi^{(n-l)}(z) B_{l,k} \left( \varphi(z), \ldots, \varphi^{(l-k+1)}(z) \right) \right| 
\leq C \| D_{\varphi,u}^m \|_{H_{p,q,\gamma}} \rightarrow \kappa_{\mu}(\varphi).
\end{align*}
(3.11)

Let
\begin{align*}
h_k(z) = z^k, \quad k = m, \ldots, n + m.
\end{align*}
(3.12)

Then clearly
\begin{align*}
\|h_k\|_{H_{p,q,\gamma}} \leq 1, \quad \text{for each } k \in \mathbb{N}.
\end{align*}
(3.13)

By formula (2.9) applied to the function } f(z) = h_m(z) we get
\begin{align*}
\left( D_{\varphi,u}^m h_m \right)^{(n)}(z) &= h_m^{(m)}(z) \sum_{l=0}^{n} C^n_l \varphi^{(n-l)}(z) B_{l,0} \left( \varphi(z), \ldots, \varphi^{(l)}(z) \right) \\
&= m! \sum_{l=0}^{n} C^n_l \varphi^{(n-l)}(z) B_{l,0} \left( \varphi(z), \ldots, \varphi^{(l)}(z) \right),
\end{align*}
(3.14)

which along with the boundedness of the operator } D_{\varphi,u}^m : H_{p,q,\gamma} \rightarrow \kappa_{\mu}(\varphi) and (3.13) implies that
\begin{align*}
m! \sup_{z \in \Omega} \mu(z) \left| \sum_{l=0}^{n} C^n_l \varphi^{(n-l)}(z) B_{l,0} \left( \varphi(z), \ldots, \varphi^{(l)}(z) \right) \right| 
\leq \left\| D_{\varphi,u}^m \left( z^m \right) \right\|_{\kappa_{\mu}} \leq \left\| D_{\varphi,u}^m \right\|_{H_{p,q,\gamma} \rightarrow \kappa_{\mu}}.
\end{align*}
(3.15)

Now assume that we have proved that for } j \in \{0,1,\ldots,k-1\} and a } k \leq n
\begin{align*}
\sup_{z \in \Omega} \mu(z) \left| \sum_{l=j}^{n} C^n_l \varphi^{(n-l)}(z) B_{l,j} \left( \varphi(z), \ldots, \varphi^{(l-j)}(z) \right) \right| 
\leq C \left\| D_{\varphi,u}^m \right\|_{H_{p,q,\gamma} \rightarrow \kappa_{\mu}}.
\end{align*}
(3.16)

Applying (2.9) to the function } f(z) = h_{m+k}(z), k \in \{0,1,\ldots,n\}, and noticing that } h_{m+k}^{(s)}(z) \equiv 0 \text{ for } s > m + k, \text{ we get
\begin{align*}
\left( D_{\varphi,u}^m h_{m+k} \right)^{(n)}(z) &= \sum_{j=0}^{k} h_{m+k}^{(m+j)}(z) \sum_{l=j}^{n} C^n_l \varphi^{(n-l)}(z) B_{l,j} \left( \varphi(z), \ldots, \varphi^{(l-j)}(z) \right) \\
&= \sum_{j=0}^{k} (m + k) \cdots (j + 1) \left( \varphi(z) \right)^{k-j} \sum_{l=j}^{n} C^n_l \varphi^{(n-l)}(z) B_{l,j} \left( \varphi(z), \ldots, \varphi^{(l-j)}(z) \right).
\end{align*}
(3.17)
From (3.17), the boundedness of the operator $D_{\psi,u}^m : H_{p,q,T} \to \mathcal{K}_\mu^{(n)}$, the fact that $\|\psi\|_{\infty} \leq 1$, the triangle inequality, noticing that $(m+k)!$ is the coefficient at $\sum_{l=k}^{n} C^m_l u^{(n-l)}(z) B_{l,k}(\psi(z), \ldots, \psi^{(l-k+1)}(z))$, and finally using hypothesis (3.16) we get

$$
\sup_{z \in \mathbb{D}} \mu(z) \left| \sum_{l=k}^{n} C^m_l u^{(n-l)}(z) B_{l,k}(\psi(z), \ldots, \psi^{(l-k+1)}(z)) \right| \leq C \left\| D_{\psi,u}^m \right\|_{H_{p,q,T} \to \mathcal{K}_\mu^{(n)}}.
$$

(3.18)

Hence by induction, (3.18) holds for each $k \in \{0, 1, \ldots, n\}$.

From (3.18), for each fixed $k \in \{0, 1, \ldots, n\}$

$$
\sup_{|\varphi(z)| \leq 1/2} \frac{\mu(z)}{\left(1 - |\varphi(z)|^2\right)^{y+1/q+1/p+k+m}} \left| \sum_{l=k}^{n} C^m_l u^{(n-l)}(z) B_{l,k}(\psi(z), \ldots, \psi^{(l-k+1)}(z)) \right| \leq C \sup_{z \in \mathbb{D}} \mu(z) \left| \sum_{l=k}^{n} C^m_l u^{(n-l)}(z) B_{l,k}(\psi(z), \ldots, \psi^{(l-k+1)}(z)) \right| \leq C \left\| D_{\psi,u}^m \right\|_{H_{p,q,T} \to \mathcal{K}_\mu^{(n)}}.
$$

(3.19)

Inequalities (3.11) and (3.19) imply

$$
\sum_{k=0}^{n} I_k \leq C \left\| D_{\psi,u}^m \right\|_{H_{p,q,T} \to \mathcal{K}_\mu^{(n)}}.
$$

(3.20)

Now assume that (3.1) holds. Then for any $f \in H_{p,q,T}$, by (2.9) and Lemma 2.2 we have

$$
\mu(z) \left| \left( D_{\psi,u}^m f \right)^{(n)}(z) \right| = \mu(z) \left| \sum_{k=0}^{n} f^{(m+k)}(\varphi(z)) \sum_{l=k}^{n} C^m_l u^{(n-l)}(z) B_{l,k}(\psi(z), \ldots, \psi^{(l-k+1)}(z)) \right|
\leq \mu(z) \left| \sum_{k=0}^{n} f^{(m+k)}(\varphi(z)) \sum_{l=k}^{n} C^m_l u^{(n-l)}(z) B_{l,k}(\psi(z), \ldots, \psi^{(l-k+1)}(z)) \right|
\leq C \left\| f \right\|_{H_{p,q,T}} \mu(z) \left| \sum_{k=0}^{n} \mu(z) \left| \sum_{l=k}^{n} C^m_l u^{(n-l)}(z) B_{l,k}(\psi(z), \ldots, \psi^{(l-k+1)}(z)) \right| \right| \left(1 - |\varphi(z)|^2\right)^{y+1/q+1/p+k+m}
\leq C \left\| f \right\|_{H_{p,q,T}} \sum_{k=0}^{n} \sup_{z \in \mathbb{D}} \mu(z) \left| \sum_{l=k}^{n} C^m_l u^{(n-l)}(z) B_{l,k}(\psi(z), \ldots, \psi^{(l-k+1)}(z)) \right| \left(1 - |\varphi(z)|^2\right)^{y+1/q+1/p+k+m}
$$

(3.21)
We also have that for each $s \in \{1, \ldots, n - 1\}$

$$
\left| \left( D_{q,u}^m f \right)^{(s)} (0) \right| = \sum_{k=0}^{s} \left| f^{(m+k)} (0) \sum_{l=k}^{s} C_{l-k}^s u^{(n-l)} (0) B_{i,k} \left( \varphi'(0), \ldots, \varphi^{(l-k+1)} (0) \right) \right|
\leq C \|f\|_{H_{p,q},\gamma} \sum_{k=0}^{s} \left| \sum_{l=k}^{s} C_{l-k}^s u^{(n-l)} (0) B_{i,k} \left( \varphi'(0), \ldots, \varphi^{(l-k+1)} (0) \right) \right| \left( 1 - |\varphi(0)|^2 \right)^{(r+1)/q+1/p+m+k},
$$

(3.24)

$$
\left| \left( D_{q,u}^m f \right)(0) \right| = |u(0)| \left| f^{(m)} (0) \right| \leq C |u(0)| \frac{\|f\|_{H_{p,q},\gamma}}{\left( 1 - |\varphi(0)|^2 \right)^{(r+1)/q+1/p+m}}.
$$

Using (3.23), (3.24), and (3.1) it follows that the operator $D_{q,u}^m : H_{p,q,Y} \to \mathcal{W}^{(n)}_{\mu}$ is bounded.

From (3.23) and (3.20) the asymptotic relation (3.2) follows.

**Theorem 3.2.** Suppose that $m, n \in \mathbb{N}$, $0 < p, q < \infty$, $-1 < \gamma < \infty$, $\varphi$ is an analytic self-map of the unit disk, $u \in H(\mathbb{D})$, and $\mu$ is a weight. Then the operator $D_{q,u}^m : H_{p,q,Y} \to \mathcal{W}^{(n)}_{\mu,0}$ is bounded if and only if $D_{q,u}^m : H_{p,q,Y} \to \mathcal{W}^{(n)}_{\mu}$ is bounded and for each $k \in \{0, 1, \ldots, n\}$

$$
\lim_{|z| \to 1} \mu(z) \left| \sum_{l=0}^{n} C_{l-k}^n u^{(n-l)} (z) B_{i,l} \left( \varphi'(z), \ldots, \varphi^{(l-k+1)} (z) \right) \right| = 0.
$$

(3.25)

**Proof.** The boundedness of $D_{q,u}^m : H_{p,q,Y} \to \mathcal{W}^{(n)}_{\mu,0}$ clearly implies that $D_{q,u}^m : H_{p,q,Y} \to \mathcal{W}^{(n)}_{\mu}$ is bounded. Applying (2.9) to the function $f(z) = h_m(z)$ and using the assumption $D_{q,u}^m (h_m) \in \mathcal{W}^{(n)}_{\mu,0}$ it follows that

$$
\mu(z) \left| \left( D_{q,u}^m h_m \right)^{(n)} (z) \right| = m! \mu(z) \left| \sum_{l=0}^{n} C_{l-k}^n u^{(n-l)} (z) B_{i,l} \left( \varphi'(z), \ldots, \varphi^{(l-k+1)} (z) \right) \right| \to 0,
$$

(3.26)

as $|z| \to 1$, which is (3.25) for $k = 0$.

Assume that we have proved the following inequalities:

$$
\lim_{|z| \to 1} \mu(z) \left| \sum_{l=0}^{n} C_{l-j}^n u^{(n-l)} (z) B_{i,l} \left( \varphi'(z), \ldots, \varphi^{(l-j+1)} (z) \right) \right| = 0,
$$

(3.27)

for $j \in \{0, 1, \ldots, k-1\}$ and a $k \leq n$.

Applying formula (2.9) to the function $f(z) = h_{m+k}(z)$, $k \in \{0, 1, \ldots, n\}$, we get (3.17). From (3.17), by using the boundedness of function $\varphi$, the triangle inequality, noticing that the coefficient at $\sum_{l=k}^{n} C_{l-k}^n u^{(n-l)} (z) B_{i,l} (\varphi(z), \ldots, \varphi^{(l-k+1)} (z))$ is independent of $z$, and finally using
hypothesis (3.27), we easily obtain

$$
\lim_{|z| \to 1} \mu(z) \left| \sum_{l=k}^{n} C_l^n u^{(n-l)}(z) B_{i,k} \left( \varphi'(z), \ldots, \varphi^{(l-k+1)}(z) \right) \right| = 0. \quad (3.28)
$$

Hence by induction we get that (3.25) holds for each \( k \in \{0, 1, \ldots, n\} \).

Now assume that \( D_{q,u}^m : H_{p,q,\mu} \to \mathcal{K}_\mu^{(n)} \) is bounded and (3.25) holds for each \( k \in \{0, 1, \ldots, n\} \). For each polynomial \( p \) we have

$$
\mu(z) \left| \left( D_{q,u}^m p \right)^{(n)}(z) \right| = \mu(z) \left| \sum_{k=0}^{n} p^{(k)}(\varphi(z)) \sum_{l=k}^{n} C_l^n u^{(n-l)}(z) B_{i,k} \left( \varphi'(z), \ldots, \varphi^{(l-k+1)}(z) \right) \right|
\leq \sum_{k=0}^{n} \left| p^{(k)}(\varphi(z)) \right| \mu(z) \left| \sum_{l=k}^{n} C_l^n u^{(n-l)}(z) B_{i,k} \left( \varphi'(z), \ldots, \varphi^{(l-k+1)}(z) \right) \right| \to 0,
$$

as \( |z| \to 1 \).

From (3.29) we have that, for each polynomial \( p \), \( D_{q,u}^m p \in \mathcal{K}_\mu^{(n)} \). The set of all polynomials is dense in \( H_{p,q,\gamma} \), so we have that for each \( f \in H_{p,q,\gamma} \), there is a sequence

$$
(\frac{p_k}{\gamma}) \in \mathcal{K}_\mu^{(n)}.
$$

Thus the boundedness of \( D_{q,u}^m : H_{p,q,\gamma} \to \mathcal{K}_\mu^{(n)} \) implies

$$
\left\| D_{q,u}^m f - D_{q,u}^m p_k \right\|_{\mathcal{K}_\mu^{(n)}} \leq \left\| D_{q,u}^m \right\|_{H_{p,q,\gamma} \to \mathcal{K}_\mu^{(n)}} \left\| f - p_k \right\|_{H_{p,q,\gamma}} \to 0, \quad \text{as } k \to \infty.
$$

Hence \( D_{q,u}^m (H_{p,q,\gamma}) \subseteq \mathcal{K}_\mu^{(n)} \), from which the boundedness of \( D_{q,u}^m : H_{p,q,\gamma} \to \mathcal{K}_\mu^{(n)} \) follows, completing the proof of the theorem.

**Theorem 3.3.** Suppose that \( m, n \in \mathbb{N}, 0 < p, q < \infty, -1 < \gamma < \infty, \varphi \) is an analytic self-map of the unit disk, \( u \in H(D) \), and \( \mu \) is a weight. Then the operator \( D_{q,u}^m : H_{p,q,\gamma} \to \mathcal{K}_\mu^{(n)} \) is compact if and only if \( D_{q,u}^m : H_{p,q,\gamma} \to \mathcal{K}_\mu^{(n)} \) is bounded and for each \( k \in \{0, 1, \ldots, n\} \)

$$
\lim_{|\varphi(z)| \to 1} \frac{\mu(z)}{1 - |\varphi(z)|^2} \left| \sum_{l=k}^{n} C_l^n u^{(n-l)}(z) B_{i,k} \left( \varphi'(z), \ldots, \varphi^{(l-k+1)}(z) \right) \right| = 0. \quad (3.31)
$$

**Proof.** First assume that \( D_{q,u}^m : H_{p,q,\gamma} \to \mathcal{K}_\mu^{(n)} \) is bounded and (3.31) holds. By Theorem 3.1 we have that for each \( k \in \{0, 1, \ldots, n\} \), (3.1) holds.
Let \( (f_i)_{i \in \mathbb{N}} \) be a sequence in \( H_{p,q} \) such that \( \sup_{i, n} \|f_i\|_{H_{p,q}} \leq L \) and \( f_i \) converges to 0 uniformly on compact subsets of \( \mathbb{D} \) as \( i \to \infty \). By the assumption, for any \( \varepsilon > 0 \), there is a \( \delta \in (0, 1) \), such that for each \( k \in \{0, 1, \ldots, n\} \) and \( \delta < |\varphi(z)| < 1 \)

\[
\frac{\mu(z) \left| \sum_{i=0}^{n} C_{i}^{n} u^{(n-i)}(z) B_{i,k}(\varphi(z), \ldots, \varphi^{(l-k+1)}(z)) \right|}{(1 - |\varphi(z)|^2)^{(y+1)/q+1/p+k+m}} < \varepsilon.
\]  

(3.32)

We have

\[
\left\| D_{\psi,u}^m f_i \right\|_{L_p^n} = \sup_{z \in \mathbb{D}} \left| D_{\psi,u}^m f_i (z) \right| + \sum_{j=0}^{n-1} \left| D_{\psi,u}^m f_i (0) \right|
\]

\[
= \sup_{z \in \mathbb{D}} \mu(z) \left| \sum_{k=0}^{n} f_i^{(m+k)}(\varphi(z)) \sum_{l=k}^{n} C_{l}^{i} u^{(n-l)}(z) B_{l,k}(\varphi(z), \ldots, \varphi^{(l-k+1)}(z)) \right|
\]

\[
+ \sum_{j=0}^{n-1} \sum_{k=0}^{n} f_i^{(m+k)}(0) \sum_{l=k}^{n} C_{l}^{i} u^{(j-l)}(0) B_{l,k}(\varphi(0), \ldots, \varphi^{(l-k+1)}(0))
\]

\[
\leq \left( \sup_{|\varphi(z)| \leq \delta} + \sup_{|\varphi(z)| > \delta} \right) \mu(z) \left| \sum_{k=0}^{n} f_i^{(m+k)}(\varphi(z)) \right| \left| \sum_{l=k}^{n} C_{l}^{i} u^{(n-l)}(z) B_{l,k}(\varphi(z), \ldots, \varphi^{(l-k+1)}(z)) \right|
\]

\[
+ \sum_{j=0}^{n-1} \sum_{k=0}^{n} f_i^{(m+k)}(0) \sum_{l=k}^{n} C_{l}^{i} u^{(j-l)}(0) B_{l,k}(\varphi(0), \ldots, \varphi^{(l-k+1)}(0)) = J_1 + J_2 + J_3.
\]  

(3.33)

Now we estimate \( J_1, J_2, \) and \( J_3 \):

\[
J_1 = \sup_{|\varphi(z)| \leq \delta} \mu(z) \left| \sum_{k=0}^{n} f_i^{(m+k)}(\varphi(z)) \right| \left| \sum_{l=k}^{n} C_{l}^{i} u^{(n-l)}(z) B_{l,k}(\varphi(z), \ldots, \varphi^{(l-k+1)}(z)) \right|
\]

\[
\leq \sum_{k=0}^{n} \sup_{|\varphi(z)| \leq \delta} f_i^{(m+k)}(z) \left| \sum_{l=k}^{n} C_{l}^{i} u^{(n-l)}(z) B_{l,k}(\varphi(z), \ldots, \varphi^{(l-k+1)}(z)) \right|
\]

\[
\leq \sum_{k=0}^{n} \sup_{|\varphi(z)| \leq \delta} f_i^{(m+k)}(z) \left| \sum_{l=k}^{n} C_{l}^{i} u^{(n-l)}(z) B_{l,k}(\varphi(z), \ldots, \varphi^{(l-k+1)}(z)) \right| \left(1 - |\varphi(z)|^2\right)^{(y+1)/q+1/p+k+m}
\]

\[
= \sum_{k=0}^{n} \sup_{|\varphi(z)| \leq \delta} f_i^{(m+k)}(z) I_k \to 0, \quad \text{as } i \to \infty,
\]  

where in (3.34) we have used the fact that from \( f_i \to 0 \) uniformly on compact subsets of \( \mathbb{D} \) as \( i \to \infty \) it follows that for each \( s \in \mathbb{N}, f_i^{(s)} \to 0 \) uniformly on compact subsets of \( \mathbb{D} \) as \( i \to \infty \).
The fact that
\[ J_3 = \sum_{j=0}^{n-1} \sum_{k=0}^i f_i^{(m+k)}(\varphi(0)) \sum_{l=k}^i C_i^l u^{(l-j)}(0) B_{l,k}(\varphi(0), \ldots, \varphi^{(l-k+1)}(0)) \rightarrow 0, \] (3.35)
as \( i \rightarrow \infty \), is proved similarly; so we omit it.

By Lemma 2.2 and (3.32) we have that
\[ J_2 \leq C \| f_i \|_{H^{\nu\mu}} \sum_{k=0}^n \sup_{|\varphi(z)| < \delta} \frac{\mu(z)}{\sum_{l=k}^n C_i^l u^{(n-l)}(z) B_{l,k}(\varphi(z), \ldots, \varphi^{(l-k+1)}(z))} < C\varepsilon(n + 1)L. \] (3.36)

From (3.34), (3.35), and (3.36) we obtain
\[ \lim_{i \rightarrow \infty} \| D_{\varphi,u}^m f_i \|_{\mathcal{K}^{(n)}_{\mu}} = 0. \] (3.37)

From this and applying Lemma 2.1 the implication follows.

Now assume that \( D_{\varphi,u}^m : H_{p,q,Y} \rightarrow \mathcal{K}^{(n)}_{\mu} \) is compact; then clearly \( D_{\varphi,u}^m : H_{p,q,Y} \rightarrow \mathcal{K}^{(n)}_{\mu} \) is bounded. Let \( (z_i)_{i \in \mathbb{N}} \) be a sequence in \( \mathbb{D} \) such that \( |\varphi(z_i)| \rightarrow 1 \) as \( i \rightarrow \infty \). If such a sequence does not exist, then the conditions in (3.31) automatically hold.

Let \( g_{w,k} \in \{0,1,\ldots,n\} \) be as in Theorem 3.1. Then the sequences \( (g_{\varphi(z_i),k})_{i \in \mathbb{N}} \) are bounded and \( g_{\varphi(z_i),k} \rightarrow 0 \) uniformly on compact subsets of \( \mathbb{D} \) as \( i \rightarrow \infty \). Since \( D_{\varphi,u}^m : H_{p,q,Y} \rightarrow \mathcal{K}^{(n)}_{\mu} \) is compact, we have that for each \( k \in \{0,1,\ldots,n\} \)
\[ \lim_{i \rightarrow \infty} \| D_{\varphi,u}^m g_{\varphi(z_i),k} \|_{\mathcal{K}^{(n)}_{\mu}} = 0. \] (3.38)

On the other hand, from (3.10) we obtain
\[ \| D_{\varphi,u}^m g_{\varphi(z_i),k} \|_{\mathcal{K}^{(n)}_{\mu}} \geq \frac{C \mu(z_i)|\varphi(z_i)|^{k+m} \sum_{l=k}^n C_i^l u^{(n-l)}(z_i) B_{l,k}(\varphi(z_i), \ldots, \varphi^{(l-k+1)}(z_i))}{(1 - |\varphi(z_i)|^2)^{(q+1)/p+k+m}}, \] (3.39)
which along with \( |\varphi(z_i)| \rightarrow 1 \) as \( i \rightarrow \infty \) and (3.38) implies that
\[ \lim_{i \rightarrow \infty} \frac{\mu(z_i) \sum_{l=k}^n C_i^l u^{(n-l)}(z_i) B_{l,k}(\varphi(z_i), \ldots, \varphi^{(l-k+1)}(z_i))}{(1 - |\varphi(z_i)|^2)^{(q+1)/p+k+m}}, \] (3.40)
for each \( k \in \{0,1,\ldots,n\} \), from which (3.31) holds in this case.
4. The Compactness of the Operator $D_{\varphi,u}^m : H_{p,q,Y} \to \mathcal{K}_{\mu,0}^{(n)}$

The compactness of $D_{\varphi,u}^m : H_{p,q,Y} \to \mathcal{K}_{\mu,0}^{(n)}$ is characterized here. The proof of the next lemma is similar to the proof of the corresponding result in [14].

**Lemma 4.1.** Suppose that $n \in \mathbb{N}_0$ and $\mu$ is a radial weight such that $\lim_{|z| \to 1} \mu(z) = 0$. A closed set $K$ in $\mathcal{K}_{\mu,0}^{(n)}$ is compact if and only if it is bounded and satisfies

$$\lim_{|z| \to 1} \sup_{f \in K} \mu(z) |f^{(n)}(z)| = 0. \quad (4.1)$$

**Theorem 4.2.** Suppose that $m, n \in \mathbb{N}$, $0 < p, q < \infty$, $-1 < \gamma < \infty$, $\varphi$ is an analytic self-map of the unit disk, $u \in H(\mathbb{D})$ and $\mu$ is a radial weight such that $\lim_{|z| \to 1} \mu(z) = 0$. Then the operator $D_{\varphi,u}^m : H_{p,q,Y} \to \mathcal{K}_{\mu,0}^{(n)}$ is compact if and only if for each $k \in \{0, 1, \ldots, n\}$

$$\lim_{|z| \to 1} \frac{\mu(z) |\sum_{l=k}^{n} C_n^l u^{(n-l)}(z)B_{l,k}(\varphi(z), \varphi^{(l-k+1)}(z))|}{\left(1 - |\varphi(z)|^2\right)^{(r+1)/q+1/p+k+m}} = 0. \quad (4.2)$$

**Proof.** First assume that $D_{\varphi,u}^m : H_{p,q,Y} \to \mathcal{K}_{\mu,0}^{(n)}$ is compact. Then it is bounded and since the test functions in (3.12) belong to $H_{p,q,Y}(\mathbb{D})$, we have that (3.25) holds. Beside this the operator $D_{\varphi,u}^m : H_{p,q,Y} \to \mathcal{K}_{\mu,0}^{(n)}$ is compact too, so that (3.31) holds. Hence, if $|\varphi|_{\infty} < 1$, from (3.25) for each $k \in \{0, 1, \ldots, n\}$ we get

$$\frac{\mu(z) |\sum_{l=k}^{n} C_n^l u^{(n-l)}(z)B_{l,k}(\varphi(z), \varphi^{(l-k+1)}(z))|}{\left(1 - |\varphi(z)|^2\right)^{(r+1)/q+1/p+k+m}} \leq \frac{\mu(z) |\sum_{l=k}^{n} C_n^l u^{(n-l)}(z)B_{l,k}(\varphi(z), \varphi^{(l-k+1)}(z))|}{\left(1 - |\varphi|_p^2\right)^{(r+1)/q+1/p+k+m}} \to 0, \quad (4.3)$$

as $|z| \to 1$, hence we obtain (4.2) in this case.

Now assume $|\varphi|_{\infty} = 1$. Let $(\varphi(z))_{i \in \mathbb{N}}$ be a sequence such that $|\varphi(z)| \to 1$ as $i \to \infty$. Then from (3.31) we have that for every $\varepsilon > 0$, there is an $r \in (0, 1)$ such that for each $k \in \{0, 1, \ldots, n\}$

$$\frac{\mu(z) |\sum_{l=k}^{n} C_n^l u^{(n-l)}(z)B_{l,k}(\varphi(z), \varphi^{(l-k+1)}(z))|}{\left(1 - |\varphi(z)|^2\right)^{(r+1)/q+1/p+k+m}} < \varepsilon \quad (4.4)$$

when $r < |\varphi(z)| < 1$, and from (3.25) there exists a $\sigma \in (0, 1)$ such that for $\sigma < |z| < 1$

$$\mu(z) \left|\sum_{l=k}^{n} C_n^l u^{(n-l)}(z)B_{l,k}(\varphi(z), \varphi^{(l-k+1)}(z))\right| < \varepsilon \left(1 - r^2\right)^{(r+1)/q+1/p+k+m}. \quad (4.5)$$
Therefore, when $\sigma < |z| < 1$ and $r < |\varphi(z)| < 1$, we have that

$$\frac{\mu(z)\left| \sum_{l=k}^{n} C_l^m u^{(n-l)}(z) B_l \varphi'(z), \ldots, \varphi^{(l-k+1)}(z) \right|}{\left( 1 - |\varphi(z)|^2 \right)^{(l+1)/q+1/p+k+m}} < \varepsilon. \quad (4.6)$$

On the other hand, if $|\varphi(z)| \leq r$ and $\sigma < |z| < 1$, from (4.5) we obtain

$$\frac{\mu(z)\left| \sum_{l=k}^{n} C_l^m u^{(n-l)}(z) B_l \varphi'(z), \ldots, \varphi^{(l-k+1)}(z) \right|}{\left( 1 - |\varphi(z)|^2 \right)^{(l+1)/q+1/p+k+m}} < \varepsilon. \quad (4.7)$$

Combining the last two inequalities we obtain (4.2), as desired.

Now assume that (4.2) holds. Taking the supremum in (3.22) over $f$ in the unit ball of $H_{p,q,1}$, then letting $|z| \to 1$ is such obtained inequality and using (4.2) we get

$$\lim_{|z| \to 1} \sup_{\|f\|_{H_{p,q,1}} \leq 1} \mu(z) \left| D_{q,u}^{m} f^{(n)}(z) \right| = 0. \quad (4.8)$$

Hence by Lemma 4.1 the compactness of the operator $D_{q,u}^{m} : H_{p,q,1} \to \mathcal{H}^{(n)}_{q,1}$ follows. $\square$

References

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