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Research Article

Iterative Schemes for Fixed Points of Relatively Nonexpansive Mappings and Their Applications

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We present two iterative schemes with errors which are proved to be strongly convergent to a common element of the set of fixed points of a countable family of relatively nonexpansive mappings and the set of fixed points of nonexpansive mappings in the sense of Lyapunov functional in a real uniformly smooth and uniformly convex Banach space. Using the result we consider strong convergence theorems for variational inequalities and equilibrium problems in a real Hilbert space and strong convergence theorems for maximal monotone operators in a real uniformly smooth and uniformly convex Banach space.

1. Introduction

Let E be a real Banach space, and E^* the dual space of E. The function $\phi : E \to E^*$ is denoted by

$$\phi(y,x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2 \tag{1.1}$$

for all $x, y \in E$, where J is the normalized duality mapping from E to E. Let C be a closed convex subset of E, and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. A point p in C is said to be an asymptotic fixed point of T [1] if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n\to\infty}(x_n - Tx_n)$ equals 0. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$. A mapping T from C into itself is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$ and nonexpansive with respect to the Lyapunov functional [2] if $\phi(Tx, Ty) \le \phi(x, y)$ for all $x, y \in C$ and it is

called relatively nonexpansive [3–6] if $\widehat{F}(T) = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of relatively nonexpansive mapping was studied in [3–6].

There are many methods for approximating fixed points of a nonexpansive mapping. In 1953, Mann [7] introduced the iteration as follows: a sequence $\{x_n\}$ is defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \tag{1.2}$$

where the initial guess element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $[0, 1]$. Mann iteration has been extensively investigated for nonexpansive mappings. One of the fundamental convergence results was proved by Reich [1]. In an infinite-dimensional Hilbert space, Mann iteration can yield only weak convergence (see [8, 9]). Attempts to modify the Mann iteration method (1.2) so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi [10] proposed the following modification of Mann iteration method (1.2) for nonexpansive mapping T in a Hilbert space: in particular, they studied the strong convergence of the sequence $\{x_n\}$ generated by

$$\begin{gathered} x_0 = x \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_n = \{z \in C : \|z - y_n\| \leq \|z - x_n\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad n = 0, 1, 2, \ldots, \end{gathered} \tag{1.3}$$

where $\{\alpha_n\} \subset [0, 1]$ and $P_{C_n \cap Q_n}$ is the metric projection from C onto $C_n \cap Q_n$.

Recently, Takahashi et al. [11] extended iteration (1.6) to obtain strong convergence to a common fixed point of a countable family of nonexpansive mappings; let C be a nonempty closed convex subset of a Hilbert space H. Let $\{T_n\}$ and $\mathcal{T}$ be families of nonexpansive mappings of C into itself such that $\emptyset \neq F(\mathcal{T}) = \bigcap_{n=1}^{\infty} F(T_n)$ and let $x_0 \in H$. Suppose that $\{T_n\}$ satisfies the NST-condition (I) with $\mathcal{T}$; that is, for each bounded sequence $\{z_n\} \subset C$, $\lim_{n \to \infty} \|z_n - T_n z_n\| = 0$ implies that $\lim_{n \to \infty} \|z_n - T_n z_n\| = 0$ for all $T \in \mathcal{T}$. For $x_1 = P_C x_0$, define a sequence $\{x_n\}$ of C as follows:

$$\begin{gathered} y_n = \alpha_n x_n + (1 - \alpha_n) T_n x_n, \\ C_n = \{z \in C : \|z - y_n\| \leq \|z - x_n\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ u_{n+1} = P_{C_n \cap Q_n} x_0, \quad n \in \mathbb{N}, \end{gathered} \tag{1.4}$$

where $0 \leq \alpha_n \leq \alpha < 1$. Then, $\{u_n\}$ converges strongly to $z_0 = P_{F(T)} x_0$.

On the other hand, Halpern [12] introduced the following iterative scheme for approximating a fixed point of T:

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T x_n \tag{1.5}$$

for all $n \in \mathbb{N}$, where $x_1 = x \in C$ and $\{\alpha_n\}$ is a sequence of $[0,1]$. Strong convergence of this type of iterative sequence has been widely studied: for instance, see [13, 14] and the references therein. In 2006, Martinez-Yanes and Xu [15] have adapted Nakajo and Takahashi's [10] idea to modify the process (1.5) for a nonexpansive mapping T in a Hilbert space:

$$\begin{gathered} x_0 = x \in C, \\ y_n = \alpha_n x_0 + (1-\alpha_n)Tx_n, \\ C_n = \left\{ z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \alpha_n \left(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle \right) \right\}, \\ Q_n = \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0 \}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \ldots, \end{gathered} \tag{1.6}$$

where $P_{C_n \cap Q_n}$ is the metric projection from C onto $C_n \cap Q_n$. They proved that if $\{\alpha_n\} \subset (0,1)$ and $\lim_{n\to\infty} \alpha_n = 0$, then the sequence $\{x_n\}$ generated by (1.6) converges strongly to $P_{F(T)}x$.

The ideas to generalize the process (1.3) and (1.6) from Hilbert space to Banach space have been studied by many authors. Matsushita and Takahashi [6], Qin and Su [16], and Plubtieng and Ungchittrakool [17] generalized the process (1.3) and (1.6) and proved the strong convergence theorems for relatively nonexpansive mappings in a uniformly convex and uniformly smooth Banach spacey; see, for instance, [2, 6, 16, 18–22] and the references therein.

Recently, Nakajo et al. [18] introduced the following condition. Let C be a nonempty closed convex subset of a Hilbert space H, let $\{T_n\}$ be a family of mappings of C into itself with $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, and let $\omega_w(z_n)$ denote the set of all weak subsequential limits of a bounded sequence $\{z_n\}$ in C. $\{T_n\}$ is said to satisfy the NST-condition (II) if, for every bounded sequence $\{z_n\}$ in C,

$$\lim_{n\to\infty} \|z_n - T_n z_n\| = 0 \text{ implies } \omega_w(z_n) \subset F. \tag{1.7}$$

Very recently, Nakajo et al. [19] introduced the more general condition so-called the NST*-condition, $\{T_n\}$ is said to satisfy the NST*-condition if, for every bounded sequence $\{z_n\}$ in C,

$$\lim_{n\to\infty} \|z_n - T_n z_n\| = \lim_{n\to\infty} \|z_n - z_{n+1}\| = 0 \text{ implies } \omega_w(z_n) \subset F. \tag{1.8}$$

It follows directly from the definitions above that if $\{T_n\}$ satisfies the NST-condition (I), then $\{T_n\}$ satisfies the NST*-condition.

Motivated and inspired by Wei and Cho [2], in this paper, we introduce two iterative schemes (3.1) and (3.14) and use the NST*-condition for a countable family of relatively nonexpansive mappings to obtain the strong convergence theorems for finding a common element of the set of fixed points of a countable family of relatively nonexpansive mappings and the set of fixed points of nonexpansive mappings in the sense of Lyapunov functional in a real uniformly smooth and uniformly convex Banach space. Using this result, we also discuss the problem of strong convergence concerning variational inequality, equilibrium, and nonexpansive mappings in Hilbert spaces. Moreover, we also apply our convergence

theorems to the maximal monotone operators in Banach spaces. The results obtained in this paper improve and extend the corresponding result of Matsushita and Takahashi [6], Qin and Su [16], Wei and Cho [2], Wei and Zhou [22], and many others.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual of E. For all $x \in E$ and $x^* \in E^*$, we denote the value of x^* at x by $\langle x, x^* \rangle$. We denote the strong convergence and the weak convergence of a sequence $\{x_n\}$ to x in E by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. We also denote the weak* convergence of a sequence $\{x_n^*\}$ to x^* in E^* by $x_n^* \rightharpoonup^* x^*$. An operator $T \subset E \times E^*$ is said to be *monotone* if $\langle x - y, x^* - y^* \rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in T$. We denote the set $\{x \in E : 0 \in Tx\}$ by $T^{-1}0$. A monotone T is said to be *maximal* if its graph $G(T) = \{(x, y) : y \in Tx\}$ is not properly contained in the graph of any other monotone operator. If T is maximal monotone, then the solution set $T^{-1}0$ is closed and convex.

The normalized duality mapping J from E to E^* is defined by

$$Jx = \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \right\} \tag{2.1}$$

for $x \in E$. By Hahn-Banach theorem, $J(x)$ is nonempty; see [23] for more details. A Banach space E is said to be strictly convex if $\|(x+y)/2\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be uniformly convex if, for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|x + y\|/2 \leq \delta$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$. Let $S(E) = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided that

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.2}$$

exists for each $x, y \in S(E)$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in S(E)$. It is well known that if E is smooth, strictly convex, and reflexive, then the duality mapping J is single valued, one-to-one, and onto.

Let E be a smooth, strictly convex, and reflexive Banach space and let C be a nonempty closed convex subset of E. Throughout this paper, we denote by ϕ the function defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2, \quad \forall x, y \in C. \tag{2.3}$$

It is obvious from the definition of the function ϕ that $(\|x\| - \|y\|)^2 \leqslant \phi(y, x) \leqslant (\|y\|^2 + \|x\|^2)$ for all $x, y \in E$.

Following Alber [24], the generalized projection P_C from E onto C is defined by

$$P_C(x) = \arg\min_{y\in C} \phi(y,x), \quad \forall x \in E. \tag{2.4}$$

If E is a Hilbert space, then $\phi(y,x) = \|y-x\|^2$ and P_C is the metric projection of H onto C.

We need the following lemmas for the proof of our main results.

Lemma 2.1 (Kamimura and Takahashi [25]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty}\phi(x_n,y_n) = 0$, then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

Lemma 2.2 (Alber [24], Kamimura and Takahashi [25]). *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E. Then*

$$\phi(x, P_C y) + \phi(P_C y, y) \le \phi(x,y), \quad \forall x \in C,\ y \in E. \tag{2.5}$$

Lemma 2.3 (Alber [24], Kamimura and Takahashi [25]). *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let $x \in E$, and let $x_0 \in C$. Then*

$$x_0 = P_C x \Longleftrightarrow \langle z - x_0, Jx_0 - Jx\rangle \ge 0, \quad \forall z \in C. \tag{2.6}$$

Lemma 2.4 (Matsushita and Takahashi [6]). *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E, and let T be a relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

Lemma 2.5 (see [2]). *Let E be a real smooth and uniformly convex Banach space. If $S : E \to E$ is a mapping which is nonexpansive with respect to the Lyapunov functional, then $F(S)$ is convex and closed subset of E.*

3. Main Results

In this section, by using the NST*-condition, we proved two strong convergence theorems for finding a common element of the set of fixed points of a countable family of relatively nonexpansive mappings and the set of fixed points of nonexpansive mappings in the sense of Lyapunov functional in a real uniformly smooth and uniformly convex Banach space.

Theorem 3.1. *Let E be a real uniformly smooth and uniformly convex Banach space, let C be a nonempty closed convex subset of E, and let $S : C \to C$ be nonexpansive with respect to the Lyapunov functional and weakly sequentially continuous. Let $\{T_n\}$ be a family of relatively nonexpansive*

mappings of C into itself such that $F := \bigcap_{n=1}^{\infty} F(T_n) \cap F(S) \neq \emptyset$ and satisfy the NST-condition. Then the sequence $\{x_n\}$ generated by*

$$\begin{gathered} x_0 \in C, \\ y_n = T_n(x_n + e_n), \\ z_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J y_n), \\ u_n = S z_n, \\ H_n = \{z \in C : \phi(z, u_n) \leqslant \phi(z, z_n) \leqslant \alpha_n \phi(z, x_n) + (1-\alpha_n)\phi(z, x_n + e_n)\}, \\ W_n = \{z \in C : \langle z - x_n, J x_0 - J x_n \rangle \leq 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x_0, \quad n = 0, 1, 2, \ldots \end{gathered} \tag{3.1}$$

converges strongly to $P_F x_0$ provided that

(i) $\{\alpha_n\} \subset [0,1]$ *with* $\alpha_n \leq 1 - \beta$ *for some* $\beta \in (0,1)$;

(ii) *the error sequence is* $\{e_n\} \subset C$ *such that* $\|e_n\| \to 0$ *as* $n \to \infty$.

Proof. We split the proof into five steps.

Step 1 (Both H_n and W_n are closed and convex subset of C). Noting the facts that

$$\begin{aligned} \phi(z, z_n) \leqslant \alpha_n \phi(z, x_n) + (1-\alpha_n)\phi(z, x_n + e_n) &\Longleftrightarrow \|z_n\|^2 - \alpha_n \|x_n\|^2 - (1-\alpha_n)\|x_n + e_n\|^2 \\ &\leq 2\langle z, J z_n - \alpha_n J x_n - (1-\alpha_n) J(x_n + e_n)\rangle, \\ \phi(z, u_n) \leqslant \phi(z, z_n) &\Longleftrightarrow \|z_n\|^2 - \|u_n\|^2 \geq 2\langle z, J z_n - J u_n\rangle, \end{aligned} \tag{3.2}$$

we can easily know that H_n is closed and convex subset of E. It is obvious that W_n is also a closed and convex subset of E.

Step 2 ($F := (\bigcap_{n=1}^{\infty} F(T_n)) \cap F(S) \subseteq H_n \cap W_n$ for all $n \in \mathbb{N} \cup \{0\}$). To observe this, take $p \in F$. Then it follows from the convexity of $\|\cdot\|^2$ that

$$\begin{aligned} \phi(p, u_n) &= \phi(Sp, S z_n) \leq \phi(p, z_n) \\ &= \|p\|^2 - 2\langle p, \alpha_n J x_n + (1-\alpha_n) J y_n\rangle + \|\alpha_n J x_n + (1-\alpha_n) J y_n\|^2 \\ &\leq \|p\|^2 - 2\alpha_n \langle p, J x_n\rangle - 2(1-\alpha_n)\langle p, J y_n\rangle + \alpha_n \|x_n\|^2 + (1-\alpha_n)\|y_n\|^2 \\ &= \alpha_n \phi(p, x_n) + (1-\alpha_n)\phi(p, y_n) \\ &= \alpha_n \phi(p, x_n) + (1-\alpha_n)\phi(p, T_n(x_n + e_n)) \\ &\leq \alpha_n \phi(p, x_n) + (1-\alpha_n)\phi(p, x_n + e_n) \end{aligned} \tag{3.3}$$

for all $n \in \mathbb{N} \cup \{0\}$. Hence $F \subset H_n$ for all $n \in \mathbb{N} \cup \{0\}$. On the other hand, it is clear that $p \in W_0 = C$. Then $p \in H_0 \cap W_0$ and $x_1 = P_{H_0 \cap W_0} x_0$ is well defined. Suppose that $p \in W_{n-1}$ for some $n \geq 1$. Then $p \in H_{n-1} \cap W_{n-1}$ and $x_n = P_{H_{n-1} \cap W_{n-1}} x_0$ is well defined. It follows from Lemma 2.3 that

$$\langle p - x_n, Jx_0 - Jx_n \rangle = \langle p - P_{H_{n-1} \cap W_{n-1}} x_0, Jx_0 - JP_{H_{n-1} \cap W_{n-1}} x_0 \rangle \leq 0, \tag{3.4}$$

which implies that $p \in W_n$. Therefore $p \in H_n \cap W_n$, and hence $x_{n+1} = P_{H_n \cap W_n} x_0$ is well defined. Then by induction, the sequence $\{x_n\}$ generated by (3.1) is well defined, for each $n \geq 0$. Moreover $F \subset H_n \cap W_n$ for each nonnegative integer n.

Step 3 ($\{x_n\}$ is bounded sequence of C). In fact, for all $p \in F \subset H_n \cap W_n \subset W_n$, it follows from Lemma 2.2 that

$$\phi(p, P_{W_n} x_0) + \phi(P_{W_n} x_0, x_0) \leq \phi(p, x_0). \tag{3.5}$$

By the definition of W_n and Lemma 2.2, we note that $x_n = P_{W_n} x_0$ and hence

$$\phi(p, x_n) + \phi(x_n, x_0) \leq \phi(p, x_0). \tag{3.6}$$

Therefore $\{x_n\}$ is bounded.

Step 4 ($\omega_w(x_n) \subset F := (\bigcap_{n=1}^{\infty} F(T_n)) \cap F(S)$). From the facts $x_n = P_{W_n} x_0$, $x_{n+1} \in W_n$, and Lemma 2.2, we have

$$\phi(x_{n+1}, x_n) + \phi(x_n, x_0) \leq \phi(x_{n+1}, x_0). \tag{3.7}$$

Therefore, $\lim_{n \to \infty} \phi(x_n, x_0)$ exists. Then $\phi(x_{n+1}, x_n) \to 0$, which implies from Lemma 2.1 that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Since $x_{n+1} \in H_n$, we have

$$\phi(x_{n+1}, u_n) \leqslant \phi(x_{n+1}, z_n) \leqslant \alpha_n \phi(x_{n+1}, x_n) + (1 - \alpha_n) \phi(x_{n+1}, x_n + e_n). \tag{3.8}$$

Note that

$$\phi(x_{n+1}, x_n + e_n) - \phi(x_{n+1}, x_n) = \|x_n + e_n\|^2 - \|x_n\|^2 + 2\langle x_{n+1}, Jx_n - J(x_n + e_n) \rangle. \tag{3.9}$$

Since J is uniform continuous on each bounded subset of E and $\|e_n\| \to 0$, we have $\phi(x_{n+1}, x_n + e_n) \to 0$ as $n \to \infty$. This implies that $\phi(x_{n+1}, u_n) \to 0$ and $\phi(x_{n+1}, z_n) \to 0$ as $n \to \infty$. Using Lemma 2.1, we obtain $\lim_{n \to \infty} \|x_{n+1} - (x_n + e_n)\| = 0$, $\lim_{n \to \infty} \|x_{n+1} - z_n\| = 0$, and $\lim_{n \to \infty} \|x_{n+1} - u_n\| = 0$. Since both $J : E \to E^*$ and $J^{-1} : E^* \to E$ are uniformly continuous on bounded subsets, it follows that $\lim_{n \to \infty} \|x_n - y_n\| = 0$.

Put $v_n := x_n + e_n$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\{x_n\}$ is bounded and $\|e_n\| \to 0$, we have $\{v_n\}$ is bounded. Note that

$$\begin{aligned} \|v_{n+1} - v_n\| &= \|(x_{n+1} + e_{n+1}) - (x_n + e_n)\| \\ &\le \|x_{n+1} - (x_n + e_n)\| + \|e_{n+1}\|, \\ \|v_n - T_n v_n\| &= \|(x_n + e_n) - T_n(x_n + e_n)\| = \|(x_n + e_n) - y_n\| \\ &\le \|(x_n + e_n) - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - y_n\| \end{aligned} \tag{3.10}$$

for all $n \in \mathbb{N} \cup \{0\}$. It implies that $\lim_{n\to\infty}\|v_{n+1} - v_n\| = \lim_{n\to\infty}\|v_n - T_n v_n\| = 0$. Therefore, we have $\omega_w(v_n) \subset \bigcap_{n=1}^{\infty} F(T_n)$ since $\{T_n\}$ satisfies NST*-condition. Note that $\|v_n - x_n\| = \|(x_n + e_n) - x_n\| = \|e_n\| \to 0$ as $n \to \infty$. Hence, we also have $\omega_w(x_n) \subset \bigcap_{n=1}^{\infty} F(T_n)$.

On the other hand, from Step 3, we know that $\omega_w(x_n) \neq \emptyset$. Then, for all $q \in \omega_w(x_n)$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup q$ as $i \to \infty$. Therefore, $u_{n_i} \rightharpoonup q$ and $z_{n_i} \rightharpoonup q$ as $i \to \infty$. Since $S : C \to C$ is weakly continuous and $u_{n_i} = Sz_{n_i}$, we have $q \in F(S)$. Hence $\omega_w(x_n) \subset F := (\bigcap_{n=1}^{\infty} F(T_n)) \cap F(S)$.

Step 5 ($x_n \to q^* = P_F x_0$, as $n \to \infty$). Let $\{x_{n_j}\}$ be any subsequence of $\{x_n\}$ which weakly converges to $q \in F$. Since $x_{n+1} = P_{H_n \cap W_n} x_0$ and $q^* \in F \subset H_n \cap W_n$, we have $\phi(x_{n+1}, x_0) \le \phi(q^*, x_0)$. Then,

$$\begin{aligned} \phi(x_n, q^*) &= \phi(x_n, x_0) + \phi(x_0, q^*) - 2\langle x_n - x_0, Jq^* - Jx_0\rangle \\ &\le \phi(q^*, x_0) + \phi(x_0, q^*) - 2\langle x_n - x_0, Jq^* - Jx_0\rangle, \end{aligned} \tag{3.11}$$

which yields

$$\limsup_{j\to\infty} \phi\left(x_{n_j}, q^*\right) \le \phi(q^*, x_0) + \phi(x_0, q^*) - 2\langle q - x_0, Jq^* - Jx_0\rangle = -2\langle q^* - q, Jq^* - Jx_0\rangle \le 0. \tag{3.12}$$

Hence $\phi(x_{n_j}, q^*) \to 0$ as $j \to \infty$. It follows from Lemma 2.1 that $x_{n_j} \to q^*$ as $j \to \infty$. Therefore, $x_n \to q^* = P_F x_0$ as $n \to \infty$. This completes the proof. □

Corollary 3.2 (see Matsushita and Takahashi [6]). *Let E be a real uniformly smooth and uniformly convex Banach space, let C be a nonempty closed convex subset of E, let T be a relatively nonexpansive mappings of C into itself, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n < 1$ and $\limsup_{n\to\infty} \alpha_n < 1$. Suppose that $\{x_n\}$ is given by*

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JTx_n), \\ H_n &= \{z \in C : \phi(z, y_n) \leqslant \phi(z, x_n)\}, \\ W_n &= \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \ge 0\}, \\ x_{n+1} &= \Pi_{H_n \cap W_n} x_0, \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{3.13}$$

where J is the duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the generalized projection from C onto $F(T)$.

Proof. Suppose that $F(T) \neq \emptyset$ and put $T_n \equiv T$, $S \equiv I$, and $e_n = 0$ for all $n \in \mathbb{N} \cup \{0\}$. Let $\{z_n\}$ be a bounded sequence in C with $\lim_{n\to\infty}\|z_n - Tz_n\| = 0$ and let $z \in \omega_w(z_n)$. Then there exists subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \rightharpoonup z$. It follows directly from the definition of T that $z \in \widehat{F}(T) = F(T)$. Hence T satisfies NST-condition; by Theorem 3.1, $\{x_n\}$ converges strongly to $P_F(T)x$. □

Theorem 3.3. *Let E be a real uniformly smooth and uniformly convex Banach space, let C be a nonempty closed convex subset of E, and let $S : C \to C$ be nonexpansive with respect to the Lyapunov functional and weakly sequentially continuous. Let $\{T_n\}$ be a family of relatively nonexpansive mappings of C into itself such that $F := \bigcap_{n=1}^{\infty} F(T_n) \cap F(S) \neq \emptyset$ and satisfy the NST*-condition. Then the sequence $\{x_n\}$ generated by*

$$\begin{gathered}
x_0 \in C, \\
y_n = T_n(x_n + e_n), \\
z_n = J^{-1}\left(\alpha_n J x_0 + (1-\alpha_n) J y_n\right), \\
u_n = S z_n, \\
H_n = \left\{z \in C : \phi(z, u_n) \leqslant \phi(z, z_n) \leqslant \alpha_n \phi(z, x_0) + (1-\alpha_n)\phi(z, x_n + e_n)\right\}, \\
W_n = \{z \in C : \langle z - x_n, J x_0 - J x_n\rangle \leq 0\}, \\
x_{n+1} = P_{H_n \cap W_n} x_0
\end{gathered} \tag{3.14}$$

converges strongly to $P_F x_0$ provided that

(i) *$\{\alpha_n\} \subset [0,1]$ is a sequence such that $\alpha_n \to 0$ as $n \to \infty$;*

(ii) *the error sequence is $\{e_n\} \subset E$ such that $\|e_n\| \to 0$ as $n \to \infty$.*

Proof. By slightly modifying the corresponding proof in Theorem 3.1, we can easily obtain the result. □

Setting $T_n \equiv T$, $S \equiv I$, and $e_n = 0$ for all $n \in \mathbb{N} \cup \{0\}$ in Theorem 3.3, we obtain the following corollary.

Corollary 3.4 (see Qin and Su [16]). *Let E be a real uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E. Let T be a relatively nonexpansive mappings of C into itself, and let $\{\alpha_n\}$ be a sequence in $(0,1)$ and $\lim_{n\to\infty}\alpha_n = 0$. Define a sequence $\{x_n\}$ in C by*

$$\begin{gathered}
x_0 \in C, \\
y_n = J^{-1}(\alpha_n J x_0 + (1-\alpha_n) J T x_n), \\
C_n = \left\{z \in C : \phi(z, y_n) \leqslant \alpha_n \phi(z, x_0) + (1-\alpha_n)\phi(z, x_n)\right\}, \\
Q_n = \{z \in C : \langle x_n - z, J x_0 - J x_n\rangle \geq 0\}, \\
x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \ldots,
\end{gathered} \tag{3.15}$$

where J is the duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the generalized projection from C onto $F(T)$.

4. Applications to the Variational Inequality Problem, Equilibrium Problem, and Fixed Points Problem of Nonexpansive Mappings in a Real Hilbert Space

In this section, using Theorems 3.1 and 3.3, we prove the strong convergence theorems for finding a common element of the set of fixed points of nonexpansive mapping, the solution set of the variational inequality problems, and the solution set of an equilibrium problems in a Hilbert space.

4.1. Common Solutions of a Fixed Point Problem and a Variational Inequality Problem

Let C be a nonempty closed convex subset of a real Hilbert space H and let A be a mapping of C into H. A classical variational inequality problem, denoted by $\mathrm{VI}(C, A)$, is to find an element $u^* \in C$ such that $\langle u - u^*, Au^* \rangle \geq 0$ for all $u \in C$. It is known that for $\lambda > 0$, $x \in C$ is a solution of the variational inequality of A if and only if $x = P_C(I - \lambda A)x$, where P_C is the metric projection from H onto C. We denote by $N_C(v)$ the normal cone for C at a point $v \in C$, that is, $N_C(v) = \{w \in H : \langle w, v - u \rangle \geq 0 \text{ for all } u \in C\}$. Define $f : H \to H$ by

$$f(v) = \begin{cases} Av + N_C(v), & \forall v \in C, \\ \emptyset, & \text{otherwise.} \end{cases} \tag{4.1}$$

Then f is maximal monotone and $f^{-1}0 = \mathrm{VI}(C, A)$ (see [26]). A mapping $A : C \to H$ is called α-inverse strongly monotone if there exists an $\alpha > 0$ such that

$$\langle Au - Av, u - v \rangle \geq \alpha \|Au - Av\|^2, \quad \forall u, v \in C. \tag{4.2}$$

It is well known that if $0 < \lambda < 2\alpha$, then $P_C(I - \lambda A)$ is nonexpansive of C into itself.

Lemma 4.1 (see [27]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let A be α-inverse strongly monotone mapping of C into H. Let $\{\lambda_n\} \subset (0, 2\alpha)$ and let P_C be a metric projection from H onto C. Let $T_n = P_C(I - \lambda_n A)$, for all $n \in \mathbb{N}$. Then $\{T_n\}$ is a sequence of nonexpansive mappings and satisfies NST-condition.*

By using Lemma 4.1 and Theorem 3.1 with $e_n = 0$, for all $n \in \mathbb{N} \cup \{0\}$, we obtain the following theorem.

Theorem 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H and let A be α-inverse strongly monotone mapping of C into H. Let S be a nonexpansive mapping of C into itself*

such that $F := F(S) \cap VI(C, A) \neq \emptyset$. Let P_C be a metric projection from H onto C. Then the sequence $\{x_n\}$ generated by

$$\begin{gathered} x_0 \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n) P_C(x_n - \lambda_n A x_n), \\ z_n = S y_n, \\ H_n = \{z \in C : \|z - z_n\| \leq \|z - x_n\|\}, \\ W_n = \{z \in C : \langle z - x_n, x_0 - x_n \rangle \leq 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x_0, \quad n = 0, 1, 2, \ldots \end{gathered} \tag{4.3}$$

converges strongly to $P_F x_0$ provided that

(i) $\{\alpha_n\} \subset [0,1]$ *with* $\alpha_n \leq 1 - \beta$ *for some* $\beta \in (0,1)$;

(ii) $\{\lambda_n\} \subset [a,b]$ *for some* $a, b \in (0, 2\alpha)$ *with* $a \leq b$.

Next, by using Lemma 4.1 and Theorem 3.3 with $e_n = 0$, for all $n \in \mathbb{N} \cup \{0\}$, we obtain the following theorem.

Theorem 4.3. *Let C be a nonempty closed convex subset of a real Hilbert space H and let A be α-inverse strongly monotone mapping of C into H. Let S be a nonexpansive mapping of C into itself such that $F := F(S) \cap VI(C, A) \neq \emptyset$. Let P_C be a metric projection from H onto C. Then the sequence $\{x_n\}$ generated by*

$$\begin{gathered} x_0 \in C, \\ y_n = \alpha_n x_0 + (1-\alpha_n) P_C(x_n - \lambda_n A x_n), \\ z_n = S y_n, \\ H_n = \left\{z \in C : \|z - z_n\|^2 \leq \|z - x_n\|^2 + \alpha_n \left(\|x_0\|^2 + 2\langle x_n - x_0, v \rangle\right)\right\}, \\ W_n = \{z \in C : \langle z - x_n, x_0 - x_n \rangle \leq 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x_0, \quad n = 0, 1, 2, \ldots \end{gathered} \tag{4.4}$$

converges strongly to $P_F x_0$ provided that

(i) $\{\alpha_n\} \subset [0,1]$ *is a sequence such that* $\alpha_n \to 0$ *as* $n \to \infty$;

(ii) $\{\lambda_n\} \subset [a,b]$ *for some* $a, b \in (0, 2\alpha)$ *with* $a \leq b$.

4.2. Common Solutions of a Fixed Point Problem and an Equilibrium Problem

Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{4.5}$$

The set of solutions of (4.5) is denoted by $EP(F)$. Numerous problems in physics, optimization, and economics reduce to find a solution of (4.5). In 1997, Combettes and Hirstoaga [15] introduced an iterative scheme of finding the best approximation to the initial data when $EP(F)$ is nonempty and proved a strong convergence theorem.

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, that is, $F(x,y) + F(y,x) \leq 0$ forall $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t \to 0} F\big((tz + (1-t)x), y\big) \leq F(x,y); \tag{4.6}$$

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [15, 28].

Lemma 4.4 (see [15, 28]). *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ in to $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{4.7}$$

Define a mapping $T_r : H \to C$ as follows:

$$T_r(x) = \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C \right\} \tag{4.8}$$

for all $z \in H$. Then, the following holds:

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle; \tag{4.9}$$

(3) *$F(T_r) = EP(F)$;*

(4) *$EP(F)$ is closed and convex.*

Lemma 4.5 (see [27]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let f be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4) and $EP(f) \neq \emptyset$. If $\{r_n\}$ is a sequence in $(0,\infty)$ satisfying $\liminf_{n\to\infty} r_n > 0$, then $\{T_{r_n}\}$ is a family of firmly nonexpansive mappings of H into C and satisfies NST-condition.*

The following theorem follows directly from Lemma 4.5 and Theorem 3.1.

Theorem 4.6. *Let C be a nonempty closed convex subset of a real Hilbert space H and let f be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let S be a nonexpansive mapping of C into itself such that $F := F(S) \cap EP(f) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by*

$$\begin{gathered} x_0 \in C, \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n) u_n, \\ z_n = S y_n, \\ H_n = \{z \in C : \|z - z_n\| \leq \|z - x_n\|\}, \\ W_n = \{z \in C : \langle z - x_n, x_0 - x_n\rangle \leq 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x_0, \quad n = 0, 1, 2, \ldots \end{gathered} \tag{4.10}$$

converges strongly to $P_F x_0$ provided that

(i) $\{\alpha_n\} \subset [0, 1]$ *with* $\alpha_n \leq 1 - \beta$ *for some* $\beta \in (0, 1)$;

(ii) $\{r_n\} \subset (0, \infty)$ *such that* $\liminf_{n \to \infty} r_n > 0$,

Proof. Put $T_n \equiv T_{r_n}$ and $e_n = 0$ for all $n \in \mathbb{N} \cup \{0\}$. Then, it follows from Lemma 4.5 and Theorem 3.1 that $\{x_n\}$ converges strongly to $P_F x_0$. □

Next, by using Lemma 4.5 and Theorem 3.3 with $e_n = 0$, for all $n \in \mathbb{N} \cup \{0\}$, we obtain the following result.

Theorem 4.7. *Let C be a nonempty closed convex subset of a real Hilbert space H and let f be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let S be a nonexpansive mapping of C into itself such that $F := F(S) \cap EP(f) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by*

$$\begin{gathered} x_0 \in C, \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_0 + (1 - \alpha_n) u_n, \\ z_n = S y_n, \\ H_n = \left\{z \in C : \|z - z_n\|^2 \leq \|z - x_n\|^2 + \alpha_n\left(\|x_0\|^2 + 2\langle x_n - x_0, v\rangle\right)\right\}, \\ W_n = \{z \in C : \langle z - x_n, x_0 - x_n\rangle \leq 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x_0, \quad n = 0, 1, 2, \ldots \end{gathered} \tag{4.11}$$

converges strongly to $P_F x_0$ provided that

(i) $\{\alpha_n\} \subset [0,1]$ *is a sequence such that* $\alpha_n \to 0$ *as* $n \to \infty$;

(ii) $\{r_n\} \subset (0,\infty)$ *such that* $\liminf_{n\to\infty} r_n > 0$,

5. Applications to Maximal Monotone Operators in Banach Space

In this section, we discuss the problem of strong convergence concerning maximal monotone operators in a real uniformly smooth and uniformly convex Banach space.

Let E be a smooth, strictly convex, and reflexive Banach space and let $T \subset E \times E^*$ be a maximal monotone operator. Then for each $r > 0$ and $x \in E$, there corresponds a unique element $x_r \in D(T)$ satisfying

$$J(x) \in J(x_r) + rT(x_r); \tag{5.1}$$

see [23]. We define the *resolvent* of T by $J_r x = x_r$. In other words, $J_r = (J + rT)^{-1} J$ for all $r > 0$. We know that J_r is relatively nonexpansive and $T^{-1}0 = F(J_r)$ for all $r > 0$ (see [6, 23]), where $F(J_r)$ denotes the set of all fixed points of J_r. We can also define, for each $r > 0$, the *Yosida approximation* of T by $A_r = \lambda^{-1}(J - JJ_r)$. We know that $(J_r x, A_r x) \in T$ for all $r > 0$.

Lemma 5.1. *Let E be a real uniformly smooth and uniformly convex Banach space and let $T : E \to E^*$ be a maximal monotone operator with $F := T^{-1}0 \neq \emptyset$ and $J_r = (J + rT)^{-1} J$ for all $r > 0$. Let $\{T_n\}$ be a sequence of relatively nonexpansive mappings of E into itself defined by $T_n \equiv J_{r_n}$ for all $n \in \mathbb{N} \cup \{0\}$, where $\{r_n\}$ is a sequence in $(0,\infty)$ such that $\liminf_{n\geq 0} r_n > 0$. Then, $\{T_n\}$ satisfies the NST-condition.*

Proof. It easy to see that $\bigcap_{n=0}^{\infty} F(T_n) = \bigcap_{n=0}^{\infty} F(J_{r_n}) = T^{-1}0$. Let $\{v_n\}$ be a bounded sequence in E such that $\lim_{n\to\infty} \|v_n - J_{r_n} v_n\| = 0$ and let $v_0 \in \omega_w(v_n)$. Then, there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that $v_{n_k} \rightharpoonup v_0$. By the uniform smoothness of E, we have $\lim_{n\to\infty} \|Jv_n - JJ_{r_n} v_n\| = 0$. Since $\liminf_{n\to\infty} r_n > 0$, we have

$$\lim_{n\to\infty} \|A_{r_n} v_n\| = \lim_{n\to\infty} \frac{1}{r_n} \|Jv_n - JJ_{r_n} v_n\| = 0. \tag{5.2}$$

Let $(u, u^*) \in T$. Then it holds from the monotonicity of T that

$$\langle u - J_{r_{n_k}} v_{n_k}, u^* - A_{r_{n_k}} v_{n_k} \rangle \geq 0 \tag{5.3}$$

for all $k \in \mathbb{N}$. Letting $k \to \infty$, we get $\langle u - v_0, u^* \rangle \geq 0$. Then, the maximality of T implies $v_0 \in T^{-1}0 := \bigcap_{n=1}^{\infty} F(J_{r_n}) = \bigcap_{n=0}^{\infty} F(T_n)$. Hence $\omega_w(v_n) \subset \bigcap_{n=0}^{\infty} F(T_n)$. Therefore, $\{T_n\}$ satisfies the NST-condition. □

Using Theorem 3.1 and Lemma 5.1, we first obtain the result of [2].

Theorem 5.2 (see [2]). *Let E be a real uniformly smooth and uniformly convex Banach space and let $S : E \to E$ be nonexpansive with respect to the Lyapunov functional and weakly sequentially*

continuous. Let $T : E \to E^*$ be a maximal monotone operator with $F := T^{-1}0 \cap F(S) \neq \emptyset$ and $J_r = (J + rT)^{-1}J$ for all $r > 0$. Then, sequence $\{x_n\}$ generated by the following scheme

$$\begin{gathered} x_0 \in E, \quad r_0 > 0, \\ y_n = J_{r_n}(x_n + e_n), \\ z_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J y_n), \\ u_n = S z_n, \\ H_n = \{z \in E : \phi(z, u_n) \leqslant \phi(z, z_n) \leqslant \alpha_n \phi(z, x_n) + (1-\alpha_n)\phi(z, x_n + e_n)\}, \\ W_n = \{z \in E : \langle z - x_n, J x_0 - J x_n \rangle \leq 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x_0, \quad n = 0, 1, 2, \ldots \end{gathered} \tag{5.4}$$

converges strongly to $P_F x_0$ provided that

(i) $\{\alpha_n\} \subset [0,1]$ *with* $\alpha_n \leq 1 - \beta$ *for some* $\beta \in (0,1)$;

(ii) $\{r_n\} \subset (0, +\infty)$ *is a sequence such that* $\liminf_{n \geq 0} r_n > 0$;

(iii) *the error sequence is* $\{e_n\} \subset E$ *such that* $\|e_n\| \to 0$ *as* $n \to \infty$.

Proof. Put $T_n \equiv J_{r_n}$ for all $n \in \mathbb{N} \cup \{0\}$. Hence by using Lemma 5.1 and Theorem 3.1, we obtain the result. □

Putting $S \equiv I$ in Theorem 5.2, we obtain the following corollary.

Corollary 5.3 (see [22]). *Let E be a real uniformly smooth and uniformly convex Banach space and let $T : E \to E^*$ be a maximal monotone operator with $F := T^{-1}0 \cap F(S) \neq \emptyset$. and $J_r = (J + rT)^{-1}J$ for all $r > 0$. Then, sequence $\{x_n\}$ generated by the following scheme*

$$\begin{gathered} x_0 \in E, \quad r_0 > 0, \\ y_n = J_{r_n}(x_n + e_n), \\ z_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J y_n), \\ H_n = \{z \in E : \phi(z, z_n) \leqslant \alpha_n \phi(z, x_n) + (1-\alpha_n)\phi(z, x_n + e_n)\}, \\ W_n = \{z \in E : \langle z - x_n, J x_0 - J x_n \rangle \leq 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x_0, \quad n = 0, 1, 2, \ldots \end{gathered} \tag{5.5}$$

converges strongly to $P_F x_0$ provided that

(i) $\{\alpha_n\} \subset [0,1]$ *with* $\alpha_n \leq 1 - \beta$ *for some* $\beta \in (0,1)$;

(ii) $\{r_n\} \subset (0, +\infty)$ *is a sequence such that* $\liminf_{n \geq 0} r_n > 0$;

(iii) *the error sequence is* $\{e_n\} \subset E$ *such that* $\|e_n\| \to 0$ *as* $n \to \infty$.

The following theorem follows directly from Lemma 5.1 and Theorem 3.3

Theorem 5.4 (see [2]). *Let E be a real uniformly smooth and uniformly convex Banach space and let $S : E \to E$ be nonexpansive with respect to the Lyapunov functional and weakly sequentially continuous. Let $T : E \to E^*$ be a maximal monotone operator with $F := T^{-1}0 \cap F(S) \neq \emptyset$. and $J_r = (J + rT)^{-1}J$ for all $r > 0$. Then, sequence $\{x_n\}$ generated by the following scheme*

$$\begin{gathered} x_0 \in E, \quad r_0 > 0, \\ y_n = J_{r_n}(x_n + e_n), \\ z_n = J^{-1}\left(\alpha_n J x_0 + (1-\alpha_n) J y_n\right), \\ u_n = S z_n, \\ H_n = \left\{z \in E : \phi(z, u_n) \leqslant \phi(z, z_n) \leqslant \alpha_n \phi(z, x_0) + (1-\alpha_n)\phi(z, x_n + e_n)\right\}, \\ W_n = \{z \in E : \langle z - x_n, J x_0 - J x_n \rangle \leq 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x_0, \quad n = 0, 1, 2, \ldots \end{gathered} \tag{5.6}$$

converges strongly to $P_F x_0$ provided that

(i) $\{\alpha_n\} \subset [0,1]$ *is a sequence such that* $\alpha_n \to 0$ *as* $n \to \infty$;

(ii) $\{r_n\} \subset (0, +\infty)$ *is a sequence such that* $\liminf_{n \geq 0} r_n > 0$;

(iii) *the error sequence is* $\{e_n\} \subset E$ *such that* $\|e_n\| \to 0$ *as* $n \to \infty$.

Putting $S \equiv I$ in Theorem 5.4, we obtain the following corollary.

Corollary 5.5 (see [22]). *Let E be a real uniformly smooth and uniformly convex Banach space, let C be a nonempty closed convex subset of E, and let $T : E \to E^*$ be a maximal monotone operator with $F := T^{-1}0 \cap F(S) \neq \emptyset$ and $J_r = (J + rT)^{-1}J$ for all $r > 0$. Then, sequence $\{x_n\}$ generated by the following scheme*

$$\begin{gathered} x_0 \in E, \quad r_0 > 0, \\ y_n = J_{r_n}(x_n + e_n), \\ z_n = J^{-1}\left(\alpha_n J x_0 + (1-\alpha_n) J y_n\right), \\ H_n = \left\{z \in E : \phi(z, z_n) \leqslant \alpha_n \phi(z, x_0) + (1-\alpha_n)\phi(z, x_n + e_n)\right\}, \\ W_n = \{z \in E : \langle z - x_n, J x_0 - J x_n \rangle \leq 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x_0, \quad n = 0, 1, 2, \ldots \end{gathered} \tag{5.7}$$

converges strongly to $P_F x_0$ provided that

(i) $\{\alpha_n\} \subset [0,1]$ *is a sequence such that* $\alpha_n \to 0$ *as* $n \to \infty$;

(ii) $\{r_n\} \subset (0, +\infty)$ *is a sequence such that* $\liminf_{n \geq 0} r_n > 0$;

(iii) *the error sequence is* $\{e_n\} \subset E$ *such that* $\|e_n\| \to 0$ *as* $n \to \infty$.

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References

[1] S. Reich, "A weak convergence theorem for the alternating method with Bregman distances," in *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*, A. G. Kartsatos, Ed., , pp. 313–318, Marcel Dekker, New York, NY, USA, 1996.

[2] L. Wei and Y. J. Cho, "Iterative schemes for zero points of maximal monotone operators and fixed points of nonexpansive mappings and their applications," *Fixed Point Theory and Applications*, vol. 2008, Article ID 168468, 2008.

[3] D. Butnariu, S. Reich, and A. J. Zaslavski, "Asymptotic behavior of relatively nonexpansive operators in Banachspaces," *Journal of Applied Analysis*, vol. 7, pp. 151–174, 2001.

[4] D. Butnariu, S. Reich, and A. J. Zaslavski, "Weak convergence of orbits of nonlinear operators in reflexive Banach spaces," *Numerical Functional Analysis and Optimization*, vol. 24, no. 5-6, pp. 489–508, 2003.

[5] Y. Censor and S. Reich, "Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization," *Optimization*, vol. 37, no. 4, pp. 323–339, 1996.

[6] S.-Y. Matsushita and W. Takahashi, "A strong convergence theorem for relatively nonexpansive mappings in a Banach space," *Journal of Approximation Theory*, vol. 134, no. 2, pp. 257–266, 2005.

[7] W. R. Mann, "Mean value methods in iteration," *Proceedings of the American Mathematical Society*, vol. 4, pp. 506–510, 1953.

[8] H. H. Bauschke, E. Matoušková, and S. Reich, "Projection and proximal point methods: convergence results and counterexamples," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 56, no. 5, pp. 715–738, 2004.

[9] A. Genel and J. Lindenstrauss, "An example concerning fixed points," *Israel Journal of Mathematics*, vol. 22, no. 1, pp. 81–86, 1975.

[10] K. Nakajo and W. Takahashi, "Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups," *Journal of Mathematical Analysis and Applications*, vol. 279, no. 2, pp. 372–379, 2003.

[11] W. Takahashi, Y. Takeuchi, and R. Kubota, "Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 1, pp. 276–286, 2008.

[12] B. Halpern, "Fixed points of nonexpanding maps," *Bulletin of the American Mathematical Society*, vol. 73, pp. 957–961, 1967.

[13] R. Wittmann, "Approximation of fixed points of nonexpansive mappings," *Archiv der Mathematik*, vol. 58, no. 5, pp. 486–491, 1992.

[14] N. Shioji and W. Takahashi, "Strong convergence of approximated sequences for nonexpansive mappings in danach spaces," *Proceedings of the American Mathematical Society*, vol. 125, no. 12, pp. 3641–3645, 1997.

[15] C. Martinez-Yanes and H.-K. Xu, "Strong convergence of the CQ method for fixed point iteration processes," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 64, no. 11, pp. 2400–2411, 2006.

[16] X. Qin and Y. Su, "Strong convergence theorems for relatively nonexpansive mappings in a Banach space," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 67, no. 6, pp. 1958–1965, 2007.

[17] S. Plubtieng and K. Ungchittrakool, "Strong convergence theorems for a common fixed point of two relatively nonexpansive mappings in a Banach space," *Journal of Approximation Theory*, vol. 149, no. 2, pp. 103–115, 2007.

[18] K. Nakajo, K. Shimoji, and W. Takahashi, "Strong convergence theorems by the hybrid method for families of nonexpansive mappings in Hilbert spaces," *Taiwanese Journal of Mathematics*, vol. 10, no. 2, pp. 339–360, 2006.

[19] K. Nakajo, K. Shimoji, and W. Takahashi, "On strong convergence by the hybrid method for families of mappings in Hilbert spaces," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 71, no. 1-2, pp. 112–119, 2009.

[20] S. Plubtieng and K. Ungchittrakool, "Hybrid iterative methods for convex feasibility problems and fixed point problems of relatively nonexpansive mappings in banach spaces," *Fixed Point Theory and Applications*, vol. 2008, Article ID 583082, 19 pages, 2008.

[21] S. Plubtieng and W. Sriprad, "Hybrid methods for equilibrium problems and fixed points problems of acountable family of relatively nonexpansive mappings in Banach spaces," *Fixed Point Theory and Applications*, vol. 2010, Article ID 962628, 17 pages, 2010.

[22] L. Wei and H. Zhou, "Strong convergence of projection scheme for zeros of maximal monotone operators," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 71, no. 1-2, pp. 341–346, 2009.

[23] W. Takahashi, *Convex Analysis and Approximation Fixed Points*, vol. 2 of *Mathematical Analysis Series*, Yokohama, Yokohama, Japan, 2000.

[24] Y. I. Alber, "Metric and generalized projection operators in Banach spaces: properties and applications," in *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*, A. G. Kartsatos, Ed., vol. 178 of *Lecture Notes in Pure and Applied Mathematics*, pp. 15–50, Marcel Dekker, New York, NY, USA, 1996.

[25] S. Kamimura and W. Takahashi, "Strong convergence of a proximal-type algorithm in a Banach space," *SIAM Journal on Optimization*, vol. 13, no. 3, pp. 938–945, 2003.

[26] R. T. Rockafellar, "On the maximality of sums nonlinear monotone operators," *Transactions of the American Mathematical Society*, vol. 149, pp. 75–88, 2006.

[27] W. Nilsrakoo and S. Saejung, "Strong convergence theorems for a countable family of quasi-Lipschitzian mappings and its applications," *Journal of Mathematical Analysis and Applications*, vol. 356, no. 1, pp. 154–167, 2009.

[28] E. Blum and W. Oettli, "From optimization and variational inequalities to equilbrium problems," *Math Students*, vol. 63, pp. 123–145, 1994.